WAVE GENERATION AND PULSATION IN STARS WITH CONVECTIVE ZONES

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Summary. Wave generation processes are classified in (1) strong and (2) weak, and (a) spontaneous and (b) stimulated processes. Then, the case (2b) operating in convective zones is discussed in detail. Both the dynamical and the thermodynamical coupling between pulsation and convection are formulated by use of the diffusion approximation for the turbulent convection. A mixing length variable with time is thereby introduced.

The work integral is transformed so that each of its terms can reveal the mechanism responsible for the stellar stability. An important destabilizing mechanism associated with the convective flux is found to exist among other known mechanisms. The mechanical work is shown to be rather important.

Wave generation processes. The genuine hydrodynamical generation of waves is due to the nonlinear Reynolds stresses (Lighthill 1952, Unno 1964). The adiabatic wave generation in an isothermal atmosphere was thoroughly studied by Stein (1967). In this case, the waves are generated spontaneously. The propagation is not isotropic, but the anisotropy is not very strong because of dominating quadrupole emissions. The wave amplitude in situ is small in subsonic turbulence, butthe effect can be appreciable after the waves propagate in the outer layers. On the other hand, if the medium is made strongly anisotropic by the presence of magnetic field or rotation, the monopole and dipole can be very important, and the wave generation can be strong. The generation of Alfvén waves from turbulence under the presence of a strong magnetic field was found to be very effective (Kato 1968, Roberts 1976). A change in the basic structure of the medium is then expected. Spiral arms in galaxies, spicules and sunspots (Parker 1974) may be the manifestation of such cases.

For waves that are trapped in some region of a star, the stimulated emission should be considered. The emitted wave and the underlying oscillation have a phase relation so that the whole process forms a self-exciting system. Therefore, in principle, the strong stimulated generation of waves may not be an inaccurate concept. Osaki (1974), however, considered that the resonance between the nonradial oscillation and the overstable convection in a fast rotating core could be the cause of the β Cephei variability. In such a case, the theory remains qualitative.

For trapped waves or pulsations, the weak stimulated emission can be accumulated and become important. The thermodynamical excitation of pulsation has been worked out by many authors (Zhevakin 1953, Baker and Kippenhahn 1962, Cox 1963, Christy 1964) as the cause of the variabilities of Cepheids and RR Lyr stars. Less investigation has been done for stars having deep convective envelopes because of the theoretical difficulties in the treatment of a convective zone.

Modulation of convection by pulsation can be calculated on the basis of the mixing-length theory (Vitense 1953) slightly generalized to include the time dependence (Unno 1967). Recently, Gabriel, Scuflaire, Noels and Boury (1975) calculated the thermodynamical coupling of the convection with the nonradial pulsation, and they demonstrated an appreciable effect of the convective flux perturbation on the stability coefficient. The dynamical coupling has been neglected so far. But, it is caused by the perturbations in the turbulent pressure, visocsity and conductivity, and its effect on the stability is not negligible as shown later.

Table 1. Classification of wave generation mechanisms

	A. SPONTANEOUS (noise, propagating)		B. STIMULATED (phase relation, trapped)
1. STRONG	STRUCTURE CHANGE		
(monopole, dipole) anisotropic	spicules sunspots (Kato, Parker)	spiral arms	β-Ceph (Osaki)
2. WEAK	Effects on Outer Layers		Excitation of Pulsation
(quadrupole) isotropic	homog. (Lighthill) isothermal (Stein)	5mn Oscill.	thermal & dynamical pulsating stars ⊙ stability

Table 1 summarizes the classification of the wave generation mechanisms discussed above. The excitation of spiral arms in galaxies (Mark 1976) is considered as an example of a strong emission mechanism. The excitation of the solar 5 min. oscillation studied by Ando and Osaki (1975) belongs to the weak stimulated emission mechanism. In these two examples, however, the coherence in spatial and temporal wave patterns is not completely perfect, and the emission mechanism may better be considered as partially spontaneous and partially stimulated. The solar stability (Dilke and Gough 1972, Boury, Gabriel, Noels, Scuflaire and Ledoux 1975, Shibahashi, Osaki and Unno 1975) is an interesting example involving the weak stimulated emission mechanism. The convection-pulsation coupling is important. A qualitative change in the underlying solar structure discussed in the present Colloquium may not take place, since the emission mechanism is weak.

Basic equations of the pulsation-convection coupling. We shall hereafter restrict ourselves to study the mechanisms of pulsational stability operating in the convective zone. At present, no complete description of the compressible inhomogeneous turbulence is available. We shall, therefore, approximate the nonlinear effects of the turbulent convection by the eddy diffusivities. Then, the conservation equations of mass, momentum and thermal energy are described by (Unno 1969)

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u} + \frac{1}{\rho}\nabla\cdot(\langle\nu\rangle\nabla\rho) \approx -\rho\nabla\cdot\mathbf{u} \quad , \tag{1}$$

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla(P+P_t) - \nabla\phi + \frac{1}{\rho}[\nabla(\langle\mu\rangle\nabla\cdot u) + (\nabla\cdot\langle\mu\rangle\nabla)\mathbf{u}] \quad , \tag{2}$$

$$\frac{dS}{dt} = \frac{1}{T}(\varepsilon_N + \varepsilon_V - \frac{1}{\rho}\nabla\cdot\mathbf{F}_R) + \frac{1}{\rho}\nabla\cdot(\langle\lambda\rangle\nabla S) \quad , \tag{3}$$

where d/dt is the Lagrangian time differentiation, ρ, T, P, S and $\mathbf{u}$ denote density, temperature, pressure, specific entropy, and the velocity, ε_N, ε_V, $\mathbf{F}_R$ and P_t denote the nuclear energy generation rate, the turbulent viscous dissipation, the radiative flux, and the turbulent pressure, respectively, and $\langle\lambda\rangle$, $\langle\mu\rangle$ and $\langle\nu\rangle$ describe the coefficients of turbulent conductivity, viscosity and diffusion that are approximated by $\langle\rho u'\ell\rangle$, ℓ being the mixing length and the prime indicating the convective fluctuation. The small scale fluctuations inside a representative convective element have been smoothed out so that the $P_t(=\langle\rho u'^2\rangle)$, $\langle\lambda\rangle$, $\langle\mu\rangle$ and $\langle\nu\rangle$ appear from their nonlinear effects. Since the turbulence has a continuous energy spectrum, the magnitude of these terms should depend also on the scale of motion under consideration (Nakano 1972), but this dependence will not be explicitly described in this paper. We shall also neglect $\langle\nu\rangle$, since mixing is efficient in the convection zone. The viscous dissipation ε_V which is dimensionally given by

$$\varepsilon_V \approx u'^3/\ell \tag{4}$$

in accordance with equation (2) should not be disregarded, since for the adiabatic convection the following relations,

$$\varepsilon_{V0} \approx (1/\rho_0)\langle\lambda\rangle_0\nabla T_0\cdot\nabla S_0 \quad , \tag{5}$$

$$\mathbf{F}_{C0} \approx -\langle\lambda\rangle_0 T_0\nabla S_0 \quad , \tag{6}$$

reduce essentially to the original Biermann formalism (1932) and then the energy conservation in steady state,

$$\rho_0\varepsilon_{N0} = \nabla\cdot(\mathbf{F}_{R0} + \mathbf{F}_{C0}) \quad , \tag{7}$$

is ensured by equation (3). Here $\mathbf{F}_C$ denotes the convective flux and the subscript 0 indicates the statistically steady undisturbed state. The equilibrium structure is determined by

$$(1/\rho_0)\nabla(P_0 + P_{t0}) + \nabla\phi_0 = 0 \tag{8}$$

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in addition to equation (7). Subtraction of these equilibrium equations from the basic equations (2) and (3) yields the equations for perturbations. The difference in spatial and temporal spectra between pulsation and convection can be used to separate the system of equations governing pulsation and convection from each other.

The work integral of the nonradial pulsation. The equations of pulsation are given by

$$- i\omega(\delta\rho/\rho_0 + \nabla\cdot\delta\mathbf{r}) = 0 \quad , \tag{9}$$

$$- \omega^2\delta\mathbf{r} + (1/\rho_0)\nabla P_1 - (\rho_1/\rho_0{}^2)\nabla P_0 = \mathcal{F}_1 \quad , \tag{10}$$

$$- i\omega\delta S = \delta\mathcal{G} \quad , \tag{11}$$

where δq and q_1 represent the Lagrangian and Eulerian perturbations of any variable q, the time differentiation $d/dt_0=\partial/\partial t+u_0{}'\cdot\nabla_0$ of the Lagrangian perturbations are replaced by $-i\omega$, and

$$\mathcal{F}_1 = - \nabla\phi_1 - \frac{1}{\rho_0}\nabla P_{t1} + \frac{\rho_1}{\rho_0{}^2}\nabla P_{t0} - \frac{i\omega}{\rho_0}[\nabla(\langle\mu\rangle_0\nabla\cdot\delta\mathbf{r}) + (\nabla\cdot\langle\mu\rangle_0\nabla)\delta\mathbf{r}] \quad , \tag{12}$$

$$\delta\mathcal{G} = \delta[T^{-1}(\varepsilon_N + \varepsilon_V - \rho^{-1}\nabla\cdot\mathbf{F}_R) + \rho^{-1}\nabla\cdot\langle\lambda\rangle\nabla S] \quad . \tag{13}$$

The effects of the modulated convection enter through P_{t1} in $\mathcal{F}_1$ and $\delta\varepsilon_V$ and $\delta\langle\lambda\rangle$ in $\delta\mathcal{G}$. The calculation of these terms will be made later.

Now we can define the pulsation energy E_P per unit volume by

$$E_P= \frac{1}{2}\rho_0[\mathbf{u}_1{}^2+\frac{P_1{}^2}{c^2\rho_0{}^2}+(- \frac{dP_0}{\rho_0 dr})(\frac{d\ln P_0}{\gamma_1 dr} - \frac{d\ln\rho_0}{dr})^{-1}(\frac{P_1}{\gamma_1 P_0} - \frac{\rho_1}{\rho_0})^2], \tag{14}$$

where $c^2=\gamma_1 P_0/\rho_0$ and $\gamma_1 =(\partial\ln P/\partial\ln\rho)_S$. It consists of the kinetic energy and the potential energies of acoustic and gravity waves. From equations (9), (10) and (11), after some manupilations, we obtain

$$\frac{\partial E_P}{\partial t} + \nabla\cdot(P_1\mathbf{u}_1 + E_P\mathbf{u}_0') = \rho_0(\mathbf{u}_1\cdot\mathcal{F}_1 + \delta T\delta\mathcal{G}) \quad . \tag{15}$$

Integrating this equation over the whole volume of a star and assuming the boundary conditions of $P_1\mathbf{u}_1=0$ and $\mathbf{u}_0'=0$ at the surface, we obtain

$$\frac{d}{dt}\int E_P dV = W = W_M + W_T \equiv \mathrm{Re}\int \rho_0(\mathbf{u}_1{}^*\cdot\mathcal{F}_1+\delta T^*\delta\mathcal{G})\,dV. \tag{16}$$

The result is independent of the quasi-adiabatic approximation and the Cowling approximation used above.

The mechanical work W_M is the sum of the turbulent pressure work W_P and the turbulent viscous stress work W_S;

$$W_P = \mathrm{Re}\int -i\omega\delta r^*\cdot\nabla\delta P_t dV = 2\omega\int P_{t0}I_m[(\delta\rho^*/\rho_0)(\delta u'/u_0')]dV \quad , \tag{17}$$

$$W_S = \mathrm{Re}\int \mathbf{u}_1^*\cdot(-i\omega)[\nabla(<\mu>_0\nabla\cdot\delta\mathbf{r}) + (\nabla\cdot<\mu>_0\nabla)\delta\mathbf{r}]dV$$

$$= -\omega^2\int<\mu>_0 (|\delta\rho/\rho_0|^2 + |\nabla\delta\mathbf{r}|^2)dV \quad . \tag{18}$$

The calculation of $I_m[(\delta\rho^*/\rho_0)(\delta u'/u_0')]$ will be given later. The thermodynamical work W_T consists of the nuclear energy work W_N, the viscous dissipation work W_V, the radiative work W_R, and the convective work W_C. After some manipulations, we obtain

$$W_N = \mathrm{Re}\int \rho_0(\delta T^*/T_0)\delta\varepsilon_N dV = \int \rho_0\varepsilon_{N0}(\varepsilon_N)_{T,ad}|\delta T/T_0|^2 dV \quad , \tag{19}$$

$$W_V = \mathrm{Re}\int \rho_0(\delta T^*/T_0)\delta\varepsilon_V dV = \int \rho_0\varepsilon_{V0}\mathrm{Re}[(\delta T^*/T_0)\{3(\delta u'/u_0')-\delta\ell/\ell_0\}]dV \quad , \tag{20}$$

$$W_R = \mathrm{Re}\int \rho_0(\delta T^*/T_0)\delta(\rho^{-1}\nabla\cdot K\nabla\ln T)dV$$

$$= -\frac{K_T}{2}L\left|\frac{\delta T}{T_0}\right|^2_{r=R} + \int\left(\frac{\nabla\cdot\mathbf{F}_{R0}}{\gamma_3-1} - \nabla\cdot\frac{K_T}{2}\mathbf{F}_{R0}\right)\left|\frac{\delta T}{T}\right|^2 dV$$

$$-\int\frac{F_{R0}H_0}{\nabla_0}\left[\left|\frac{\partial}{\partial r}\left(\frac{\delta T}{T_0}\right)\right|^2 + \frac{\ell(\ell+1)}{r^2}\left|\frac{\delta T}{T_0}\right|^2\right]dV + \int 2\nabla\cdot\mathbf{F}_{R0}\mathrm{Re}\left[\frac{\delta T^*}{T_0}\frac{\partial\delta r}{\partial r}\right]dV$$

$$+\mathrm{Re}\int F_{R0}\frac{\delta T^*}{T_0}\left(\frac{\partial^2\delta r}{\partial r^2} - \frac{2}{r}\frac{\partial\delta r}{\partial r} + \frac{2-\ell(\ell+1)}{r^2}\delta r\right)dV \quad , \tag{21}$$

$$W_C = \mathrm{Re}\int \rho_0 (\delta T^*/T_0)\delta(T\rho^{-1}\nabla\cdot<\lambda>\nabla S)dV$$

$$= \int\left(\frac{\nabla\cdot\mathbf{F}_{C0}}{\gamma_3-1} - \nabla\cdot\frac{\mathbf{F}_{C0}/2}{\gamma_3-1} - T_0\nabla\cdot\frac{\mathbf{F}_{C0}}{T_0}\right)\left|\frac{\delta T}{T_0}\right|^2 dV$$

$$+\mathrm{Re}\int\frac{F_{C0}}{T_0}\frac{\partial\delta T^*}{\partial r}\left(\frac{\delta u'}{u_0'} + \frac{\delta\ell}{\ell_0}\right)dV + \int 2T_0\nabla\cdot\frac{\mathbf{F}_{C0}}{T_0}\mathrm{Re}\left[\frac{\delta T^*}{T_0}\frac{\partial\delta r}{\partial r}\right]dV$$

$$+\mathrm{Re}\int F_{C0}\frac{\delta T^*}{T_0}\left(\frac{\partial^2\delta r}{\partial r^2} - \frac{2}{r}\frac{\partial\delta r}{\partial r} + \frac{2-\ell(\ell+1)}{r^2}\delta r\right)dV \quad , \tag{22}$$

where

$$\gamma_3-1 = \left(\frac{\partial\ln T}{\partial\ln\rho}\right)_S \quad , \quad \nabla_0 = \frac{d\ln T_0}{d\ln P_0} \quad , \quad H_0 = -\frac{dr}{d\ln P_0} \quad ,$$

$$K = \frac{4acT^4}{3\kappa\rho} \quad , \quad K_T = \left(\frac{\partial\ln K}{\partial\ln T}\right)_S \quad , \quad (\varepsilon_N)_{T,ad} = \left(\frac{\partial\ln\varepsilon_N}{\partial\ln T}\right)_S \quad ,$$

L denotes the surface luminosity, and equation (6) has been used. The radiative conductivity K appears in the expression of the radiative flux $\mathbf{F}_R$, $\mathbf{F}_R = -K\nabla\ln T$. In calculating W_R and W_C, the following identity has been used,

$$\delta(\nabla\cdot A_0\nabla B_0) = -A_0\frac{\partial B_0}{\partial r}\left(\frac{\partial^2\delta r}{\partial r^2} - \frac{2}{r}\frac{\partial\delta r}{\partial r} + \frac{2-\ell(\ell+1)}{r}\delta r\right) - 2\nabla\cdot(A_0\nabla B_0)\frac{\partial\delta r}{\partial r} \quad ,$$

where the spherical harmonics $Y_\ell^m(\theta,\phi)$ for the angular dependence of a normal mode has been assumed. The work integrals will be transformed to more convenient forms for revealing physical processes, after the convection modulated by the pulsation is discussed in the following.

Time dependent convection. Equations of the time dependent convection can be separated out from the basic equations (1), (2) and (3) as follows,

$$d\rho'/dt_1 + \nabla\cdot(\rho\mathbf{u}') \approx \nabla\cdot(\rho\mathbf{u}') \approx 0 \quad , \tag{23}$$

$$\left(\frac{d}{dt_1} + \frac{1}{\tau_V}\right)\mathbf{u}' + \frac{1}{\rho}\nabla P' - \frac{\rho'}{\rho^2}\nabla P = -\nabla\phi' - \frac{1}{\rho}\nabla P_t{}' + \frac{\rho'}{\rho^2}\nabla P_t \tag{24}$$

$$dS'/dt_1 + \mathbf{u}'\cdot\nabla S = [T^{-1}(\varepsilon_N + \varepsilon_V - \rho^{-1}\nabla\cdot\mathbf{F}_R) + \rho^{-1}\nabla\cdot\langle\lambda\rangle\nabla S]' \quad , \tag{25}$$

when $d/dt_1=\partial/\partial t+\mathbf{u}_1\cdot\nabla$, $\tau_V\approx\ell/u'$, the primed quantities are the Eulerian perturbations due to convection, and the coordinate and the quantities without prime include the Lagrangian perturbations due to pulsation in addition to the equilibrium values.

Now we neglect the right hand side of equation (24) and use the Boussinesq approximation instead of equation (23). Then, using the lateral component of equation (24) to eliminate P' from the radial component, we obtain

$$\left(\frac{d}{dt_1} + \frac{1}{\tau_V}\right)u_r{}' = \frac{k_\perp^2}{k^2}\frac{\rho'}{\rho^2}\frac{\partial P}{\partial r} \approx \frac{k_\perp^2}{k^2}\,|\rho_T|\,\frac{T'}{T}\left(-\frac{\partial P}{\rho\partial r}\right) \quad , \tag{26}$$

where $\rho_T=(\partial\ln\rho/\partial\ln T)_P$ and a spatial dependence of $\exp(i\mathbf{k}\cdot\mathbf{r})$ has been assumed for the convection variables. Also, equation (25) can be rewritten as

$$\left(\frac{d}{dt_1} + \frac{1}{\tau_C} + \frac{1}{\tau_R} - \frac{1}{\tau_N}\right)S' + u_r{}'\frac{\partial S_0}{\partial r} = 0 \quad , \tag{27}$$

where τ_C, τ_N, and τ_R denotes the time scales of the turbulent conduction, the heating by nuclear burning, and the radiative cooling of a convective element. We shall hereafter write u' for $u_r{}'$ and τ for τ_V and τ_C without distinguishing the different numerical factors of the order of unity that are unimportant in the stability analysis.

Equations (26) and (27) have to be supplemented by the definition of the mixing length ℓ which is modulated by pulsation. We assume that a convective element born at time t' has a mixing length equal to the instantaneous scale height H ($=-\partial r/\partial\ln P$) initially and evolves according to the law $\rho\ell^3$=const. during its life time τ. Then, the relative excess $\delta\ell/\ell_0$ is given by $([\delta H]/H_0)e^{-i\omega t'}$ at its birth and will be further increased by $(-1/3)([\delta\rho]/\rho_0)(e^{-i\omega t}-e^{-i\omega t'})$ during the time interval from t' to t, the time dependence in δH and $\delta\rho$ being expressed explicitly and $[\delta H]$ and $[\delta\rho]$ being the compressible amplitudes. For the average convective element, we obtain

$$\frac{[\delta\ell]}{\ell_0} e^{-i\omega t} = \int_0^\infty [\frac{[\delta H]}{H_0} e^{-i\omega t'} - \frac{1}{3}\frac{[\delta\rho]}{\rho_0} (e^{-i\omega t} - e^{-i\omega t'})] \exp(-\frac{t-t'}{\tau}) \frac{d(t-t')}{\tau} ,$$

or

$$\frac{\delta\ell}{\ell_0} = \frac{1}{1-i\omega\tau} (\frac{\delta H}{H_0} + \frac{i\omega\tau}{3}\frac{\delta\rho}{\rho_0}) , \quad (28)$$

assuming a constant birth rate and constant life time for the convective elements. A more precise theory for the time-dependent mixing length has been developed by Gough (1976).

For equilibrium ($d/dt_1=0$), equations (26) and (27) reduce essentially to the Vitense (1953) formalism. The modulation of convection by pulsation can be calculated from the Lagrangian variations of equations (26) and (27), (Unno 1967, Gabriel, Scuflaire, Noels and Boury 1975). For qualitative study, we now consider an envelope ($\tau_N/\tau_R \to \infty$) in which convection takes place almost adiabatically ($\tau_R/\tau \to \infty$). The result comes out to be :

$$\frac{\delta T'}{T_0} = \frac{1}{1-i\omega\tau} (\frac{\delta\ell}{\ell_0} - \frac{\partial\delta r}{\partial r}) \quad (29)$$

$$\frac{\delta u'}{u_0} = \frac{1}{1-i\omega\tau} (\frac{\delta\ell}{\ell_0} - \frac{\partial\delta r}{\partial r}) + \frac{H_0}{2-i\omega\tau}\frac{\partial}{\partial r}(\frac{\delta P}{P_0}) , \quad (30)$$

where the variations of ρ_T and of the mean molecular weight have been neglected. The value of $\delta\ell/\ell_0$ is calculated from equation (28) in which by definition ($H=-dr/d\ln P$),

$$\delta H/H_0 = \partial\delta r/\partial r + H_0(\partial/\partial r)(\delta P/P_0) . \quad (31)$$

Thus, the modulation of convection to be used for the calculation of the work integral is now expressed in terms of $(\delta T/T_0)$ and $(\partial\delta r/\partial r)$, on account that $\nabla_{ad}(\delta P/P_0)=(\gamma_3-1)(\delta\rho/\rho_0)=\delta T/T_0$.

Some useful formulae. Readers can skip this section, unless mathematical details are of interest.

The work integrals W_R and W_C should be simplified further by use of the equations of adiabatic oscillations. Neglecting $\mathcal{F}_1$ and $\delta\mathcal{G}$, we obtain from equations (9), (10) and (11) after some manipulations,

$$\frac{\partial}{\partial r}(r^2\delta r) - \frac{\ell(\ell+1)GM_r}{\omega^2 r^2}\delta r = - r^2 (1 - \frac{\ell(\ell+1)c^2}{\omega^2 r^2}) \frac{\delta\rho}{\rho_0} , \quad (32)$$

$$\frac{\partial}{\partial r}(\frac{GM_r}{r^2}\delta r) - \omega^2\delta r = - \frac{1}{\rho_0}\frac{\partial\delta P}{\partial r} - \frac{GM_r}{r^2}\frac{\delta\rho}{\rho_0} . \quad (33)$$

Multiplying equation (33) by r^2, differentiating, and using equation (32), we obtain

$$\frac{\partial^2\delta r}{\partial r^2} - \frac{2}{r}\frac{\partial\delta r}{\partial r} + \frac{2-\ell(\ell+1)}{r^2}\delta r = -\frac{r^2}{GM_r}\left(\omega^2 - \frac{\ell(\ell+1)c^2}{r^2}\right)\frac{\delta\rho}{\rho_0} - \frac{\partial}{\partial r}\left[\frac{r^2}{GM_r\rho_0}\frac{\partial\delta P}{\partial r} + \frac{\delta\rho}{\rho_0}\right] \quad , \qquad (34)$$

where the variation of M_r has been neglected as in the Cowling approximation (ϕ_1=0). Thus, we have a formula appearing in equations (21) and (22) :

$$\mathrm{Re}\left[\frac{\delta T^*}{T_0}\left(\frac{\partial^2\delta r}{\partial r^2} - \frac{2}{r}\frac{\partial\delta r}{\partial r} + \frac{2-\ell(\ell+1)}{r^2}\delta r\right)\right]$$

$$= -\frac{\omega^2/r}{(\gamma_3-1)\omega_g^2}\left|\frac{\delta T}{T_0}\right|^2 + \frac{H_0}{\nabla_{ad}}\left[\left|\frac{\partial}{\partial r}\left(\frac{\delta T}{T_0}\right)\right|^2 + \frac{\ell(\ell+1)}{r^2}\left|\frac{\delta T}{T_0}\right|^2\right]$$

$$- \frac{\partial}{2\partial r}\left[\frac{H_0}{\nabla_{ad}}\frac{\partial}{\partial r}\left|\frac{\delta T}{T_0}\right|^2 - \left\{\frac{\gamma_1-1}{\gamma_3-1} - H_0\frac{\partial}{\partial r}\left(\frac{1}{\nabla_{ad}}\right)\right\}\left|\frac{\delta T}{T_0}\right|^2\right]$$

$$+ \frac{\partial}{2\partial r}\left\{\frac{\gamma_1-1}{\gamma_3-1} - H_0\frac{\partial}{\partial r}\left(\frac{1}{\nabla_{ad}}\right)\right\}\left|\frac{\delta T}{T_0}\right|^2 \quad , \qquad (35)$$

where $\omega_g{}^2 \equiv GM_r/r^3$. (36)

Next we shall derive the expression of $\partial\delta r/\partial r$. Differentiating equation (32) with r and using equation (33), we obtain

$$\partial^2(r^2\delta r)/\partial r^2 - \ell(\ell+1)\delta r = -(\partial/\partial r)(r^2\delta\rho/\rho_0) - \ell(\ell+1)\omega^{-2}A\delta P/\rho_0 \quad , \qquad (37)$$

where $A=(d\ln\rho_0/dr)-(1/\gamma_1)(d\ln P_0/dr)$. The operation : $[(37)/r^2-(34)]$ gives the expression of $\partial\delta r/\partial r$. Unfortunately, the resulting expression is not so simple. We shall be satisfied with a crude approximation in which we introduce an effective number of nodes n such that $\partial^2(r^2\delta r)/\partial r^2 \approx -n^2\delta r$ and $\partial^2(r^2\delta\rho/\rho_0)\partial r^2 \approx -n^2\delta\rho/\rho_0$. If we neglect A, we obtain from equation (37),

$$\partial\delta r/\partial r \approx -\{n^2/[n^2+\ell(\ell+1)]\}(\delta\rho/\rho_0) \quad . \qquad (38)$$

Physical processes of stability. With the help of the results in the preceding two sections, all the integrands in the work integrals given by equations (17)-(22) can be transformed into the forms proportional to the power of pulsation.

After a number of partial integrations, we obtain finally,

$$W_M = \omega\int\frac{P_{t0}}{\gamma_3-1}\left[2u_T{}^I - \frac{d}{dr}\ln\left(\frac{r^2P_{t0}}{\gamma_3-1}\right)u_T{}'^I + \frac{du_T{}'^I}{dr}\right]\left|\frac{\delta T}{T_0}\right|^2 dV - \omega^2\int\langle\mu\rangle_0\left(\left|\frac{\delta\rho}{\rho_0}\right|^2 + |\nabla\delta\mathbf{r}|^2\right)dV \quad , \qquad (39)$$

$$W_T = (1/2)L(\nabla_{ad}{}^{-1} - K_T)|\delta T/T_0|^2_{r=R}$$

$$- \int(1/2)\nabla\cdot[K_T\mathbf{F}_{R0} + \{(\gamma_3-1)^{-1} - 2 + u_T{}^R + \ell_T{}^R\}\mathbf{F}_{C0}]|\delta T/T_0|^2 dV$$

https://doi.org/10.1017/S0252921100112527 Published online by Cambridge University Press

$$+\int \rho_0\varepsilon_{NO}[(\varepsilon_N)_{T,ad} + (\Gamma_3-1)^{-1}+2\delta_T-\nabla_{ad}^{-1}(1-\nabla_{ad,P}+\varepsilon_{N,P}/2)]|\delta T/T_0|^2 dV$$

$$+\int \rho_0\varepsilon_{VO}[2(u_T{}^R-\ell_T{}^R+\delta_T{}^R)-1-\{\frac{d}{dr}\ln(r^2\rho_0\varepsilon_{VO})+\frac{d}{dr}\}(u_T'^R-\ell_T'^R)]|\delta T/T_0|^2 dV$$

$$+\int [F_{RO}(\nabla_{ad}{}^{-1}-\nabla_0{}^{-1})+F_{CO}\nabla_{ad}{}^{-1}]H_0[|\partial(\delta T/T_0)/\partial r|^2+\{\ell(\ell+1)/r^2\}|\delta T/T_0|^2]dV$$

$$+\int \{(F_{RO}+F_{CO})[-\frac{\omega^2 H_0}{\nabla_{ad}c^2}+\frac{d}{dr}(\frac{1-\nabla_{ad,P}}{\nabla_{ad}})\}|\frac{\delta T}{T_0}|^2+F_{CO}(u_T'^R+\ell_T'^R)|\frac{\partial}{\partial r}(\frac{\delta T}{T_0})|^2\}dV \quad , \qquad (40)$$

where approximations (5) and (6) have been used, δ_T, H_T etc. are defined by, [equations (28)-(31)],

$$\frac{\partial\delta r}{\partial r} = \delta_T\frac{\delta T}{T_0} \approx -\frac{n^2/(\Gamma_3-1)}{n^2+\ell(\ell+1)}\frac{\delta T}{T_0} \quad ,$$

$$\frac{\delta H}{H_0} = H_T\frac{\delta T}{T_0} + H_T'\frac{\partial}{\partial r}(\frac{\delta T}{T_0}) = (\delta_T + \frac{\nabla_{ad,P}}{\nabla_{ad}})\frac{\delta T}{T_0} + \frac{H_0}{\nabla_{ad}}\frac{\partial}{\partial r}(\frac{\delta T}{T_0}) \quad ,$$

$$\frac{\delta\ell}{\ell_0} = \ell_T\frac{\delta T}{T_0} + \ell_T'\frac{\partial}{\partial r}(\frac{\delta T}{T_0}) = \frac{1}{1-i\omega\tau}[(H_T + \frac{i\omega\tau/3}{\Gamma_3-1})\frac{\delta T}{T_0} + H_T'\frac{\partial}{\partial r}(\frac{\delta T}{T_0})]$$

$$\frac{\delta u'}{u_0} = u_T\frac{\delta T}{T_0} + u_T'\frac{\partial}{\partial r}(\frac{\delta T}{T_0}) = (\frac{\ell_T-\delta_T}{1-i\omega\tau} + \frac{\nabla_{ad}{}^{-1}\nabla_{ad,P}}{2-i\omega\tau})\frac{\delta T}{T_0} + (\frac{\ell_T'}{1-i\omega\tau} + \frac{H_0\nabla_{ad}{}^{-1}}{2-i\omega\tau})\frac{\partial}{\partial r}(\frac{\delta T}{T_0}) \quad ,$$

and the other symbols have been defined before except that

$$\varepsilon_{N,P} = d(\ln\varepsilon_{NO})/d\ln P_0 \quad , \quad \text{and} \quad \nabla_{ad,P} = d(\ln\nabla_{ad})/d\ln P_0 \quad .$$

The turbulent pressure work represented by the first integral in equation (39) can be positive or negative depending upon the mode (ℓ and n) and the relative convection time $\omega\tau$, while the turbulent stress work represented by the second integral is always negative. The ratio between them is of the order of $(\omega\tau)^{-1}$ which can be larger or smaller than unity depending upon the mode and the stellar structure. If the integrands of the thermodynamical work W_T are estimated to be of the order of $(F_{CO}/\ell_0)|\delta T/T_0|^2$ and $F_{CO}\sim\rho_0 u_0'^3$ in convection zones, the ratio W_M/W_T becomes to be of the order of $\omega\tau$ or $(\omega\tau)^2$. The mechanical work should, therefore, be taken into account in the stellar stability calculation.

In equation (40), the first line on the right hand side shows the spherical effect and the κ-mechanism. If κ increases strongly with increasing temperature and density (negative K_T), larger radiation loss results at lower density higher entropy state, causing the destabilization of the oscillation. The sphericity has a similar effect, though not so pronounced. The second line shows the correction to the κ and other effects. The third line shows primarily the ε mechanism which is the destabilizing effect due to high

https://doi.org/10.1017/S0252921100112527 Published online by Cambridge University Press

temperature sensitivity of the nuclear burning process. The fourth line shows the similar effect by the viscous dissipation, but its work like the turbulent pressure work can be positive or negative. The term proportional to F_{R0} in the fifth line describes the Cowling-Souffrin-Spiegel (or radiative Cowling) mechanism (Graff 1976), and the term proportional to F_{C0} describes the turbulent conduction mechanism (or convective Cowling mechanism) which seems to be found by the present analysis. The sixth line shows the corrections to these two mechanisms. Because of the latter corrections, the radiative Cowling mechanism may not work for high frequency modes, while the convective Cowling mechanism may remain effective.

If a fluid element moves up (or down) slowly (small ω) in a superadiabatic layer, the temperature inside becomes higher (or lower) than outside, and the density inside becomes lower (or higher) than outside because of the pressure balance. Both the radiation and the turbulent conduction decrease (or increase) the thermal energy inside where the entropy is higher (or lower). This explains the radiative and convective Cowling mechanisms. Both mechanisms are due to the superadiabaticity, but they differ greatly in efficiency, because the gain (or loss) in thermal energy is of the same order as the convective flux for the convective Cowling mechanism while it is smaller by a factor of $(\nabla_0-\nabla_{ad})$ for the radiative Cowling mechanism. Although none of the mechanisms can be neglected, the κ mechanism and the convective Cowling mechanism are the main destabilizing mechanisms of pulsation in a convective star.

Various approximations have been employed in the derivation of equations (39) and (40). Many of them should be easily avoided in numerical work of stellar stability.

The author thanks Dr Y. Osaki for discussion.

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