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ON A PROBLEM OF BARNES AND DUNCAN

by R. G. McLEAN

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Consider the free monoid on a non-empty set P, and let R be the quotient monoid determined by the relations:

$$p^2 = p \quad \forall p \in P.$$

Let R have its natural involution $*$ in which each element of P is Hermitian. We show that the Banach $*$-algebra $\ell^1(R)$ has a separating family of finite dimensional $*$-representations and consequently is $*$-semisimple. This generalizes a result of B. A. Barnes and J. Duncan (*J. Funct. Anal.* **18** (1975), 96–113.) dealing with the case where P has two elements.

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We equip R with its natural involution $*$ in which each element of P is Hermitian. When P contains exactly two elements Barnes and Duncan [**2**] have shown that the Banach $*$-algebra $\ell^1(R)$ has a separating family of finite dimensional $*$-representations. We show that this result is in fact true for an arbitrary P. It follows that $\ell^1(R)$ is $*$-semisimple.

Let S be a monoid i.e. a semigroup with an identity element 1. By a representation π of S we shall mean a bounded map π from S into the set of all bounded linear operators on a (real or complex) Hilbert space H such that $\pi(1)$ is the identity operator and $\pi(xy) = \pi(x)\pi(y)$ for all $x, y \in S$. Each representation of S has a unique extension to a representation of the Banach algebra $\ell^1(S)$. Tensor products and direct sums may be formed in a similar way to those of group representations.

Definition. A representation π of a monoid S on a Hilbert space H will be called *formally real* is there is an orthonormal basis $\{e_i | i \in I\}$ of H for which $\langle \pi(x)e_s | e_r \rangle$ is real for all $x \in S$ and all $r, s \in I$.

It is known that if the $*$-representations of S are separating then the Banach $*$-algebra $\ell^1(s)$ is $*$-semisimple. This follows from Theorem 3.4 of [**1**] (a direct proof may be found in [3]). The following lemma shows that under suitable conditions a separating set of representations of S gives rise to a separating set of representations of $\ell^1(S)$.

Lemma. *Let $\mathscr{Q}$ be a set of formally real representations of a monoid S and let $\mathscr{R}$ be the set of representations of the Banach $*$-algebra $\ell^1(S)$ consisting of the one dimensional identity representation together with all extensions of finite tensor products of members of $\mathscr{Q}$, If $\mathscr{Q}$ separates points of S, then $\mathscr{R}$ separates points of $\ell^1(S)$.*

Proof. Let $\mathscr{A}$ be the set of all functions $f: S \to \mathbb{F}$ with

$$f(x) = \langle \pi(x)\phi | \psi \rangle \qquad \forall x \in S.$$

for some representation π of S on a Hilbert space H and some vectors $\phi, \psi \in H$, where π is a finite direct sum of representations from $\mathscr{R}$. Then $\mathscr{A}$ is a self-conjugate unital subalgebra of $\ell^\infty(S)$ which separates points of S so by Theorem 1 of [3], if f is a non-zero element of $\ell^1(S)$, then there is a $g \in \mathscr{A}$ with

$$\sum_{x \in S} f(x)g(x) \neq 0.$$

It follows that there is a $\pi \in \mathscr{R}$ with $\pi(f) \neq 0$. □

Theorem. *The Banach $*$-algebra $\ell^1(R)$ has a separating family of finite dimensional $*$-representations.*

Proof. Let T denote the quotient of the free monoid on $\{u, v\}$ determined by the equations $u^2 = u$ and $v^2 = v$; then T has an injective two dimensional $*$-representation. For if π is the $*$-representation, considered in [2], defined by

$$\pi(u) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi(v) = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

then for any $n \in \mathbb{N}$:

$$\pi((uv)^n) = 2^{-n}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \pi((uv)^{n-1}u) = 2^{-(n-1)}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\pi((vu)^n) = 2^{-n}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \pi((vu)^{n-1}v) = 2^{-(n-1)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and it follows that π is injective.

For each ordered pair $(p,q)\in P^2$ with $p\neq q$, let $\psi_{(p,q)}$ be the surjective $*$-morphism from R to T determined by

$$\psi_{(p,q)}(w)=\begin{cases} u & \text{if } w=p \\ v & \text{if } w=q \\ 1 & \text{if } w=1 \text{ or } w\in P\backslash\{u,v\} \end{cases}$$

and let $\pi_{(p,q)}$ be the $*$-representation $\pi\circ\psi_{(p,q)}$. We show that the set $\mathcal{Q}=\{\pi_{(p,q)}| p\in P, q\in P, p\neq q\}$ separates points of R.

Each element of $R\backslash\{1\}$ may be written as a word in the alphabet P in which no two adjacent letters are equal. Let x and y be distinct elements of R. If one of these is the identity let p be the first letter of the other, then $\psi_{(p,q)}(x)\neq\psi_{(p,q)}(y)$ for any $q\in P$.

If one word, say x, is a prefix of the other, then there are $p,q\in P$ and $a,b\in R$ with

$$x=ap \quad \text{and} \quad y=apqb$$

and now

$$\psi_{(p,q)}(x)\neq\psi_{(p,q)}(y).$$

Otherwise let $p,q\in P$ be the first two letters in which the words x and y differ; then clearly

$$\psi_{(p,q)}(x)\neq\psi_{(p,q)}(y).$$

Since π is injective it follows that $\mathcal{Q}$ separates points of R. Also since π is two dimensional it follows from the lemma that $\ell^1(R)$ has a separating family of finite dimensional $*$-representations. □

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Department of Mathematics
The City University
Northampton Square
London EC1V 0HB
United Kingdom