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A modulus of 3-dimensional vector fields

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Abstract. In this paper, we prove that μ/λ is a modulus for a Šilnikov system with eigenvalues λ and $-\mu \pm i\omega$. To prove this we define a number using knot and link invariants of periodic orbits, which is related to the ratio of eigenvalues μ/λ.

1. *Introduction*

Let X be a C^r-vector field on $\mathbb{R}^3$, $r > 0$. A hyperbolic stationary point p is of Šilnikov type if

(i) $DX(p)$ has eigenvalues λ, $-\mu \pm i\omega$ with λ and μ positive and ω non zero;

(ii) $\mu < \lambda$;

(iii) $W^s(p)$ and $W^u(p)$ intersect at a homoclinic orbit Γ.

The purpose of this paper is to prove μ/λ is a modulus, that is:

THEOREM. *Let the C^r-vector field X_j $(j = 1, 2)$ have a C^1-linearizable stationary point p_j of Šilnikov type with eigenvalues λ_j, $-\mu_j \pm i\omega_j$. Suppose that $\mu_1/\lambda_1 \neq \mu_2/\lambda_2$. Then there is no homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(p_1) = p_2$, and h maps each orbit of X_1 onto an orbit of X_2.*

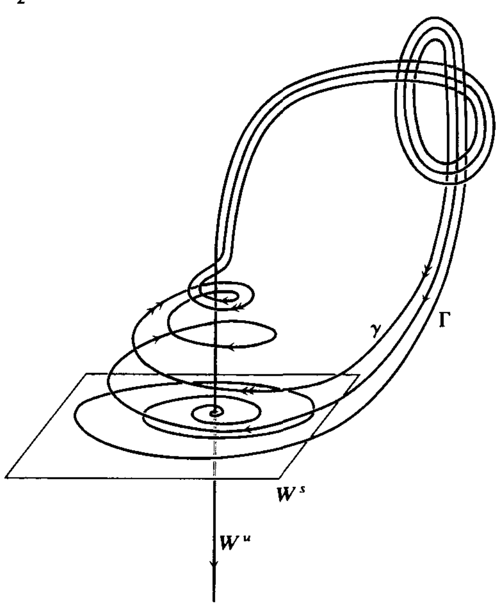

FIGURE 1

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To prove the theorem, we consider the 'period 2 (as the Poincaré map) orbits'; see figure 1. The theorem of Šilnikov implies that there are infinitely many such orbits arbitrarily close to the homoclinic orbit Γ. We use some knot invariants of the periodic orbits to count the number of twists around the homoclinic orbit Γ. The ratio of twists in the first and the second turns determines the ratio of eigenvalues λ and μ. Therefore μ/λ is determined by the topological data of the phase portrait, that is, μ/λ is a modulus.

For diffeomorphisms on n-manifolds it is known that if $W^s(p)$ and $W^u(p)$ intersect non-transversally, where p and q are stationary points, then there exists a modulus; see **[8]** and **[9]**. For vector fields the same question can be studied and in fact for the codimension one case without cycles the existence or not of moduli has been settled through the work in **[13]**, **[14]**, **[12]** and **[1]**. However the techniques have to be somewhat different from the ones for diffeomorphisms: the main reason for this is that the topological equivalence does not require to preserve the 'time length'. In our three dimensional case, we can use some knot invariants to measure the time length.

It is one of the peculiarities of the three dimensional systems that the embedding problem of the periodic orbits has a definite meaning. Joan S. Birman and R. F. Williams have been studying the knotted periodic orbits in the Lorenz systems in **[2]** and **[3]**. See also **[7]** for an application to periodic systems.

The viewpoint of knot theory would be also useful for the study of the perturbations of Šilnikov type systems. The homoclinic orbit Γ can be a complicated knot, since by small perturbations one can make subsidiary homoclinic orbits (see **[5]** and **[6]**) whose knots are of cable type about Γ, and from them one can make more complicated homoclinic orbits. One can also make attracting periodic orbits by small perturbations, if $\mu<\lambda<2\mu$. It would be particularly interesting to study the knot types of these attracting periodic orbits, since it would be used to define a modulus of Šilnikov bifurcations which depends only on the embedding types of attractors.

After I submitted this manuscript, I learned about the work of several people on moduli. In particular Van Strien, who told me that his methods in **[12]** could be extended to provide another proof of the present result (it seems that his method could be extended even to the higher dimensional cases, possibly without the assumption $\mu<\lambda$). However the knot theoretical approach would have an advantage on the application to attractor problem.

2. *A topological invariant*

In this section we shall define a topological invariant $\eta_{X,p}$.

A regular neighbourhood of the homoclinic orbit Γ is a pair (V, ϕ), where

(i) V is a compact neighbourhood of p on which X is topologically linearizable;

(ii) $\phi: D^2\times(T_-, T_+)\to\mathbb{R}^3$, $T_-<0<T_+$, is a homeomorphism onto its image which maps each line $\{(x, t)\mid T_-<t<T_+\}$, $x\in D^2$, into an orbit; and

(iii) $V\cup \operatorname{image}\phi$ is a neighbourhood of Γ.

Let $\mathcal{T}$ be the set of regular neighbourhoods of Γ. For each $(V, \phi)\in\mathcal{T}$, let $P^2(V, \phi)$

be the set of periodic orbits γ of X such that

(i) $\gamma \subseteq V \cup \text{image } \phi$; and

(ii) γ intersects the section $\{\phi(x, 0) \mid x \in D^2\}$ at just two points.

Let link (γ, Γ) denote the linking number of γ and Γ. For a positive number n, let $P_n^2(V, \phi)$ be the set of periodic orbits γ in $P^2(V, \phi)$ such that link $(\gamma, \Gamma) > n$.

Let $\Delta_\gamma(t)$ be the Alexander polynomial of the knot γ, and let deg $(\Delta_\gamma(t))$ be the degree of this polynomial (see [4] and [**10**]). We set

$$\eta_n(V, \phi) = \inf \frac{\deg(\Delta_\gamma(t))}{\operatorname{link}(\gamma, \Gamma)},$$

where the infimum is over all γ's in $P_n^2(V, \phi)$. Let

$$\eta(V, \phi) = \lim_{n\to\infty} \eta_n(V, \phi).$$

It may seem that $\eta(V, \phi)$ depends on (V, ϕ). However, it follows from the following proposition that this is not the case. We can therefore denote $\eta(V, \phi)$ by $\eta_{X,p}$, and get a topological invariant.

PROPOSITION 1. *Let (V_1, ϕ_1) and (V_2, ϕ_2) be in $\mathcal{T}$. Then $P_n^2(V_1, \phi_1) = P_n^2(V_2, \phi_2)$ for sufficiently large n.*

Proof. There exists a neighbourhood V' of p such that if $\gamma \in P^2(V_1, \phi_1)$ runs through V' at both the first and second turns, then $\gamma \in P^2(V_2, \phi_2)$. Then we can choose a smaller neighbourhood V'' of p such that if $\gamma \in P^2(V_1, \gamma_1)$ runs through V'' once, then it runs through V' in the next turn. Therefore every $\gamma \in P^2(V_1, \gamma_1) - P^2(V_2, \phi_2)$ lies outside V''.

Now we show that link (γ, Γ) is bounded from above in $P^2(V_1, \phi_1) - P^2(V_2, \phi_2)$. Notice that the distance between γ and Γ is bounded from below by some positive number, since γ lies outside V''. Notice also that the length of γ is bounded from above. Using the analytic definition of linking number [**10**],

$$\operatorname{link}(\gamma, \Gamma) = \frac{1}{4\pi} \int_\gamma \int_\Gamma \frac{(x'-x)(dy\,dz' - dz\,dy') + (y'-y)(dz\,dx' - dx\,dz') + (z'-z)(dx\,dy' - dy\,dx')}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{3/2}},$$

we therefore see that link (γ, Γ) is bounded from above in $P^2(V_1, \phi_1) - P^2(V_2, \phi_2)$. Thus $P_n^2(V_1, \phi_1) - P_n^2(V_2, \phi_2)$ is empty for sufficiently large n. Similarly, $P_n^2(V_2, \phi_2) - P_n^2(V_1, \phi_1)$ is empty for sufficiently large n. Thus we conclude $P_n^2(V_1, \phi_1) = P_n^2(V_2, \phi_2)$ for sufficiently large enough n. □

3. *Šilnikov's theorem*

Without loss of generality we suppose that X is linear in the unit cylinder

$$S = \{(r, \theta, z) \mid 0 \le r \le 1, \theta \in \mathbb{R}, |z| \le 1\}.$$

X gives a linear differential equation in S,

$$\dot{r} = -\mu r, \quad \dot{\theta} = \omega, \quad \dot{z} = \lambda z.$$

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The solution of this equation with initial value (r_0, θ_0, z_0) is

$$r = r_0 e^{-\mu t}, \quad \theta = \theta_0 + \omega t, \quad z = z_0 e^{\lambda t}.$$

The orbit which starts at the point $(1, \theta_0, z_0)$, $z_0 > 0$, leaves the cylinder at the point $(e^{-\mu T}, \theta_0 + \omega T, 1)$, where T is given by $z_0 e^{\lambda T} = 1$. So we can define a map

$$L: \{(1, \theta, z) \,|\, \theta \in \mathbb{R}, 0 < z < 1\} \to \{(r, \theta, 1) \,|\, 0 < r \leq 1, \theta \in \mathbb{R}\}$$
$$L: (1, \theta, z) \to (z^{\mu/\lambda}, \theta - (\omega/\lambda) \log z, 1).$$

Notice that L maps each vertical line ($\theta = \text{const.}$) to a spiral and each horizontal line ($z = \text{const.}$) to a circle. Notice also that the orbit which starts from a point $(1, \theta, z)$ twists around the z-axis about $-(\omega/2\pi\lambda) \log z$ times in the cylinder.

Let

$$\Sigma^2 = \{(r, \theta, 1) \,|\, |r| < d_2, \theta \in \mathbb{R}\},$$

where d_2 is a sufficiently small positive number such that there exists a regular neighbourhood (V, ϕ) with:

$$\Sigma^2 = \{\phi(x, 0) \,|\, x \in D^2\},$$
$$V = \text{the unit cylinder } S.$$

Let $f: \Sigma^2 \to \{(1, \theta, z) \,|\, \theta \in \mathbb{R}, |z| \leq 1\}$ be the diffeomorphism onto its image defined by the flow in the tube. Let

$$\Sigma^1 = \{(1, \theta, z) \,|\, |\theta| \leq \pi/4, 0 < z \leq d_1\},$$

where d_1 is sufficiently small such that L maps Σ^1 into Σ^2. We use the same letter L for the restricted map $L|\Sigma^1$.

We now need more notation. Let

$$\Sigma^1(\varepsilon_1, \varepsilon_2) = \{(1, \theta, z) \,|\, |\theta| \leq \pi/4, \varepsilon_1 > z > \varepsilon_2\},$$
$$\Sigma^1(\varepsilon_1) = \Sigma^1(\varepsilon_1, 0),$$
$$l(\varepsilon) = \{(1, \theta, \varepsilon) \,|\, |\theta| \leq \pi/4\},$$

where ε_1, ε_2 and ε are positive numbers smaller than d_1.

Now we can state the theorem of Šilnikov [**11**] in the following form:

THEOREM (Šilnikov). *Under the above conditions, there exist positive numbers a_0, K and ε_0 such that for any ε, $0 < \varepsilon < \varepsilon_0$, the following conditions are satisfied*:

(1) *let $a_n = a_0 \, e^{-\pi n\lambda/\omega}$, and let N_1 and N_2 be integers such that*

$$a_{N_1 - 1} \leq \varepsilon < a_{N_1 - 2}, \quad a_{N_2 + 1} < K\varepsilon^{\lambda/\mu} \leq a_{N_2},$$

then

$$(f \circ L) l(a_n) \cap \Sigma^1(\varepsilon) = \varnothing$$

for all $N_1 - 1 \leq n \leq N_2$;

(2) *the dynamics of $f \circ L$ restricted to the invariant set in $\Sigma^1(\varepsilon, K\varepsilon^{\lambda/\mu})$ is a horse shoe with the 'horizontal strips' $H_{N_1}, H_{N_1+1}, \ldots, H_{N_2}$, where each H_n, $N_1 \leq n \leq N_2$ is the connected component of $(f \circ L)^{-1}(\Sigma^1(\varepsilon, K\varepsilon^{\lambda/\mu}))$ in $\Sigma^1(a_{n-1}, a_n)$.*

The proof of this theorem of Šilnikov is contained in [15]. For each periodic orbit γ in $P^2(U, \phi)$, let $M(\gamma)$ and $m(\gamma)$ be positive numbers such that

$$\gamma \cap \{(1, \theta, z) \mid \theta \in R, 0 < z \leq 1\} = \{(1, \theta_1, M(\gamma)), (1, \theta_2, m(\gamma))\}$$

for some θ_1 and θ_2. We assume $m(\gamma) \leq M(\gamma)$.

COROLLARY. $\lim_{\varepsilon \to 0} \inf (\log M(\gamma))/\log m(\gamma) = \mu/\lambda$, *where the infimum is over all* γ*'s in* $P^2(V, \phi)$ *with* $M(\gamma) < \varepsilon$.

Proof. The inequality

$$\lim_{\varepsilon \to 0} \inf (\log M(\gamma))/\log m(\gamma) \geq \mu/\lambda$$

follows from

$$M(\gamma) \leq (\sup \|Df\|)(m(\gamma))^{\mu/\lambda}$$

where supremum is over Σ^2, so we shall prove the opposite direction.

Given ε, $\varepsilon < \varepsilon_0$, let $H_{N_1}, \ldots, H_{N_2}$ be the horizontal strips as in the theorem above. Then there exists a periodic orbit γ which corresponds to the period-2-bisequence $\ldots H_{N_1}, H_{N_2}, H_{N_1}, H_{N_2} \ldots$, that is, γ intersects Σ^1 on H_{N_1} and H_{N_2}. Thus

$$a_{N_1} \leq M(\gamma) \leq a_{N_1 - 1}, \qquad a_{N_2} \leq m(\gamma) \leq a_{N_2 - 1},$$

and

$$\frac{\log a_{N_1}}{\log a_{N_2 - 1}} \geq \frac{\log M(\gamma)}{\log m(\gamma)}.$$

By the definition of the sequence a_n, we get

$$a_{N_2+1} \leq K\varepsilon^{1/\mu}, \quad \text{and} \quad a_{N_2+1} = e^{-2\pi\lambda/\omega} a_{N_2-1}.$$

Thus

$$a_{N_2-1} \leq K\, e^{2\pi\lambda/\omega} \varepsilon^{\lambda/\mu},$$

therefore

$$-\log a_{N_2-1} \geq -(\lambda/\mu) \log \varepsilon - \log K - 2\pi\lambda/\omega.$$

On the other hand,

$$a_{N_1-2} \geq \varepsilon \quad \text{and} \quad a_{N_1} = e^{-2\pi\lambda/\omega} a_{N_1-2}.$$

Hence

$$-\log a_{N_1} < -\log \varepsilon + 2\pi\lambda/\omega$$

From these inequalities we get

$$\frac{-\log \varepsilon + 2\pi\lambda\omega}{-(\lambda/\mu) \log \varepsilon - \log K - 2\pi\lambda/\omega} \geq \frac{\log a_{N_1}}{\log a_{N_2-1}} \geq \frac{\log M(\gamma)}{\log m(\gamma)}$$

As ε goes to zero, we get the required inequality. □

4. *Calculation of* $\eta(V, \phi)$

In this section we shall investigate the knots and links of orbits, and prove the theorem stated in the introduction.

Consider an orbit of X in the unit cylinder S which starts at a point $(1, \theta, z) \in \Sigma^1$. We define the number $N(z)$ by

$$N(z) = -(\omega/2\pi\lambda) \log z.$$

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Roughly speaking, $N(z)$ measures how many times the orbit twists around the z-axis. The linking number, link (γ, Γ), is determined not only by the number of linking in the cylinder, which is about $N(M(\gamma))-N(m(\gamma))$, but also by the behaviour of the flow in the tube. However, the flow is regular in the tube, thus we have a constant C such that

$$|N(M(\gamma))+N(m(\gamma))-\text{link}\,(\gamma,\Gamma)|\leq C$$

for all γ in $P^2(V, \phi)$. Hence we have:

PROPOSITION 2. *For any $\varepsilon>0$, there exists a number N_0 such that*

$$\left|\frac{\text{link}\,\gamma}{N(M(\gamma))+N(m(\gamma))}-1\right|<\varepsilon$$

for all $\gamma\in P^2(V,\phi)$ with $N(M(\gamma))\geq N_0$.

We now investigate the knot type of $\gamma\in P^2(V,\phi)$. First suppose that Γ is unknotted. Notice that the twists in the turn with $m(\gamma)$ can be unknotted and the knot of γ is determined only by the turn with $M(\gamma)$. See figure 2.

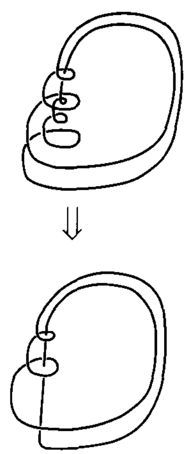

FIGURE 2

Hence there exists a constant C, which is determined by the twists in the tube, such that γ has the knot type of torus knot $T_{2,n}$ with

$$|N(M(\gamma))-n|\leq C.$$

In the general case in which Γ is knotted, we have a constant C such that γ is the $(2, n)$-cable about Γ, $K_{2,n}$, with

$$|N(M(\gamma))-n|\leq C.$$

Let $\Delta_\gamma(t)$ be the Alexander polynomial of the knot γ. Then

$$\Delta_\gamma(t)=\Delta_\Gamma(t^2)\Delta_{T_{2,n}}(t),$$

see [**4**, p. 144], and degree of $\Delta_{T_{2,n}}$ is $n-1$. Thus we have:

PROPOSITION 3. *For any $\varepsilon < 0$, there exists a number N_0 such that*

$$\left|\frac{\deg \gamma}{N(M(\gamma))} - 1\right| < \varepsilon$$

for all $\gamma \in P^2(V, \phi)$ with $N(M(\gamma)) \geq N_0$.

Now the theorem stated in the introduction follows from these propositions and the corollary of the theorem of Šilnikov. Since $M(\gamma) \to 0$ if and only if $\gamma \in P_n^2(V, \phi)$ with $n \to \infty$, we get

$$\begin{aligned}\eta(V, \phi) &= \liminf \deg \gamma / \mathrm{link}\,(\gamma, \Gamma) \\ &= \liminf\,(\log M(\gamma))/(\log m(\gamma) + \log m(\gamma)) \\ &= \frac{\mu/\lambda}{1 + \mu/\lambda}.\end{aligned}$$

Therefore μ/λ is determined by the topological invariant $\eta(V, \phi)$.

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