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NORMAL INVARIANTS OF LENS SPACES

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ABSTRACT. We show that normal and stable normal invariants of polarized homotopy equivalences of lens spaces $M = L(2^m; r_1, \ldots, r_n)$ and $N = L(2^m; s_1, \ldots, s_n)$ are determined by certain ℓ-polynomials evaluated on the elementary symmetric functions $\sigma_i(r_1^2, \ldots, r_n^2)$ and $\sigma_i(s_1^2, \ldots, s_n^2)$. Each polynomial ℓ_k appears as the homogeneous part of degree k in the Hirzebruch multiplicative L-sequence. When $n = 8$, the elementary symmetric functions alone determine the relevant normal invariants.

1**. Introduction.** The development of surgery theory and the study of lens spaces was largely motivated by questions concerning free actions of groups on spheres. For example, which groups admit such actions? Is it possible to classify all free actions on spheres in each of the smooth, piecewise linear and topological categories? If so, then what invariants are available to make this classification explicit?

Consider free actions of a cyclic group G. By the Lefschetz Fixed Point Theorem, only $\mathbf{Z}_2$ can act freely on an even-dimensional sphere. We therefore assume G to be acting on S^{2n-1}. If T is a generator of G, then the isomorphism $G \cong \pi_1(S^{2n-1}/G)$ determines a corresponding generator of the fundamental group of the quotient space. Fixing this generator and the orientation induced by S^{2n-1}, we obtain a *polarization* of S^{2n-1}/G. A map of polarized spaces which preserves the individual polarizations is itself said to be polarized.

Now let $\mathcal{C}$ denote one of the smooth, piecewise linear or topological categories. By definition, two actions $\mu_1, \mu_2 : G \times S^{2n-1} \longrightarrow S^{2n-1}$ are equivalent in $\mathcal{C}$ if there is an isomorphism (*i.e.*, diffeomorphism, PL homeomorphism or homeomorphism) φ of S^{2n-1} such that $\varphi \circ \mu_1(T, x) = \mu_2\big(T, \varphi(x)\big)$.

Since μ_1 and μ_2 are equivalent in $\mathcal{C}$ if and only if there is a polarized isomorphism of S^{2n-1}/μ_1 and S^{2n-1}/μ_2 in $\mathcal{C}$, one may hope to classify free actions of G on S^{2n-1} by classifying all polarized quotients S^{2n-1}/G.

If $G = \mathbf{Z}_d$ is a finite cyclic group, this quotient space is called a *fake lens space*. It is characterized by the fact that its universal cover is a sphere and its fundamental group is cyclic of finite order.

Free linear actions of $G = \mathbf{Z}_d$ on S^{2n-1} are defined in the following manner. Let $s_1, \ldots, s_n$ be integers coprime to d. For each $j = 1, \ldots, n$, let t^{s_j} be the free 2-dimensional real orthogonal representation of $\mathbf{Z}_d$ which is given by rotation through $\frac{2\pi s_j}{d}$ radians. As $s_1, \ldots, s_n$ are chosen prime to d, the $2n$-dimensional representation $t^{s_1} + \cdots + t^{s_n}$ is free and

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induces a free action of $\mathbf{Z}_d$ on the sphere $S(t^{s_1} + \cdots + t^{s_n}) \approx S^{2n-1}$. The resulting quotient space is called a *classical lens space* and is denoted $L(d; s_1, \ldots, s_n)$. The topology of this space depends on the choice of weights $\underline{s} = (s_1, \ldots, s_n)$; in particular, $\underline{s}$ determines a polarization of $L(d; s_1, \ldots, s_n)$. There is a polarized homotopy equivalence h between two lens spaces $L(d; r_1, \ldots, r_n)$ and $L(d; s_1, \ldots, s_n)$ if and only if $\prod r_j \equiv \prod s_j \bmod d$ (see [O]).

Wall [W] showed that every fake lens space is homotopy equivalent to a classical lens space. Hence the problem of classifying free actions of $\mathbf{Z}_d$ on S^{2n-1} is equivalent to that of classifying manifolds having the homotopy type of a classical lens space. Problems of this nature are addressed by surgery theory; hence it is important to compute the sets and maps occurring in the surgery exact sequence of $N = L(d; s_1, \ldots, s_n)$:

$$\cdots \longrightarrow S^C(N) \xrightarrow{\eta^C} T^C(N) \xrightarrow{\theta^C} L^h_{2n-1}(\pi_1 N).$$

Here $S^C(N)$ is the *structure set* of N, consisting of h-cobordism classes of manifolds homotopy equivalent to N, and $T^C(N)$ is the set of *normal invariants* or *normal cobordism classes* of degree 1 normal maps with target N. The basic elements of surgery may be found in [W], [tDH] and [Br].

Throughout the present paper, two lens spaces M and N will be referred to as normally cobordant if there exists a polarized homotopy equivalence $h\colon M \longrightarrow N$ such that (M, h) is normally cobordant to (N, id_N). In the topological category, normal cobordism classes of pairs (M, h) with target N are classified by homotopy classes of maps of N into a space F/Top (see [MM]). The latter space behaves well under localization at 2 and away from 2, thereby allowing one to compute the class of (M, h) as a combination of its even and odd parts. For an explanation of homotopy theoretic localization of spaces, the reader is referred to [Su] and [A].

Let $L(\cdot)$ denote the Hirzebruch multiplicative L-sequence evaluated on the Pontrjagin classes of a manifold. By a theorem of Szczarba [Sz], the integral Pontrjagin classes of the lens space $N = L(2^m; s_1, \ldots, s_n)$ are given by the formula:

$$p_i(N) = \sigma_i(s_1^2, \ldots, s_n^2) \cdot \omega^{2i}$$

where σ_i denotes the i-th elementary symmetric polynomial and $\omega \in H^2(N; \mathbf{Z})$ is a generator. We let

$$\ell_k(\underline{s}) = \ell_k\Big(\sigma_1(s_1^2, \ldots, s_n^2), \ldots, \sigma_k(s_1^2, \ldots, s_n^2)\Big),$$

where ℓ_k is the homogeneous polynomial of degree k appearing in the multiplicative sequence L. We shall also abbreviate $\sigma_i(s_1^2, \ldots, s_n^2)$ to $\sigma_i(\underline{s})$. Then, for example, the first three ℓ-polynomials are:

$$\ell_1(\underline{s}) = \frac{1}{3}\sigma_1(\underline{s})$$
$$\ell_2(\underline{s}) = \frac{1}{45}\Big(7\sigma_2(\underline{s}) - \big(\sigma_1(\underline{s})\big)^2\Big)$$
$$\ell_3(\underline{s}) = \frac{1}{945}\Big(62\sigma_3(\underline{s}) - 13\sigma_2(\underline{s}) \cdot \sigma_1(\underline{s}) + 2\big(\sigma_1(\underline{s})\big)^3\Big).$$

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Moreover, $L(N) = 1 + \sum_{k>0} \ell_k(\underline{s}) \cdot \omega^{2k}$.

Our main result concerns normal cobordism of classical lens spaces.

THEOREM 1.1. *Let $M = L(2^m; r_1, \ldots, r_n)$ and $N = L(2^m; s_1, \ldots, s_n)$. Suppose that $r_j \equiv s_j \equiv 1 \bmod 4$ for all j and further, that $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. Let $h: M \longrightarrow N$ be a polarized homotopy equivalence. Then (M, h) is normally cobordant to (N, id_N) if and only if $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ for $1 \leq k < \frac{2n-1}{4}$.*

REMARK. If M and N are homotopy equivalent polarized lens spaces with $\pi_1 M = \pi_1 N = \mathbf{Z}_{2^m}$ then, by Lemma 2.1 and its proof, we can always find weights $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ so that the hypotheses of Theorem 1.1 are satisfied. Therefore we may freely apply Theorem 1.1 to any classical lens spaces having fundamental group of order 2^m.

Following [CSSWW], M and N are said to be *stably normally cobordant* if the stabilizations $M_\rho = S(t^{r_1} + \cdots + t^{r_n} + \rho)/\mathbf{Z}_d$ and $N_\rho = S(t^{s_1} + \cdots + t^{s_n} + \rho)/\mathbf{Z}_d$ are normally cobordant for all free representations ρ of $\mathbf{Z}_d$. As a consequence of Theorem 1.1, we have:

THEOREM 1.2. *Let $M = L(2^m; r_1, \ldots, r_n)$ and $N = L(2^m; s_1, \ldots, s_n)$. Suppose that $r_j \equiv s_j \equiv 1 \bmod 4$ for all j and that $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. Then M and N are stably normally cobordant if and only if $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ for all $k > 0$.*

Let $\alpha(i)$ denote the number of non-trivial terms in the dyadic expression of the positive integer i. Using Theorem 3.1, we are able to classify stable normal cobordism classes of 15-dimensional lens spaces with 2-primary fundamental group in terms of the Pontrjagin classes of these spaces.

COROLLARY 1.3. *Let $n = 8$. Suppose $r_j \equiv s_j \equiv 1 \bmod 4$ for $j = 1, 2, \ldots, 8$ and further, that $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. The 15-dimensional lens spaces $M = L(2^m; r_1, \ldots, r_n)$, $N = L(2^m; s_1, \ldots, s_n)$ are stably normally cobordant if and only if $2^{\alpha(i)-1}\sigma_i(\underline{r}) \equiv 2^{\alpha(i)-1}\sigma_i(\underline{s}) \bmod 2^{m+3}$ for $1 \leq i \leq 8$.*

In particular, it is not necessary for the Pontrjagin classes of stably normally cobordant lens spaces to be congruent modulo 2^{m+3}. Theorem 1.2 thus corrects an earlier calculation of these stable normal invariants made in [CSSWW]. It would seem that the five authors neglected some powers of 2 which appear in the numerators of the ℓ-polynomials.

Our results only determine the normal invariants of homotopy equivalence between classical lens spaces. Therefore Theorem 1.1 does not completely compute the kernel of $\eta: S^{\mathrm{Top}}(N) \longrightarrow T^{\mathrm{Top}}(N)$ when $d = 2^m$. As far as we are aware, the homeomorphism classification of fake lens spaces is still unknown. The classification for d odd was completed by Browder, Petrie and Wall [BPW]: in this case, fake lens spaces are determined up to homeomorphism by their Reidemeister torsion and ρ-invariants.

2. **Proof of Theorem 1.1.** Henceforth M and N will denote classical lens spaces whose fundamental groups are of order 2^m, $m \in \mathbf{N}$. If $m = 1$, then M and N are simply projective spaces. If $m = 2$, then there is a polarized homotopy equivalence $h: M \xrightarrow{\simeq} N$ if and only if M and N are diffeomorphic. Therefore we will assume that m is at least 3.

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LEMMA 2.1. *Let* $M = L(2^m; r_1, \ldots, r_n)$ *and* $N = L(2^m; s_1, \ldots, s_n)$ *be lens spaces such that* $\pi_1(M) = \pi_1(N)$ *is cyclic of order* 2^m, $m \geq 3$. *Given a polarized homotopy equivalence* $h: M \longrightarrow N$, *there exist lens spaces* M_1, N_1 *and a polarized homotopy equivalence* $h_1: M_1 \longrightarrow N_1$ *such that*

i) M,N *are 8-fold covers of* M_1,N_1 *respectively*
ii) h *covers* h_1 *up to homotopy.*

PROOF. By the homotopy classification of lens spaces (see [O, Section 8]), $\prod r_j \equiv \prod s_j \bmod 2^m$. Therefore, up to simultaneous reversal of orientation, we may assume that $r_j \equiv s_j \equiv 1 \bmod 4$ for $j = 1, 2, \ldots, n$. Let z be an integer such that $\prod r_j - \prod s_j = 2^m z$ and choose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{Z}$ so that the relation

$$(1)\quad b_1 s_2 \cdots s_n + \cdots + s_1 s_2 \cdots s_{n-1} b_n - a_1 r_2 \cdots r_n - \cdots - r_1 \cdots r_{n_1} a_n \equiv z \pmod 8$$

is satisfied. For example, because $s_j \equiv 1 \pmod 4$ for all j, we may take $b_1 = z s_2 \cdots s_n$ and $b_2 = \cdots = b_n = a_1 = \cdots = a_n = 0$.

Let $M_1 = L(2^{m+3}; r_1 + 2^m a_1, \ldots, r_n + 2^m a_n)$ and $N_1 = L(2^{m+3}; s_1 + 2^m b_1, \ldots, s_n + 2^m b_n)$. Observe that

$$\begin{aligned} \prod (r_j + 2^m a_j) &= \prod r_j + 2^m (a_1 r_2 \cdots r_n + \cdots + r_1 \cdots r_{n-1} a_n) \\ &\qquad + \text{ terms involving higher powers of } 2^m. \end{aligned}$$

As $m \geq 3$, this implies that

$$\prod_j (r_j + 2^m a_j) \equiv \prod_j r_j + 2^m (a_1 r_2 \cdots r_n + \cdots + r_1 \cdots r_{n-1} a_n) \bmod 2^{m+3}$$

and hence that

$$\begin{aligned} \prod_j (r_j + 2^m a_j) - \prod_j (s_j + 2^m b_j) &\equiv \prod r_j - \prod s_j + 2^m (a_1 r_2 \cdots r_n + \cdots - b_1 s_2 \cdots s_n - \cdots) \\ &\equiv 2^m z + 2^m (a_1 r_2 \cdots r_n + \cdots - b_1 s_2 \cdots s_n - \cdots) \bmod 2^{m+3}. \end{aligned}$$

By equation (1), this last quantity is $0 \bmod 2^{m+3}$. It follows that $\prod (r_j + 2^m a_j) \equiv \prod (s_j + 2^m b_j) \bmod 2^{m+3}$ and that there exists a polarized homotopy equivalence $h_1: M_1 \longrightarrow N_1$. Any lifting of h_1 to a map $M \longrightarrow N$ will preserve generators of the fundamental groups and will have degree one. By [O], this lifting will be homotopic to h. ∎

PROOF OF 1.1. Our proof follows Cappell and Shaneson's proof of a similar result (see [CS, Theorem 1.1]).

Let BF denote the classifying space for stable spherical fibrations, and BTop the classifying space for stable topological Euclidean space bundles. The homotopy fibre of the natural map BTop $\longrightarrow BF$ is denoted $F/$Top; it is the classifying space for stable fibre homotopy trivialisation of topological Euclidean space bundles. Thus the normal invariant of $h: M \longrightarrow N$ is an element $\eta(h) \in [N, F/\mathrm{Top}]$. For additional material on $F/$Top, we suggest the reader turn to [MM], [Su], [KS] and [N].

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The space F/Top carries two distinct infinite loop space structures [N], each of whose corresponding H-space structures is homotopy commutative. In either case, $[N, F/\mathrm{Top}]$ is an abelian group, and equals the pullback over $[N, F/\mathrm{Top}] \otimes \mathbf{Q}$ of its localisations $[N, F/\mathrm{Top}]_{(2)}$ and $[N, F/\mathrm{Top}]_{(\mathrm{odd})}$. Since F/Top is simply connected, Sullivan's results on localisation yield, for any set τ of primes, an equivalence of functors: $[-, F/\mathrm{Top}]_{(\tau)} = [-, F/\mathrm{Top}_{(\tau)}]$. Therefore it is sufficient for us to show that the image of $\eta(h)$ in each of $[N, F/\mathrm{Top}_{(2)}]$ and $[N, F/\mathrm{Top}_{(\mathrm{odd})}]$ is trivial.

One further consequence of Sullivan's work is a description of the homotopy type of F/Top:

$$F/\mathrm{Top}_{(\mathrm{odd})} \simeq \mathrm{BO}_{(\mathrm{odd})}$$
$$F/\mathrm{Top}_{(2)} \simeq \prod_{i>0}\bigl(K(\mathbf{Z}_2, 4i-2) \times K(\mathbf{Z}_{(2)}, 4i)\bigr)$$

The splitting at 2 is specified by the Kervaire class $\mathcal{K} \in H^{4*+2}(F/\mathrm{Top}_{(2)}; \mathbf{Z}_2) = [F/\mathrm{Top}_{(2)}, K(\mathbf{Z}_2, 4*+2)]$ and by the class $\mathcal{L} \in H^{4*}(F/\mathrm{Top}_{(2)}; \mathbf{Z}_{(2)}) = [F/\mathrm{Top}_{(2)}, K(\mathbf{Z}_{(2)}, 4*)]$ defined by Morgan and Sullivan in [MoSu].

Let $\pi_N\colon S^{2n-1} \longrightarrow N$ denote the natural projection and $Tr\colon \widetilde{\mathrm{KO}}^0(S^{2n-1}) \longrightarrow \widetilde{\mathrm{KO}}^0(N)$ the Atiyah transfer in real K-theory. By a theorem of Becker and Gottlieb [BG], the composite

$$(Tr \circ \pi_N^*) \otimes 1\colon \widetilde{\mathrm{KO}}^0(N) \otimes \mathbf{Z}\left[\frac{1}{2^m}\right] \longrightarrow \widetilde{\mathrm{KO}}^0(N) \otimes \mathbf{Z}\left[\frac{1}{2^m}\right]$$

is an isomorphism. Hence $\pi_N^*\colon [N, \mathrm{BO}_{(\mathrm{odd})}] \longrightarrow [S^{2n-1}, \mathrm{BO}_{(\mathrm{odd})}]$ is a monomorphism. By Bott Periodicity (see [A, Section 5.1]), the latter groups are trivial and it follows that $[N, F/\mathrm{Top}_{(\mathrm{odd})}] = [N, \mathrm{BO}_{(\mathrm{odd})}] = 0$.

In order to analyse the 2-primary part of $\eta(h)$, note that we have an isomorphism

$$[N, F/\mathrm{Top}_{(2)}] \cong \bigoplus_{i>0}\bigl(H^{4i-2}(N; \mathbf{Z}_2) \oplus H^{4i}(N; \mathbf{Z}_{(2)})\bigr)$$

defined by the universal classes $\mathcal{K}, \mathcal{L}$ in the cohomology of $F/\mathrm{Top}_{(2)}$. That part of $\eta(h)$ lying in $H^{4*-2}(N; \mathbf{Z}_2)$ is seen to be zero as follows. Let $h'\colon \mathbf{R}P^{2n-1} \longrightarrow \mathbf{R}P^{2n-1}$ be a homotopy equivalence covering $h\colon M \longrightarrow N$ for which $\pi_N^*\eta(h) = \eta(h')$. By [O], h' is homotopic to a diffeomorphism and so has $\eta(h') = 0$. Now the natural map $K(\mathbf{Z}_2, 1) \longrightarrow K(\mathbf{Z}_{2^m}, 1)$ induces isomorphisms in even-dimensional $\mathbf{Z}_2$-cohomology (see [E, Section 2]). As $\mathbf{R}P^{2n-1}$ and N are the $(2n-1)$-skeleta of these classifying spaces, it follows that $\pi_N^*\colon H^{4i-2}(N; \mathbf{Z}_2) \longrightarrow H^{4i-2}(\mathbf{R}P^{2n-1}; \mathbf{Z}_2)$ is an isomorphism for $4i-2 < 2n-1$. Thus $\eta(h) = (\pi_N^*)^{-1}\eta(h') = 0$ in $H^{4i-2}(N; \mathbf{Z}_2)$, as required.

It remains to calculate $\eta(h)^*\mathcal{L}$. Let M_1, N_1 and $h_1\colon M_1 \longrightarrow N_1$ be as in Lemma 2.1. By [MoSu],

(2) $$8\eta(h_1)^*\mathcal{L} + 1 = (h_1^*)^{-1}\mathcal{L}_{M_1} \cdot \mathcal{L}_{N_1}^{-1}$$

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where $\mathcal{L}_{M_1}$, $\mathcal{L}_{N_1}$ are the transversality characteristic classes of M_1, N_1 respectively. By [Sz], the stable tangent bundles of M_1 and N_1 decompose as Whitney sums of SO_2-bundles. Following Cappell and Shaneson's line of reasoning [CS], we conclude that $\mathcal{L}_{M_1} = L(M_1)^{-1}$ and $\mathcal{L}_{N_1} = L(N_1)^{-1}$. Then (2) becomes

$$8\eta(h_1)^*\mathcal{L} + 1 = (h_1^*)^{-1}L(M_1)^{-1} \cdot L(N_1).$$

The map $\mu^*: H^{4i}(N_1; \mathbf{Z}_{(2)}) \longrightarrow H^{4i}(N; \mathbf{Z}_{(2)})$ induced by the covering $N \xrightarrow{\mu} N_1$ is surjective with kernel $\{x|8x = 0\}$. Hence $\eta(h)^*\mathcal{L} = \mu^*\eta(h_1)^*\mathcal{L}$ is trivial if and only if $8\eta(h_1)^*\mathcal{L} = 0$, that is, if and only if $h_1^*L(N_1) = L(M_1)$. As the normal invariant of h is measured entirely by $\eta(h)^*\mathcal{L}$, this proves 1.1. ∎

We now have an algorithm for determining when two homotopy equivalent lens spaces M and N, with $\pi_1 M = \pi_1 N = \mathbf{Z}_{2^m}$, are normally cobordant. First normalize the weights as in the proof of Lemma 2.1 so that $r_j \equiv s_j \equiv 1 \bmod 4$ and $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. By Theorem 1.1, M and N are normally cobordant if and only if $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ for $1 \leq k \leq \frac{2n-2}{4}$. The ℓ_k are computable as known polynomials in the elementary symmetric functions $\sigma_1(r_1^2, \ldots, r_n^2), \ldots, \sigma_n(r_1^2, \ldots, r_n^2)$. Alternatively, one may make use of the multiplicativity of $L = \{\ell_k\}_{k>0}$ and compute $\ell_k(\underline{r})$ as the coefficient of x^{2k} in the power series expansion of $\prod_{j=1}^{n} \frac{r_j x}{\tanh r_j x}$.

Now let N be a lens space and $\pi_1 N$ of arbitrary order. Wall [W, Section 14E] showed that the odd torsion part of the normal cobordism class of N is determined by the ρ-invariant of N. By passing to intermediate covering lens spaces, Wall's theorem and Theorem 1.1 combine to prove:

COROLLARY 2.2. *Let $m \geq 0$ and let q be odd. Then $M = L(2^m q; r_1, \ldots, r_n)$ and $N = L(2^m q; s_1, \ldots, s_n)$ are topologically normally cobordant if and only if $\rho(M)$ and $\rho(N)$ agree and $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ for all $k < \frac{2n-1}{4}$.*

3. **Stable Results.** Let $\rho_1 = t^{r_1} + \cdots + t^{r_n}$ denote the free representation of $\mathbf{Z}_{2^m}$ associated to the lens spaces $M = L(2^m; r_1, \ldots, r_n)$, and likewise let ρ_2 be the representation defining $N = L(2^m; s_1, \ldots, s_n)$. Let $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$ and let $h: M \longrightarrow N$ be a polarized homotopy equivalence. If $\rho = t^{c_1} + \cdots + t^{c_q}$ is an arbitrary free representation of $\mathbf{Z}_{2^m}$, the direct sum representations $\rho_1 + \rho$, $\rho_2 + \rho$ define free actions of $\mathbf{Z}_{2^m}$ on the sphere $S^{2(n+q)-1} = S^{2n-1} * S^{2q-1}$. Let $M_\rho = L(2^m; r_1, \ldots, r_n, c_1, \ldots, c_q)$ and $N_\rho = L(2^m; s_1, \ldots, s_n, c_1, \ldots, c_q)$ denote the lens spaces obtained from these actions and let $\tilde{h}: S^{2n-1} \longrightarrow S^{2n-1}$ be the $\mathbf{Z}_{2^m}$-equivariant homotopy equivalence covering h. Then $\tilde{h} * \mathrm{id}_{S^{2q-1}}: S^{2n-1} * S^{2q-1} \longrightarrow S^{2n-1} * S^{2q-1}$ is a $\mathbf{Z}_{2^m}$-homotopy equivalence and covers a homotopy equivalence $h_\rho: M_\rho \xrightarrow{\simeq} N_\rho$. We say that the *stable* normal invariant of $h: M \longrightarrow N$ vanishes if the normal invariant of h_ρ vanishes for all free representations ρ of $\mathbf{Z}_{2^m}$.

THEOREM 3.1. *Let $M = L(2^m; r_1, \ldots, r_n)$ and $N = L(2^m; s_1, \ldots, s_n)$. Suppose that $r_j \equiv s_j \equiv 1 \bmod 4$ for all j and that $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. Then the stable topological normal invariant of $h: M \xrightarrow{\simeq} N$ vanishes if and only if $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ for all $k > 0$.*

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REMARK. By Hilbert's Nullstellensatz, the last condition is guaranteed by a certain *finite* number of congruences, $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$, $k \leq K$. It would be useful to have an explicit expression for K in terms of the weights.

PROOF. By Theorem 1.1, $\eta(h)$ is stably trivial if and only if $(h_\rho)^* L(N_\rho) = L(M_\rho) \bmod 2^{m+3}$ for all free representations ρ of $\mathbf{Z}_{2^m}$. Any free representation may be decomposed as a sum of free 2-dimensional representations, so we may assume $\rho = t^c$ for some $c \equiv 1 \bmod 4$. Let $\omega \in H^2(M_\rho; \mathbf{Z}_{(2)})$ be the preferred generator. Then

$$\begin{aligned} L(M_\rho) &= L\Big((1 + c^2\omega^2) \cdot \prod(1 + r_j^2\omega^2)\Big) \\ &= L(1 + c^2\omega^2) \cdot L\Big(\prod(1 + r_j^2\omega^2)\Big) \\ &= \Big[1 + \sum_{k>0} \ell_k(\underline{c})\omega^{2k}\Big] \cdot \Big[1 + \sum_{i>0} \ell_i(\underline{r})\,\omega^{2i}\Big] \end{aligned}$$

and similarly $(h_\rho)^* L(N_\rho) = [1 + \sum_{k>0} \ell_k(\underline{c})\omega^{2k}] \cdot [1 + \sum_{i>0} \ell_i(\underline{s})\omega^{2i}]$. It follows that

$$L(M_\rho) - (h_\rho)^* L(N_\rho) = \Big[1 + \sum_{k>0} \ell_k(\underline{c})\,\omega^{2k}\Big] \cdot \Big[\sum_{i>0}\Big(\ell_i(\underline{r}) - \ell_i(\underline{s})\Big)\omega^{2i}\Big]$$

and the homogeneous part of dimension $2n$ is

$$\sum_{4i+4k=2n} \Big(\ell_i(\underline{r}) - \ell_i(\underline{s})\Big) \cdot \ell_k(\underline{c})\omega^{2n}.$$

If n is odd, this sum is zero. If $n = 2n'$, then

$$\sum_{4i+4k=4n'} \Big(\ell_i(\underline{r}) - \ell_i(\underline{s})\Big) \cdot \ell_k(\underline{c}) = \ell_{n'}(\underline{r}) - \ell_{n'}(\underline{s}),$$

for the (unstable) vanishing of $\eta(h)$ implies that $\ell_i(\underline{r}) - \ell_i(\underline{s}) \equiv 0 \bmod 2^{m+3}$ for $i < \frac{2n-1}{4}$. Therefore $(h_\rho)^* L(N_\rho) \equiv L(M_\rho) \bmod 2^{m+3}$ if and only if $\ell_{n'}(\underline{r}) \equiv \ell_{n'}(\underline{s}) \bmod 2^{m+3}$, as required. ∎

We contrast Theorem 3.1 with the following assertion of Cappell, Shaneson, Steinberger, Weinberger and West [CSSWW]:

Let $M = L(2^m; r_1, \ldots, r_n)$, $N = (2^m; s_1, \ldots, s_n)$ *and suppose that* $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. *Then the stable normal invariant of* $h: M \xrightarrow{\simeq} N$ *is trivial if and only if* $\sigma_i(r_1^2, \ldots, r_n^2) \equiv \sigma_i(s_1^2, \ldots, s_n^2) \bmod 2^{m+3}$ *for* $1 \leq i \leq n$.

The last condition would imply that the Pontrjagin classes of N and any stabilization N_ρ pull back via h to those of M and its stabilization M_ρ. It would then follow that $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ for all $k > 0$. When dealing with rational coefficients, the converse is true, for the Pontrjagin classes may be expressed over $\mathbf{Q}$ as polynomials in the ℓ-classes. Over $\mathbf{Z}_{(2)}$ however, the example given below demonstrates that equivalence of the ℓ-classes does not imply equivalence of the Pontrjagin classes.

Let us first fix some notation. The i-th elementary symmetric function of n indeterminates $y_1, y_2, \ldots, y_n$ is, as before, denoted $\sigma_i(y_1, \ldots, y_n)$. If $i > n$, then $\sigma_i(y_1, \ldots, y_n) = 0$.

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Let $I = (i_1, \ldots, i_t)$ be a partition of $k > 0$. Then S_I will denote the unique polynomial in k variables satisfying

$$S_I\big(\sigma_1(y_1, \ldots, y_n), \ldots, \sigma_k(y_1, \ldots, y_n)\big) = \sum y_1^{i_1} \cdots y_t^{i_t}$$

where the summation is over all distinct monomials of the form $y_1^{i_1} \cdots y_t^{i_t}$. For example, if $I = (1, \ldots, 1)$, then $S_I(\sigma_1, \ldots, \sigma_k) = \sum y_1 \cdots y_k = \sigma_k(y_1, \ldots, y_n)$. If $I = (k)$, then $S_k(\sigma_1, \ldots, \sigma_k) = \sum_{j=1}^n y_j^k$, the k-th Newton polynomial in $y_1, \ldots, y_n$. The polynomials S_k satisfy the Newton formula:

$$S_k - S_{k-1} \cdot \sigma_1 + \cdots \pm S_1 \cdot \sigma_{k-1} \mp k\sigma_k = 0 \tag{3}$$

See [MiSt, Section 16], for a thorough discussion of symmetric functions and the polynomials S_I.

For brevity, we shall write $\sigma_i(\underline{r})$ (resp. $\sigma_i(\underline{s})$) in place of $\sigma_i(r_1^2, \ldots, r_n^2)$ (resp. $\sigma_i(s_1^2, \ldots, s_n^2)$).

By [MiSt, Section 19] or [H, Section 1], the ℓ-polynomials are given by the formula:

$$\ell_k(\underline{r}) = \sum S_I(b_1, \ldots, b_k)\sigma_{i_1}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}) \tag{4}$$

to be summed over all partitions $I = (i_1, \ldots, i_t)$ of k. Here b_k, $k > 0$, is the coefficient of x^k in the power series expansion of

$$\frac{\sqrt{x}}{\tanh\sqrt{x}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} B_k}{(2k)!} x^k$$

and B_k is the k-th Bernoulli number (see [MiSt, appendix B]). Note that in all cases, the numerator of B_k is odd whereas its denominator contains a single factor of 2.

Let $\nu(\cdot)$ denote 2-adic valuation: any integer q can uniquely be written in the form $q = 2^v \cdot w$, with w odd and $v \geq 0$. Then $\nu(q) = v$. Moreover, q is divisible by 2 if and only if $\nu(q) > 0$.

LEMMA 3.2. *Let $q \in \mathbf{N}$ and let $\alpha(q)$ denote the number of non-zero terms in the dyadic expression for q. Then $\nu(q!) = q - \alpha(q)$.*

PROOF. The positive integer q determines $\varepsilon_i \in \{0, 1\}$, $i \geq 0$, such that $q = \sum_{i\geq 0} \varepsilon_i 2^i$. Then a straightforward counting argument shows:

$$\begin{aligned} \nu(q!) &= \sum_{j>0} \left[\frac{q}{2^j}\right] = \sum_{j\geq 1}\sum_{i\geq j} \varepsilon_i 2^{i-j} = \sum_{1\leq j\leq i} \varepsilon_i 2^{i-j} \\ &= \sum_{1\leq i} \varepsilon_i \cdot \Big[\sum_{1\leq j\leq i} 2^{i-j}\Big] = \sum_{1\leq i} \varepsilon_i (2^i - 1) = q - \alpha(q). \end{aligned}$$

∎

LEMMA 3.3. *For all $k > 0$, $\nu(b_k) = \alpha(k) - 1$.*

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PROOF. As $b_k = \frac{2^{2k}}{(2k)!} B_k$, it is easily seen that $\nu(b_k) = (2k) - \bigl(2k - \alpha(2k)\bigr) - 1 = \alpha(k) - 1$. ∎

LEMMA 3.4. $S_k(b_1, \ldots, b_k) = (-1)^{k-1}(2^{2k-1} - 1) b_k$.

The proof is an application of Cauchy's Identity,

$$f(x)\frac{d}{dx}\left(\frac{x}{f(x)}\right) = 1 - x\frac{d \log f(x)}{dx} = 1 + \sum (-1)^k S_k(c_1, \ldots, c_k) x^k$$

with $f(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}$ (see [MiSt, problems 19B, 19C]).

We shall need to pay special attention to partitions I of the form $(q, p, p, \ldots, p)$, that is, I consists of an integer $q \neq p$ followed by several copies of p. Such a partition shall be written as (q, p^μ), where $\mu \geq 0$ is the multiplicity of p in I.

LEMMA 3.5. *Let* $\mu \geq 0$.

i) For $q \equiv 3, 5, 6 \bmod 8$, $S_{(q,8^\mu)} \equiv 0 \bmod 2$.
ii) For $q \equiv 7 \bmod 8$, $S_{(q,8^\mu)} \equiv 0 \bmod 4$ *and* $S_{(q,4,8^\mu)} \equiv 0 \bmod 2$.

PROOF. Let $\mu \geq 0$ and $q \leq 8$. There exist $\varepsilon_i, \varepsilon_i' \in \{0, 1\}$ such that $\mu = \sum_{i\geq 0} \varepsilon_i 2^i$ and $q = \sum_{i\geq 0} \varepsilon_i' 2^i$. As $q \leq 8$, $\varepsilon_i' = 0$ for $i \geq 3$. Therefore

$$q + 8\mu = (\varepsilon_0' + \varepsilon_1' \cdot 2 + \varepsilon_2' \cdot 2^2 + \sum_{i\geq 3} \varepsilon_{i-3} \cdot 2^i,$$

whence $\nu(q + 8\mu) \geq \nu(q)$ (with equality if and only if $\mu = 0$).

Suppose now that $q \equiv 3, 5$ or $6 \bmod 8$. We proceed by induction on μ. The case $\mu = 0$ is trivial, for $S_q \equiv b_q \pmod 2$ and $\nu(b_q) = \alpha(q) - 1 > 0$. For general values of μ, observe that $S_{(q,8^\mu)} = S_{(q)} \cdot S_{(8^\mu)} - S_{(q+8,8^{\mu-1})}$. By Lemmas 3.4 and 3.3, $S_{(q)}$ is even. By the inductive hypothesis, so too is $S_{(q+8,8^{\mu-1})}$.

Next, suppose that $q \equiv 7 \bmod 8$. Then $S_q \equiv 0 \bmod 4$ and $S_{(q,4)} = S_q \cdot S_4 - S_{(q+4)} \equiv 0 \bmod 2$. Observing that

$$\begin{aligned} S_7 \cdot S_{8^\mu} &= S_{(7,8^\mu)} + S_{(15,8^{\mu-1})} \\ S_7 \cdot S_{(4,8^\mu)} &= S_{(7,4,8^\mu)} + S_{(11,8^\mu)} + S_{(15,4,8^{\mu-1})} \end{aligned}$$

our conclusion is now reached by the same inductive argument. ∎

THEOREM 3.6. *Let* $n = 8$ *and suppose* $r_j \equiv s_j \equiv 1 \bmod 4$ *for all* $j = 1, 2, \ldots, 8$. *Then* $\ell_k(\underline{r}) \equiv \ell_k(\underline{s}) \bmod 2^{m+3}$ *for all* $k > 0$ *if and only if* $2^{\alpha(i)-1}\sigma_i(\underline{r}) \equiv 2^{\alpha(i)-1}\sigma_i(\underline{s}) \bmod 2^{m+3}$ *for each* $i = 1, 2, \ldots, 8$.

PROOF. By assumption, $r_j \equiv 1 \bmod 4$, so $r_j^2 \equiv 1 \bmod 8$ for each $j = 1, \ldots, 8$. Therefore $\sigma_i(\underline{r}) = \sum_{j_1<\cdots<j_i} r_{j_1}^2 \cdots r_{j_i}^2 \equiv (\sum_{j_1<\cdots<j_i} 1) \equiv \binom{8}{i} \bmod 8$. In particular, $\sigma_4(\underline{r}) \equiv 2 \bmod 4$ and $\sigma_i(\underline{r}) \equiv 0 \bmod 4$ for $i \in \{1, 2, 3, 5, 6, 7\}$. The same holds for $\sigma_i(\underline{s})$, $1 \leq i \leq 8$.

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Sufficiency: Suppose $2^{\alpha(i)-1}\sigma_i(\underline{r}) \equiv 2^{\alpha(i)-1}\sigma_i(\underline{s})$ for each $i = 1, 2, \ldots, 8$. Then

$$z\sigma_i(\underline{r}) \equiv z\sigma_i(\underline{s}) \bmod 2^{m+3} \text{ for any integer } z \equiv 0 \bmod 2^{\alpha(i)-1}. \tag{5}$$

We claim that

$$S_I(b_1, \ldots, b_k) \cdot \sigma_{i_1}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}) \equiv S_I(b_1, \ldots, b_k) \cdot \sigma_{i_1}(\underline{s}) \cdots \sigma_{i_t}(\underline{s}) \bmod 2^{m+3}$$

for all $k > 0$ and all partitions I of k. By (4), this will prove the congruences $\ell_k(\underline{r}) \equiv \ell_k(\underline{s})$, $k > 0$.

If $I = (q, 8^\mu)$ and $q = 3, 5$ or 6, then by Lemma 3.5, $S_I(b_1, \ldots, b_k) \equiv 0 \bmod 2$. From (5), we conclude that

$$\begin{aligned} S_I(b_1, \ldots, b_k)\sigma_q(\underline{r})\big(\sigma_8(\underline{r})\big)^\mu &\equiv S_I(b_1, \ldots, b_k)\sigma_q(\underline{s})\big(\sigma_8(\underline{r})\big)^\mu \bmod 2^{m+3} \\ &\equiv S_I(b_1, \ldots, b_k)\sigma_q(\underline{s})\big(\sigma_8(\underline{s})\big)^\mu \bmod 2^{m+3}. \end{aligned}$$

If $I = (7, 8^\mu)$ then by Lemma 3.5, $S_I(b_1, \ldots, b_k) \equiv 0 \bmod 4$ and a similar argument shows that $S_{(7,8^\mu)}\sigma_7(\underline{r})\big(\sigma_8(\underline{r})\big)^\mu \equiv S_{(7,8^\mu)}\sigma_7(\underline{s})\big(\sigma_8(\underline{s})\big)^\mu$.

If $I = (7, 4, 8^\mu)$ then $S_I(b_1, \ldots, b_k) \equiv 0 \bmod 2$. Furthermore, $\sigma_4(\underline{r}) \equiv 0 \bmod 2$, and so $S_I(b_1, \ldots, b_k)\sigma_4(\underline{r}) \equiv 0 \bmod 4$. Again, (5) implies that

$$\begin{aligned} S_I\sigma_4(\underline{r})\sigma_7(\underline{r})\big(\sigma_8(\underline{r})\big)^\mu &\equiv S_I\sigma_4(\underline{r})\sigma_7(\underline{s})\big(\sigma_8(\underline{r})\big)^\mu \\ &\equiv S_I\sigma_4(\underline{s})\sigma_7(\underline{s})\big(\sigma_8(\underline{s})\big)^\mu \bmod 2^{m+3}. \end{aligned}$$

Now for any remaining partition I, we have $\sigma_{i_1}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}) \equiv \sigma_{i_1}(\underline{s}) \cdots \sigma_{i_t}(\underline{s}) \bmod 2^{m+3}$. For example, if $I = (q, i_2, \ldots, i_t)$ and $q = 3, 5$ or 6, then it may be supposed that $1 \le i_2 \le 7$. As $\sigma_{i_2}(\underline{r}) \equiv 0 \bmod 2$, we find by (5):

$$\sigma_q(\underline{r})\sigma_{i_2}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}) \equiv \sigma_q(\underline{s})\sigma_{i_2}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}).$$

Likewise, if $I = (7, i_2, \ldots, i_t)$ then either we may assume $i_2 \in \{1, 2, 3, 5, 6, 7\}$ or we may assume $i_2 = i_3 = 4$. But $\sigma_{i_2}(\underline{r}) \equiv \big(\sigma_4(\underline{r})\big)^2 \equiv 0 \bmod 4$, so once again we have $\sigma_7(\underline{r})\sigma_{i_2}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}) \equiv \sigma_7(\underline{s})\sigma_{i_2}(\underline{s}) \cdots \sigma_{i_t}(\underline{s})$.

Necessity: Suppose $\ell_k(\underline{r}) \equiv \ell_k(\underline{s})$ for all k. By (4), $\ell_1(\underline{r}) = \frac{1}{3}\sigma_1(\underline{r})$. Hence $\sigma_1(\underline{r}) \equiv \sigma_1(\underline{s}) \bmod 2^{m+3}$.

Assume now that $2^{\alpha(i)-1}\sigma_i(\underline{r}) \equiv 2^{\alpha(i)-1}\sigma_i(\underline{s})$ for all $i < k < 8$. If $I = (i_1, \ldots, i_t)$ is a partition of k which itself does not equal (k), then $i_1, \ldots, i_t < k$. As a result, $2^{\alpha(i_j)-1}\sigma_{i_j}(\underline{r}) \equiv 2^{\alpha(i_j)-1}\sigma_{i_j}(\underline{s})$ for each $j = 1, \ldots, t$. Arguing as above, we find that $S_I \cdot \sigma_{i_1}(\underline{r}) \cdots \sigma_{i_t}(\underline{r}) \equiv S_I \cdot \sigma_{i_1}(\underline{s}) \cdots \sigma_{i_t}(\underline{s})$ for all partitions $I \not\equiv (k)$. Therefore $0 \equiv \ell_k(\underline{r}) - \ell_k(\underline{s}) \equiv S_k \cdot \big(\sigma_k(\underline{r}) - \sigma_k(\underline{s})\big) \bmod 2^{m+3}$ By Lemmas 3.4 and 3.3, $\nu(S_k) = \alpha(k) - 1$, so $2^{\alpha(k)-1}\sigma_k(\underline{r}) \equiv 2^{\alpha(k)-1}\sigma_k(\underline{s})$ as desired. ∎

COROLLARY 3.7. *Let $n = 8$ and let $r_j \equiv s_j \equiv 1 \bmod 4$ for $j = 1, 2, \ldots, 8$. Suppose further that $\prod r_j \equiv \prod s_j \bmod 2^{m+3}$. The 15-dimensional lens spaces $M = L(2^m; r_1, \ldots, r_8)$, $N = L(2^m; s_1, \ldots, s_8)$ are stably normally cobordant if and only if $2^{\alpha(i)-1}\sigma_i(\underline{r}) \equiv 2^{\alpha(i)-1}\sigma_i(\underline{s})$ for $1 \le i \le 8$.*

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Simple calculations now show that, for $\underline{r} = (1, 1, 1, 1, 2297, 3271, 3449, 3769)$, and $\underline{s} = (3, 3, 3, 3, 35, 181, 2243, 4005)$, the lens spaces

$$M = L(2^{11}; 1, 1, 1, 1, 2297, 3449, 3769, 4921)$$
$$N = L(2^{11}; 3, 3, 3, 3, 35, 181, 2243, 4005)$$

are stably normally cobordant, but $\sigma_i(\underline{r}) \not\equiv \sigma_i(\underline{s}) \bmod 2^{14}$ for $i = 3, 7$.

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