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EMBEDDING THE AFFINE COMPLEMENT OF THREE INTERSECTING LINES IN A FINITE PROJECTIVE PLANE

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Abstract

An $(r, 1)$-design is a pair (V, F) where V is a v-set and F is a family of non-null subsets of V (b in number) which satisfy the following.

(1) Every pair of distinct members of V is contained in precisely one member of F.

(2) Every member of V occurs in precisely r members of F.

A pseudo parallel complement $PPC(n, \alpha)$ is an $(n+1, 1)$-design with $v = n^2 - \alpha n$ and $b \leqq n^2 + n - \alpha$ in which there are at least $n - \alpha$ blocks of size n. A pseudo intersecting complement $PIC(n, \alpha)$ is an $(n+1, 1)$-design with $v = n^2 - \alpha n + \alpha - 1$ and $b \leqq n^2 + n - \alpha$ in which there are at least $n - \alpha + 1$ blocks of size $n - 1$. It has previously been shown that for $\alpha \geqq 4$, every $PIC(n, \alpha)$ can be embedded in a $PPC(n, \alpha - 1)$ and that for $n > (\alpha^4 - 2\alpha^3 + 2\alpha^2 + \alpha - 2)/2$, every $PPC(n, \alpha)$ can be embedded in a finite projective plane of order n. In this paper we investigate the case of $\alpha = 3$ and show that any $PIC(n, 3)$ is embeddable in a $PPC(n, 2)$ provided $n \geqq 14$.

1. Introduction

An (r, λ)-*design* is a pair (V, F) where V is a finite set of V elements (called varieties) and F is a family of non-null subsets of V (called blocks) such that

(1) every variety of V occurs in precisely r blocks of F.

(2) every pair of distinct varieties occurs in precisely λ blocks of F.

The symbol b is used to denote the number of blocks in F and the word cardinality and size are used interchangeably. We refer to an $(n+1, 1)$-design with $v = n^2 - \alpha n$ and $b \leqq n^2 + n - \alpha$ which contains at least $n - \alpha + 1$ lines of size n as a pseudo parallel complement $PPC(n, \alpha)$. An $(n+1, 1)$-design with $v = n^2 - \alpha n + \alpha - 1$ varieties and $b \leqq n^2 + n - \alpha$ blocks, at least $n - \alpha + 1$ of which are of size $n-1$ is called a *pseudo intersecting complement* $PIC(n, \alpha)$. We say that an $(n+1, 1)$-design D_1 can be *embedded* in an $(n+1, 1)$-design D_2 if there exists a subset W of the variety set of D_2 such that the restriction of D_2 to W is isomorphic to D_1. We note that a $PIC(n, \alpha)$ can be obtained by deleting α intersecting lines from a finite affine plane of order n. The following theorem is proved in Mullin and Vanstone (1976).

THEOREM 1.1. *Let D be any $PIC(n,\alpha)$ with $\alpha \geqq 4$. Then D can be embedded in a $PPC(n, \alpha-1)$. Further, if $n > (\alpha^4 - 2\alpha^3 + 2\alpha^2 + \alpha - 2)/2$, then D can be embedded in an affine plane of order n.*

In this paper we investigate the validity of this theorem for $\alpha = 3$. We briefly discuss the case $\alpha = 2$ which has been considered by deWitte.

2. Main result

For the purposes of this section let D be a $PIC(n, 3)$. It can easily be shown (Vanstone (1973)) that the longest block in an $(n+1, 1)$-design has cardinality less than or equal to $n+1$.

LEMMA 2.1. *D contains*

(i) $n-2$ blocks of size $n-1$.

(ii) $3(n-1)$ blocks of size $n-2$.

(iii) n^2-3n+2 blocks of size $n-3$.

PROOF. Let b_i be the number of blocks of size i in D. D contains $n^2+n-3-l$ blocks where l is a nonnegative integer. Elementary counting arguments yield the following.

$$\sum_{i=1}^{n+1} b_i = n^2+n-3-l.$$

$$\sum_{i=1}^{n+1} i b_i = (n+1)(n^2-3n+2)$$

$$\sum_{i=1}^{n+1} i(i-1)b_i = (n^2-3n+2)(n^2-3n+1).$$

Eliminating b_{n-2} and b_{n-3} from these equations gives

$$\sum_{i=1}^{n+1} [i-(n-2)][i-(n-3)]b_i = 2(n-2)-(n-2)(n-3)l.$$

Since $b_{n-1} \geqq n-2$ and all of the coefficients on the left side of this equation are nonnegative, we conclude that $l = b_1 = b_2 = \cdots = b_{n-4} = b_n = b_{n+1} = 0$. Solving for b_{n-2} and b_{n-3} gives the desired result.

LEMMA 2.2. *Suppose D contains a block B_1 of size $(n-i)$ and a block B_2 of size $(n-j)$ such that $B_1 \cap B_2 \neq \varnothing$ then B_i and B_j are mutually disjoint from $ij+i+j-3$ blocks of D for $1 \leqq i, j \leqq 3$.*

PROOF. Immediate.

LEMMA 2.3. *If $n \geqq 7$ then D contains two blocks of size $n-1$ which are disjoint.*

PROOF. Any block of size $n-1$ is disjoint from precisely $2n-4$ blocks of D. Suppose no two blocks of size $n-1$ are disjoint; then, the sets of blocks disjoint from each block of size $n-1$ must all be disjoint. Hence

$$(n-2)(2n-4)+n-2 \leqq n^2+n-3.$$

This implies $n \leqq 6$ and completes the proof.

THEOREM 2.1. (Main result). *For $n \geqq 14$ the blocks of size $n-1$ in D are mutually disjoint.*

PROOF. By Lemma 2.3 D contains blocks B and B^* of size $n-1$ which are disjoint. Let U be the set of $2n-4$ blocks of D which are disjoint from B. Clearly $B^* \in U$. If $x \in V \backslash B$, x must occur with each element of B precisely once and since x must occur in $n+1$ blocks of D, x must be contained in two blocks of U. Hence, if V is the variety set of D then every element of $V \backslash B$ is contained in precisely 2 blocks of U.

Let T^* be the blocks of U other than B^* which contain an element of B^* and T be the $n-4$ blocks of U which are disjoint from B^*. If T^* has a block B_1 of size $n-1$ then by Lemma 2.2 B^* and B_1 are mutually disjoint from no blocks of D which would contradict the fact that they are disjoint from B. Hence, T^* contains only blocks of size $n-2$ and $n-3$.

Let C be a block of T^* of size $n-i$, $i=2$ or 3. C is disjoint from precisely $n+i-5$ blocks of U. Since C and B^* intersect they are mutually disjoint from $2i-2$ blocks of D. But C and B^* are disjoint from B and hence C is disjoint from at most $2i-3$ blocks of T. Thus C is disjoint from at least $n+i-5-2i+3$ or $n-i-2$ blocks of T^*.

Now suppose T^* contains a block E of size $n-2$. E is disjoint from at least $n-4$ blocks of T^*. Suppose E intersects a block F of T^* other than itself. Since F is disjoint from at least $n-i-2$ blocks of T^* where $i=2$ or 3 then E and F are mutually disjoint from at least $n-i-3$ blocks of T^*. But by Lemma 2.2 they are disjoint from at most $3i-2$ blocks of T^*. Therefore

$$n - i - 3 \leqq 3i - 2$$

or

$$n \leqq 4i + 1 \leqq 13.$$

Hence if $n > 13$ then any block of size $n - 2$ in T^* is disjoint from all blocks of T^*.

Suppose C is a block of size $n - 2$ in T^* and let $x \in B^* \cap C$. By the above arguments, since $n > 13$, C is disjoint from all other blocks of T^* and so the element of $C \backslash \{x\}$ must each occur in T once and no two in a common block. Since $|C \backslash \{x\}| = n - 3$ and $|T| = n - 4$, this is impossible. Thus T^* contains only blocks of size $n - 3$. Simple counting then shows that all the blocks of T are of size $n - 1$. Lemma 2.2 implies that any blocks of size $n - 1$ in T must be disjoint. This completes the proof.

With this result we establish the following.

THEOREM 2.2. *For $n \geqq 14$, D is embeddable in a $PPC(n, 2)$. Moreover, D is embeddable in a finite projective plane of order n.*

With Theorem 2.1, the proof of this result follows exactly as that given in Mullin and Vanstone (1976) for the case of $\alpha > 3$ and so we omit the proof.

It should be noted that for $n = 5$ there are two examples of $PIC(5, 3)$ which are not embeddable in a finite projective plane of order 5. One such example can be found in Mullin and Vanstone (1976). We display both examples in Section 4.

3. The $\alpha = 2$ case

As mentioned earlier P. deWitte (private communication) has considered the case of a $PIC(n, 2)$ and has determined when it is embeddable. For completeness, we make several observations about this case.

If D is a $PIC(n, 2)$ then D has $n^2 + n - 2$ blocks, $n^2 - 2n + 1$ varieties and at least $n - 1$ blocks of size $n - 1$. From this one can deduce as in Lemma 2.1 that D contains $3(n - 1)$ blocks of size $n - 1$ and $n^2 - 2n + 1$ blocks of size $n - 2$, and that any variety of D is contained in 3 blocks of size $n - 1$ and $n - 2$ of size $n - 2$. Deleting a block B of size $n - 1$ and all of its varieties from D gives a $PIC(n, 3)$. Theorem 2.1 then implies that for $n \geqq 14$, B and the $n - 2$ blocks of size $n - 1$ disjoint from it are mutually disjoint and that D is embeddable in a $PPC(n, 1)$.

Instead of using the results of $PIC(n, 3)$ to prove results on $PIC(n, 2)$ we could prove a sequence of results analogous to those of Section 2 which would yield the following.

THEOREM 3.1. *Let B be a block of size $n-1$ in a $PIC(n,2)$ for $n>7$. Then B and the $n-2$ blocks of size $n-1$ in D disjoint from B are mutually disjoint.*

This implies that any $PIC(n,2)$ for $n>7$ is embeddable in a finite projective plane of order n.

4. Some observations

In this section we record some observations which may be useful in settling the case $\alpha=3$ and $n \leqq 13$. Using the standard terminology of geometry, we say that two blocks of an $(n+1,1)$-design are parallel if they do not intersect.

LEMMA 4.1. *In a $PIC(n,3)$, parallelism is an equivalence relation on the blocks of size $n-1$.*

PROOF. It is clear that parallelism is reflexive and symmetric. We show that it is transitive. Let B_1, B_2, B_3 be three blocks of size $n-1$ in the $PIC(n,3)$ D, and suppose B_1 is parallel to B_2 and B_2 is parallel to B_3. Suppose B_1 is not parallel to B_3. By Lemma 2.2, B_1 and B_3 are mutually disjoint from no other blocks in D. This contradicts the fact that they are mutually disjoint from B_2. Therefore, B_1 is parallel to B_3 and the proof is complete.

This lemma ensures us that the $n-2$ blocks of size $n-1$ in D partition into classes (called parallel classes) $P_1, P_2, \cdots, P_t$ such that any two distinct blocks in P_i $(1 \leqq i \leqq t)$ are parallel and any block in P_i intersects any block in P_j for $i \neq j$. Theorem 2.1 essentially proves that, for $n \geqq 14$, the $(n-1)$ blocks partition into only one parallel class in a $PIC(n,3)$. The next result shows that, for $6 \leqq n \leqq 13$, the $n-1$ blocks partition into at most 2 parallel classes.

THEOREM 4.1. *Let D be a $PIC(n,3)$ where $n \geqq 6$. Then, the blocks of size $n-1$ in D partition into at most two parallel classes.*

PROOF. Suppose the $n-1$ blocks of D partition into parallel classes $P_1, P_2, \cdots, P_t$ and let $|P_i| = \alpha_i$, $1 \leqq i \leqq t$. Consider any block B of size $n-1$ in D. B is disjoint from precisely $2n-4$ other blocks, denoted T_B. Let a, b and c be the number of blocks of size $n-1$, $n-2$, and $n-3$ respectively in T_B. Thus

$$a+b+c=2n-4$$

and since every variety, excluding those in B, is contained in 2 blocks of T_B.

$$(n-1)a+(n-2)b+(n-3)c=2(n-1)(n-3).$$

From these equations,

$$2a + b = 2(n - 3). \tag{4.1}$$

If B and B' are any two intersecting blocks of size $n - 1$, then by Lemma 2.2, T_B is disjoint from $T_{B'}$.

Let B_i be any block in P_i for $1 \leqq i \leqq t$. Counting the number of blocks of size $n - 2$ in all T_{B_i}, $1 \leqq i \leqq t$, we get

$$\sum_{i=1}^{t} 2(n-3) - 2(\alpha_i - 1).$$

But D contains precisely $3(n - 1)$ blocks of size $n - 2$. Hence,

$$\sum_{i=1}^{t} 2(n-3) - 2(\alpha_i - 1) \leqq 3(n-1).$$

Since $\sum_{i=1}^{t} \alpha_i = n - 2$, we obtain

$$n \leqq \frac{4t-7}{2t-5} \leqq 5$$

unless $t \leqq 2$. This completes the proof.

For $n = 5$, the $n - 1$ blocks in a $PIC(5, 3)$ can partition into at most 3 nonempty classes. For 3 classes, each must contain precisely one block, for 2 classes, one contains 2 blocks the other 1 and of course there may be only one class. All three of these situations occur. An example of the last case mentioned can be obtained from the affine plane of order 5. Examples of the other two cases are given below. They are unique up to isomorphism.

1234	36T	1TE	49E	78V	27	47	1V	5T
1567	37E	39V	5VE	79T	35	18	2E	69
2589	46V	48T	68E	2TV	45	19	38	26
1234	38T	1EV	4TE	36V	16	5E	6T	9V
5678	279	26E	89E	7TV	17	25	47	39
159T	469	37E	45V	28V	18	35	48	2T.

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