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ENDPOINT ESTIMATES FOR MULTILINEAR FRACTIONAL INTEGRALS

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Abstract

We study the boundedness for multilinear fractional integrals on spaces as Morrey spaces and Lipschitz spaces.

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1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $(\mathbb{R}^n)^m = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the m-fold product space. The multilinear fractional integral is defined by

$$I_{\alpha,m}(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{|(x - y_1, \ldots, x - y_m)|^{mn-\alpha}} \, dy_1 \cdots dy_m,$$

where $0 < \alpha < mn,\ m \in \mathbb{N}$.

Obviously, the multilinear fractional integral $I_{\alpha,m}$ is a natural generalization of the classical fractional integral I_α. Kenig and Stein [6] as well as Grafakos and Kalton [3] considered the boundedness of a family of related fractional integrals. The main purpose of this paper is to establish some endpoint estimates for the multilinear fractional integral. Before stating our results, let us first introduce some notation.

We first recall the definition of the Morrey space [7]. For $1 \le q \le p < \infty$, the Morrey space $M_q^p(\mathbb{R}^n)$ defined as the sets of functions $f(x) \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_q^p(\mathbb{R}^n)} := \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} \frac{1}{r^{n/q-n/p}} \left(\int_{B(x_0,r)} |f(y)|^q \, dy \right)^{1/q} < \infty,$$

where $B(x_0, r)$ denotes the ball in $\mathbb{R}^n$ with center x_0 and radius r.

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REMARK. It is easy to see that the relation $M^p_{q_1}(\mathbb{R}^n) \subset M^p_{q_2}(\mathbb{R}^n)$ holds with $1 \le q_2 \le q_1 \le p < \infty$, and $M^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. In addition, we know that $L^{p,\infty}(\mathbb{R}^n)$ is contained in $M^p_q(\mathbb{R}^n)$ with $1 \le q < p < \infty$ (see [5, Lemma 1.7]). More precisely, $\|f\|_{M^p_q(\mathbb{R}^n)} \le C\|f\|_{L^{p,\infty}(\mathbb{R}^n)}$ with $1 \le q < p < \infty$, here and in what follows, the letter C will denote a constant, not necessarily the same in different occurrences, and let p' satisfy $1/p + 1/p' = 1$ with $p \ge 1$.

We say that f belongs to weak Morrey space $WM^p_q(\Omega)$ if $1 \le q \le p < \infty$,

$$\|f\|_{WM^p_q(\Omega)} := \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} \frac{1}{r^{n/q-n/p}} \sup_{\lambda>0}(\lambda^q |\{x \in B(x_0, r) \cap \Omega : |f(x)| > \lambda\}|)^{1/q} < \infty.$$

Let $0 \le \beta < 1/n$, we say that $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathcal{L}(\beta)$ if there exists some constant C_1 such that for any ball B,

$$\frac{1}{|B|^{1+\beta}} \int_B |f(x) - m_B(f)|\, dx \le C_1,$$

where $m_B(f) = (1/|B|) \int_B f(x)\, dx$. The smallest constant C_1 will be denoted by $\|f\|_{\mathcal{L}(\beta)}$; see [7].

Let us observe that for $\beta = 0$, the space $\mathcal{L}(\beta)$ coincides with the version $BMO(\mathbb{R}^n)$ space. Moreover, $\mathcal{L}(\beta)$ coincides with *Lip*(β) (Lipschitz integral space) when $0 < \beta < 1/n$; see [7].

Let us now formulate our results as follows.

THEOREM 1.1. *Let* $m \in \mathbb{N}$, $(m-1)n < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $1/q = 1/q_1 + \cdots + 1/q_m$ *with* $1 \le q_i < p_i < \infty$ *for* $i = 1, \ldots, m$. *If* $p = n/\alpha$, *then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{BMO(\mathbb{R}^n)} \le C \prod_{j=1}^m \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)}.$$

THEOREM 1.2. *Let* $m \in \mathbb{N}$, $0 < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $1/q = 1/q_1 + \cdots + 1/q_m$ *with* $1 \le q_i < p_i < \infty$ *for* $i = 1, \ldots, m$. *If* $n/\alpha < p$ *and* $0 < \alpha - n/p < 1$, *then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{Lip(\alpha-n/p)} \le C \prod_{j=1}^m \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)}.$$

In the endpoint case $p = n/\alpha$, Strichartz [9] proved the exponential integrability of $I_{\alpha,m}$ when $m = 1$. The following result extends Strichartz's result to the case $m \ge 2$.

THEOREM 1.3. *Let* $m \in \mathbb{N}$, $0 < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m = \alpha/n$ *with* $1 < p_i < \infty$ *for* $i = 1, \ldots, m$. *Let* B *be a ball of radius* R *in* $\mathbb{R}^n$ *and let* $f_j \in L^{p_j}(B)$

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be supported in B. Then there exist constants k_1, k_2 depending only on n, m, α, p and the p_j such that

$$\int_B \exp\left(k_1 \left(\frac{|I_{\alpha,m}(f_1, \ldots, f_m)(x)|}{\prod_{j=1}^{k} \|f_j\|_{L^{p_j}(B)}}\right)^{n/(mn-\alpha)}\right) dx \le k_2 R^n.$$

THEOREM 1.4. *Suppose that $m \in \mathbb{N}$, $0 < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_i \le p_i \le \infty$ for $i = 1, \ldots, m$. Assume that $1/s = 1/q - \alpha/n$ and $p < n/\alpha$.*

(a) *If each $q_i > 1$, then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{M_s^{pn/n-p\alpha}(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)}.$$

(b) *If $1 \le q_i < p_i$ and $s(1/q - 1/p) > q_i(1/q_i - 1/p_i)$ for $i = 1, \ldots, m$, then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{WM_s^{pn/(n-p\alpha)}(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)}.$$

2. The proofs of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. Given $\vec{f} = (f_1, \ldots, f_m)$, for any ball $B = B(x_0, r)$, it suffices to prove that the following inequality

$$\frac{1}{|B|} \int_B |I_{\alpha,m}(\vec{f})(x) - m_B(I_{\alpha,m}(\vec{f}))|\, dx \le C \prod_{j=1}^{m} \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)} \tag{2.1}$$

holds.

Define $f_j^0 = f_j \chi_{2B}$ and $f_j^\infty = f - f_j^0$ for $j = 1, \ldots, m$. Then

$$\begin{aligned}
&\frac{1}{|B|} \int_B |I_{\alpha,m}(\vec{f})(x) - m_B(I_{\alpha,m}(\vec{f}))|\, dx \\
&\quad \le \sum_{r_1,\ldots,r_m \in \{0,\infty\}} \frac{1}{|B|} \int_B |I_{\alpha,m}(f_1^{r_1}, \ldots, f_m^{r_m})(x) - m_B(I_{\alpha,m}(f_1^{r_1}, \ldots, f_m^{r_m}))|\, dx.
\end{aligned}$$

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First we estimate the term of the set corresponding to $r_1 = \cdots = r_m = 0$. Then

$$\begin{aligned}
\frac{1}{|B|}&\int_B |I_{\alpha,m}(f_1^0, \ldots, f_m^0)(x) - m_B(I_{\alpha,m}(f_1^0, \ldots, f_m^0))|\,dx \\
&\le \frac{C}{|B|}\int_B |I_{\alpha,m}(f_1^0, \ldots, f_m^0)(x)|\,dx \\
&\le \frac{C}{|B|}\int_B |x-y_1|^{\alpha-mn}\,dx \prod_{j=1}^m \int_{2B} |f_j(y_j)|\,dy_j \\
&\le C\prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)}, \qquad (2.2)
\end{aligned}$$

since $0 < mn - \alpha < n$.

Consider first the case where exactly l of the r_j are ∞ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the argument below by permuting the indices. We now estimate the term

$$\begin{aligned}
I := |I_{\alpha,m}&(f_1^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)(x) \\
&- I_{\alpha,m}(f_1^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)(y)|.
\end{aligned}$$

In fact, let $\alpha = \sum_{i=1}^l \alpha_i$ and $0 < \alpha_i < (mn+1)/l - n(1-1/p_i)$ for $i = 1, \ldots, m$. For any $x,\ y \in B$, then

$$\begin{aligned}
|I| \le C &\prod_{j=l+1}^m \int_{2B} |f_j(y_j)|\,dy_j \int_{(\mathbb{R}^n)^l} \frac{|x-y|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha+1}} \\
&\times \prod_{k=1}^l f_k^\infty(y_k)|\,dy_1 \cdots dy_l \\
&\le Cr \prod_{j=l+1}^m r^{n(1-1/p_j)} \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)} \prod_{k=1}^l \int_{\mathbb{R}^n \setminus 2B} \frac{|f_k(y_k)|}{|x-y_k|^{(mn+1)/l-\alpha_k}}\,dy_k \\
&\le C\prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)}. \qquad (2.3)
\end{aligned}$$

It remains to estimate the last term

$$II := |I_{\alpha,m}(f_1^\infty, \ldots, f_m^\infty)(x) - I_{\alpha,m}(f_1^\infty, \ldots, f_m^\infty)(y)|.$$

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Let $\alpha = \sum_{i=1}^m \alpha_i$ with $\alpha_i = n/p_i$ for $i = 1, \ldots, m$. For any $x,\ \ y \in B$, then

$$\begin{aligned}|II| &\le \int_{(\mathbb{R}^n)^m} \frac{|x-y|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha+1}} \prod_{j=1}^m f_j^\infty(y_j)|\, dy_1 \cdots dy_l \\ &\le Cr \prod_{j=1}^m \int_{\mathbb{R}^n \setminus 2B} \frac{|f_j(y_j)|}{|x-y_j|^{(mn+1)/m-\alpha_i}}\, dy_j \\ &\le C \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)}. \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), then (2.1) holds. Thus, Theorem 1.1 is proved.

From Theorem 1.1 and Remark , we have the following result.

COROLLARY 2.1. *Let $\alpha,\ \ p,\ \ p_j$ be as in Theorem 1.1, then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{BMO(\mathbb{R}^n)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j,\infty}(\mathbb{R}^n)}.$$

PROOF OF THEOREM 1.2. For any $x,\ \ y \in \mathbb{R}^n$, we only need to prove

$$|I_{\alpha,m}(f_1, \ldots, f_m)(x) - I_{\alpha,m}(f_1, \ldots, f_m)(y)| \le C|x-y|^{\alpha-n/p} \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mathbb{R}^n)}. \tag{2.5}$$

Let $B = B(x, r)$ with $r = |x-y|$. Define $f_j^0 = f_j \chi_{2B}$ and $f_j^\infty = f - f_j^0$ for $j = 1, \ldots, m$. Then

$$\begin{aligned}&|I_{\alpha,m}(f_1, \ldots, f_m)(x) - I_{\alpha,m}(f_1, \ldots, f_m)(y)| \\ &\quad \le \sum_{r_1,\ldots,r_m \in \{0,\infty\}} |I_{\alpha,m}(f_1^{r_1}, \ldots, f_m^{r_m})(x) - I_{\alpha,m}(f_1^{r_1}, \ldots, f_m^{r_m})(y)|.\end{aligned}$$

First we estimate the term of the set corresponding to $r_1 = \cdots = r_m = 0$. Then

$$\begin{aligned}&|I_{\alpha,m}(f_1^0, \ldots, f_m^0)(x) - I_{\alpha,m}(f_1^0, \ldots, f_m^0)(y)| \\ &\quad \le \int_{(\mathbb{R}^n)^m} \frac{|f_1^0(y_1) \cdots f_m^0(y_m)|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}}\, dy_1 \cdots dy_m \\ &\qquad + \int_{(\mathbb{R}^n)^m} \frac{|f_1^0(y_1) \cdots f_m^0(y_m)|}{|(y-y_1, \ldots, y-y_m)|^{mn-\alpha}}\, dy_1 \cdots dy_m \\ &\quad := E_1 + E_2.\end{aligned}$$

Since the proof of E_1 is similar to E_2, we only give a proof of E_1. Let $\alpha = \sum_{i=1}^m \alpha_i$ with $n/p_i < \alpha_i < n$ for $i = 1, \ldots, m$. Then

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$$
\begin{aligned}
E_1 &\leq C \prod_{j=1}^{m} \int_{2B} \frac{|f_j(y_j)|}{|x-y_j|^{n-\alpha_j}}\, dy_j \\
&\leq C \prod_{j=1}^{m} r^{\alpha_j - n/p_j} \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)} \\
&\leq C r^{\alpha - n/p} \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)}. \qquad (2.6)
\end{aligned}
$$

Consider first the case where exactly l of the r_j are ∞ for some $1 \leq l < m$. We give the arguments for one of these cases. The rest are similar. Similar to the proof of I in Theorem 1.1, let $\alpha = \sum_{i=1}^{l} \alpha_i$ and $0 < \alpha_i < (mn+1)/l - n(1-1/p_i)$ for $i = 1, \ldots, m$, then

$$
\begin{aligned}
&|I_{\alpha,m}(f_1^{\infty}, \ldots, f_l^{\infty}, f_{l+1}^{0}, \ldots, f_m^{0})(x) - I_{\alpha,m}(f_1^{\infty}, \ldots, f_l^{\infty}, f_{l+1}^{0}, \ldots, f_m^{0})(y)| \\
&\quad \leq C \prod_{j=l+1}^{m} \int_{2B} |f_j(y_j)|\, dy_j \int_{(\mathbb{R}^n)^l} \frac{|x-y|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha+1}} \\
&\qquad \times \prod_{k=1}^{l} |f_k^{\infty}(y_k)|\, dy_1 \cdots dy_l \\
&\quad \leq Cr \prod_{j=l+1}^{m} r^{n(1-1/p_j)} \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)} \prod_{k=1}^{l} \int_{\mathbb{R}^n \setminus 2B} \frac{|f_k(y_k)|}{|x-y_k|^{(mn+1)/l - \alpha_i}}\, dy_k \\
&\quad \leq C r^{\alpha - n/p} \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)}. \qquad (2.7)
\end{aligned}
$$

Now we estimate the last term. Let $\alpha = \sum_{i=1}^{m} \alpha_i$ and $0 < \alpha_i - n/p_i < 1/n$ for $i = 1, \ldots, m$. Then

$$
\begin{aligned}
&|I_{\alpha,m}(f_1^{\infty}, \ldots, f_m^{\infty})(x) - I_{\alpha,m}(f_1^{\infty}, \ldots, f_m^{\infty})(y)| \\
&\quad \leq Cr \prod_{j=1}^{m} \int_{\mathbb{R}^n \setminus 2B} \frac{|f_j(y_j)|}{|x-y_j|^{(mn+1)/m - \alpha_i}}\, dy_j \\
&\quad \leq C r^{\alpha - n/p} \prod_{j=1}^{m} \|f_j\|_{M^{p_j}_{q_j}(\mathbb{R}^n)}. \qquad (2.8)
\end{aligned}
$$

Combining (2.6), (2.7) and (2.8), then (2.5) holds. Thus, Theorem 1.2 is proved.

We remark that when $p = \infty$ in Theorem 1.2, the conclusion also holds. Indeed, the proof is similar to the case $p < \infty$, moreover the proof is simpler.

From Theorem 1.2 and Remark , we have the following result.

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COROLLARY 2.2. *Let α, p, p_j be same as Theorem 1.2, then*

$$\|I_{\alpha,m}(f_1,\ldots,f_m)\|_{Lip(\alpha-n/p)} \le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\infty}(\mathbb{R}^n)}.$$

3. The proofs of Theorems 1.3 and 1.4

PROOF OF THEOREM 1.3. Let us first assume $\|f_j\|_{L^{p_j}(B)}=1$, $j=1,\ldots,m$, then for any $\delta>0$ and $x\in B$,

$$\begin{aligned}
&|I_{\alpha,m}(f_1,\ldots,f_m)(x)|\\
&\quad\le\int_{|(x-y_1,\ldots,x-y_m)|<\delta}\frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m\\
&\qquad+\int_{|(x-y_1,\ldots,x-y_m)|\ge\delta}\frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m\\
&\quad:=F_1+F_2.
\end{aligned}$$

For F_1, let $\alpha=\sum_{i=1}^m\alpha_i$ with $\alpha_i=n/p_i$ for $i=1,\ldots,m$. Then

$$\begin{aligned}
F_1 &\le C\prod_{j=1}^{m}\int_{|x-y_j|<\delta}\frac{|f_j(y_j)|}{|x-y_j|^{n-\alpha_j}}\,dy_j\\
&\le C\prod_{j=1}^{m}\delta^{\alpha_j}M(f_j)(x)\\
&:=C_1\delta^{\alpha}\prod_{j=1}^{m}M(f_j)(x),
\end{aligned}\tag{3.1}$$

where M denotes the standard Hardy–Littlewood maximal function.

For F_2, if $(y_1,\ldots,y_m)$ satisfies $|x-y_1|^2+\cdots+|x-y_m|^2\ge\delta^2$, then for some j, say $j=m$,

$$|x-y_j|=|x-y_m|\ge\delta/\sqrt{m}.$$

Thus,

$$F_2\le\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}}\frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m.$$

Define $f_j^0=f_j\chi_{B(x,\delta/\sqrt{m})}$ and $f_j^\infty=f-f_j^0$ for $j=1,\ldots,m$. Then

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$$F_2 \le C \sum_{r_1,\ldots,r_m\in\{0,\infty\}} \int_{\delta/\sqrt{m}\le|x-y_m|\le 2R} \int_{(\mathbb{R}^n)^{m-1}} \frac{|f_1^{r_1}(y_1)\cdots f_m^{r_m}(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} \times dy_1\cdots dy_m.$$

We first consider the case $r_1=\cdots=r_{m-1}=0$. Then,

$$\begin{aligned}
&\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R} \int_{(\mathbb{R}^n)^{m-1}} \frac{|f_1^0(y_1)\cdots f_{m-1}^0(y_{m-1})f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m\\
&\le C\prod_{j=1}^{m-1}\int_{|x-y_j|<\delta}|f_j(y_j)|\,dy_j \int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\frac{|f_m(y_m)|}{|x-y_m|^{mn-\alpha}}\,dy_m\\
&\le C\prod_{j=1}^{m-1}\delta^{n/p_j'}\|f_j\|_{L^{p_j}(B)}\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\frac{|f_m(y_m)|}{|x-y_m|^{mn-\alpha}}\,dy_m\\
&\le C\prod_{j=1}^{m-1}\delta^{n/p_j'}\|f_j\|_{L^{p_j}(B)}\delta^{\alpha-mn+n/q_m'}\|f_m\|_{L^{p_m}(B)}\\
&\le C. \qquad (3.2)
\end{aligned}$$

Consider first the case where exactly l of the r_j are ∞ for some $1\le l<m$. We only give the argument for one of these cases. Let $\alpha=\sum_{i=1}^m\alpha_i$ with $\alpha_i=n/p_i$ for $i=1,\ldots,m$. Then

$$\begin{aligned}
&\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R} \int_{(\mathbb{R}^n)^{m-1}} \frac{|f_1^\infty(y_1)\cdots f_l^\infty(y_l)f_{l+1}^0(y_{l+1})\cdots f_m^0(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m\\
&\le C\prod_{j=l+1}^{m-1}\delta^{n/p_j'}\|f_j\|_{L^{p_j}(B)}\prod_{k=1}^{l}\int_{\delta/\sqrt{m}\le|x-y_k|\le 2R}\frac{|f_m(y_k)|}{|x-y_k|^{n-\alpha_k}}\,dy_k\\
&\quad\times\int_{\delta/\sqrt{m}\le|x-y_k|\le 2R}\frac{|f_m(y_m)|}{|x-y_m|^{(m-l)n-\sum_{k=l+1}^m\alpha_k}}\,dy_m\\
&\le C\prod_{j=l+1}^{m-1}\delta^{n/p_j'}\|f_j\|_{L^{p_j}(B)}\prod_{k=1}^{l}\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{1/p_k'}\|f_k\|_{L^{p_k}(B)}\\
&\quad\times\delta^{-[(m-l)n-\sum_{k=l+1}^m\alpha_k+n/p_m']}\|f_m\|_{L^{p_m}(B)}\\
&\le C\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{\sum_{k=1}^l\frac{1}{p_k'}}\prod_{j=1}^m\|f_j\|_{L^{p_j}(B)}\\
&\le C\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n}. \qquad (3.3)
\end{aligned}$$

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Now, we estimate the last term. Let α_i as above, then

$$\begin{aligned}
&\int_{\delta/\sqrt{m}\le|x-y_m|\le 2R}\int_{(\mathbb{R}^n)^{m-1}} \frac{|f_1^\infty(y_1)\cdots f_m^\infty(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m\\
&\le C\prod_{j=1}^m \int_{\delta/\sqrt{m}\le|x-y_k|\le 2R} \frac{|f_j(y_j)|}{|x-y_j|^{n-\alpha_j}}\,dy_j\\
&\le C\prod_{j=1}^m \left[\log\frac{2\sqrt{m}R}{\delta}\right]^{1/q_j'} \|f_j\|_{L^{p_j}(B)}\\
&= C\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n}. \qquad (3.4)
\end{aligned}$$

Combining (3.2), (3.3) and (3.4), we obtain

$$F_2 \le C_2\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n}.$$

Thus, by (3.1) and the above, we obtain

$$|I_{\alpha,m}(f_1,\ldots,f_m)(x)| \le C_1\delta^\alpha\prod_{j=1}^m M(f_j)(x) + C_2\left[\log\frac{2\sqrt{m}R}{\delta}\right]^{(mn-\alpha)/n} \qquad (3.5)$$

provided $x\in B$ and $0<\delta\le 2\sqrt{m}R$. In particular, the choice of $\delta = 2\sqrt{m}R$ yields for all $x\in B$,

$$|I_{\alpha,m}(f_1,\ldots,f_m)(x)| \le C_1\delta^\alpha\prod_{j=1}^m M(f_j)(x).$$

Hence, the election of

$$\delta=\delta(x)=\epsilon\left[|I_{\alpha,m}(f_1,\ldots,f_m)(x)|/C_1\prod_{j=1}^m M(f_j)(x)\right]^{1/\alpha}$$

will satisfy $\delta\le 2\sqrt{m}R$ for all $\epsilon\le 1$. Now, (3.5) implies

$$\begin{aligned}
&|I_{\alpha,m}(f_1,\ldots,f_m)(x)| \le \epsilon^\alpha|I_{\alpha,m}(f_1,\ldots,f_m)(x)|\\
&\quad + C_2\left[\frac{1}{n}\log\left(\frac{(2\sqrt{m}R)^n C_1^{n/\alpha}[\prod_{j=1}^m M(f_j)(x)]^{n/\alpha}}{\epsilon^n|I_{\alpha,m}(f_1,\ldots,f_m)(x)|^{n/\alpha}}\right)\right]^{(mn-\alpha)/n}. \qquad (3.6)
\end{aligned}$$

If we use the notation $C_3=(1-\epsilon^\alpha)^{n/(mn-\alpha)}$, (3.6) is equivalent to

$$k_1|I_{\alpha,m}(f_1,\ldots,f_m)(x)|^{n/(mn-\alpha)} \le \log\left(\frac{C_4[\prod_{j=1}^m M(f_j)(x)]^{n/\alpha}}{|I_{\alpha,m}(f_1,\ldots,f_m)(x)|^{n/\alpha}}\right), \qquad (3.7)$$

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where $k_1 = nC_3/C_1^{n/(mn-\alpha)}$ and $C_4 = (2\sqrt{m}R)^n \epsilon^{-n} C_1^{n/\alpha}$. By exponentiating (3.7), we obtain

$$\exp(k_1|I_{\alpha,m}(f_1, \ldots, f_m)(x)|^{n/(mn-\alpha)}) \le \frac{C_4[\prod_{j=1}^m M(f_j)(x)]^{n/\alpha}}{|I_{\alpha,m}(f_1, \ldots, f_m)(x)|^{n/\alpha}}. \tag{3.8}$$

Let $B_1 = \{x \in B : |I_{\alpha,m}(f_1, \ldots, f_m)(x)| \ge 1\}$ and $B_2 = B \setminus B_1$. By (3.8),

$$\begin{aligned}\int_{B_1} \exp(k_1|I_{\alpha,m}(f_1, \ldots, f_m)(x)|^{n/(mn-\alpha)})\,dx &\le C_4 \int_{B_1} \left(\prod_{j=1}^m M(f_j)(x)\right)^{n/\alpha} dx \\ &\le C_4 \left(\prod_{j=1}^m \|M(f_j)\|_{L^{p_j}(B)}\right)^{n/\alpha} \\ &\le C_5 R^n,\end{aligned}$$

where $C_5 = C(2\sqrt{m})^n \epsilon^{-n} C_1^{n/\alpha}$. On the other hand,

$$\int_B \exp(k_1|I_{\alpha,m}(f_1, \ldots, f_m)(x)|^{n/(mn-\alpha)})\,dx \le \exp(k_1)|B_2| \le C_6 R^n. \tag{3.9}$$

Thus, adding the integrals above over B_1 and B_2,

$$\int_{B_2} \exp(k_1|I_{\alpha,m}(f_1, \ldots, f_m)(x)|^{n/(mn-\alpha)})\,dx \le K_2 R^n.$$

Let us now turn to the general $\vec{f} = (f_1, \ldots, f_m)$. If $\|f_j\|_{L^{p_j}(B)} \ne 1$ for $j = 1, \ldots, m$, then we use the notation $g_j = f_j/\|f_j\|_{L^{p_j}(B)}$ and $\vec{g} = (g_1, \ldots, g_m)$. Obviously,

$$I_{\alpha,m}(\vec{g}) = I_{\alpha,m}(\vec{f}) \Big/ \prod_{j=1}^m \|f_j\|_{L^{p_j}(B)}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\int_B \exp\left(k_1 \left(\frac{I_{\alpha,m}(\vec{f})(x)}{\prod_{j=1}^k \|f_j\|_{L^{p_j}(B)}}\right)^{n/(mn-\alpha)}\right) dx \le k_2 R^n,$$

which is our assertion. Thus, Theorem 1.3 is proved.

The proof of Theorem 1.4 is rather trivial. It is a simple consequence of Hölder's inequality on weak L^p spaces which is a standard fact. Hence, we omit the details here.

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References

[1] F. Chiarenza and M. Frasca, 'Morrey spaces and Hardy–Littlewood maximal function', *Rend. Mat. Appl.* **7**(3–4) (1987), 273–279.

[2] L. Grafakos, 'On multilinear fractional integrals', *Studia Math.* **102** (1992), 49–56.

[3] L. Grafakos and N. Kalton, 'Some remarks on multilinear maps and interpolation', *Math. Ann.* **319** (2001), 151–180.

[4] L. Grafakos and R. Torres, 'Multilinear Calderón–Zygmund theory', *Adv. Math.* **165** (2002), 124–164.

[5] H. Kozono and M. Yamazaki, 'Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data', *Comm. Partial Differential Equations* **19** (1994), 959–1014.

[6] C. Kenig and E. Stein, 'Multilinear estimates and fractional integration', *Math. Reseach. Letter* **6** (1999), 1–15.

[7] J. Peetre, 'On the theory of $L^{p,\lambda}$ spaces', *J. Funct. Anal.* **4** (1969), 71–87.

[8] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* (Princeton University Press, Princeton, NJ, 1993).

[9] R. S. Strichartz, 'A note on Trudinger's extension of Sobolev's inequalities', *Indiana Univ. Math. J.* **21** (1972), 841–842.

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