A CONSTRUCTION FOR WYTHOFFIAN POLYTOPES

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1. Introduction. This paper contains an account of a simple method of deriving the coordinates of the vertices of any uniform polytope or honeycomb (degenerate polytope) whose symmetry group is generated by reflections.

Polytopes and honeycombs of this type have been described by many authors, amongst whom must be mentioned Schläfli **(10)**, Gosset **(8)**, Mrs. Boole Stott **(14)**, Schoute **(12**; **13)**, Elte **(7)**, Robinson **(9)**, and Coxeter **(1**; **2**; **3**; **5)**. The whole theory of uniform polytopes was unified by Coxeter **(4**; **6**, pp.86, 196**)**, who adapted Wythoff's construction **(15)** to obtain a general geometrical method for obtaining all the uniform polytopes whose symmetry groups are generated by reflections.[1] His discussion was elegantly illustrated by the use of a graphical notation **(7**, p. 191; **4**, p. 329**)**.

One of the most comprehensive discussions of uniform polytopes in analytical terms is that of Schoute **(11**; **12)**, whose paper, in four parts, comprises a commentary of 190 pages on Mrs. Boole Stott's geometrical methods. As Professor Coxeter remarked to me in a letter, "it is sad to think how much unnecessary work Schoute did, through not anticipating Wythoff's construction."

This present paper is concerned with an analytical account of the Wythoffian polytopes and is based principally on the geometrical ideas of Coxeter's paper **(4)**. After the determination, for each group, of a set of *basic vectors*, the coordinates of the vertices of any uniform polytope associated with that group may be written down. A modified form of the same method can be applied to determining the coordinates of the vertices of the Wythoffian honeycombs.

2. Finite groups. Suppose that $\mathfrak{G}$ is a finite n-dimensional group generated by reflections in n primes whose point of concurrency is O, the origin of the (cartesian) coordinate system. Considering the reflections as operating on an $(n-1)$-dimensional sphere whose centre is O, the fundamental region of the group may be taken to be a spherical simplex whose bounding figures are the intersections of the sphere with a specially chosen set of primes, reflections in which generate the group **(6**, pp. 188–191**)**.

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[1]It is convenient to call these polytopes *Wythoffian*. In **(4)** Coxeter uses the word Wythoffian in a different sense to include some uniform polytopes whose symmetry groups are not generated by reflections, namely the "snub" polytopes in three and four dimensions.

If we represent the simplex (and therefore the reflection group) by a Coxeter graph (**4**, p. 329), then we may suppose that the ith node of the graph (the nodes being numbered in some arbitrary manner) corresponds to the prime $\mathbf{p}_i$ which intersects the sphere in the bounding figure $P_1 P_2 \ldots P_{i-1} P_{i+1} \ldots P_n$ of the simplex $P_1 P_2 \ldots P_n$.

Now define n basic vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ in the following manner: $\mathbf{r}_i$ is in direction OP_i and the distance of its end point from $\mathbf{p}_i$ is $\frac{1}{2}$. Thus, considering

2.1 $$\mathbf{r}_i = (r_{i1}, r_{i2}, \ldots, r_{in})$$

as the coordinate vector of the point R_i, then the reflection $\mathbf{r}_i^*$ of $\mathbf{r}_i$ in $\mathbf{p}_i$ is the coordinate vector of R_i^* which is at unit distance from R_i.

The following is the basic result:

2.2 *One vertex of any Wythoffian polytope (of unit edge length) derived from the group $\mathfrak{G}$ has the coordinate vector*

$$\epsilon_1\mathbf{r}_1 + \epsilon_2\mathbf{r}_2 + \ldots + \epsilon_n\mathbf{r}_n,$$

where $\epsilon_i = 1$ if the ith node of the graph is ringed, and $\epsilon_i = 0$ if the ith node of the graph is not ringed.

The other vertices of the polytope may be found by applying the operations of $\mathfrak{G}$, that is, by repeated reflections in the primes $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$. Consequently, when we have found the set of basic vectors for $\mathfrak{G}$, we can immediately determine the coordinates of any polytope found by applying Wythoff's construction to $\mathfrak{G}$.

2.3 *Example*: $\mathfrak{G} = C_3$, the symmetry group of the cube (order 48).

Graph	Number of node	Basic vector	Reflecting plane
 4	1	$\mathbf{r}_1 = \frac{1}{2}(1, 1, 1)$	$x_3 = 0$
	2	$\mathbf{r}_2 = \frac{1}{2}(\sqrt{2}, \sqrt{2}, 0)$	$x_2 - x_3 = 0$
	3	$\mathbf{r}_3 = \frac{1}{2}(\sqrt{2}, 0, 0)$	$x_1 - x_2 = 0$

The operations of this group correspond to permuting the coordinates in every way and also to changing the sign of any one. We write the values of the ϵ_i in the form $\langle\epsilon_1, \epsilon_2, \epsilon_3\rangle$ and so derive coordinates for the seven Wythoffian derivatives of C_3, as in Table 2.4.

The proof of the basic result follows immediately from Coxeter's account of Wythoff's construction. Evidently the "first vertex" is left invariant by a

2.4 *Table of polyhedra associated with C_3.*

(All the coordinates in this table have been multiplied by 2, and so relate to polyhedra of edge length 2.)

$\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$	First vertex	Number of vertices	Coordinates of vertices[2]	Name	
$\langle 1, 0, 0 \rangle$	$(1, 1, 1)$	8	$(\pm 1, \pm 1, \pm 1)$	Cube	$t_0\gamma_3 = \gamma_3$
$\langle 0, 1, 0 \rangle$	$(\sqrt{2}, \sqrt{2}, 0)$	12	$(\pm\sqrt{2}, \pm\sqrt{2}, 0)'$	Cuboctahedron	$t_1\gamma_3$
$\langle 0, 0, 1 \rangle$	$(\sqrt{2}, 0, 0)$	6	$(\pm\sqrt{2}, 0, 0)'$	Octahedron	$t_2\gamma_3$
$\langle 0, 1, 1 \rangle$	$(2\sqrt{2}, \sqrt{2}, 0)$	24	$(\pm 2\sqrt{2}, \sqrt{2}, 0)'$	Truncated octahedron	$t_{1,2}\gamma_3$
$\langle 1, 0, 1 \rangle$	$(1+\sqrt{2}, 1, 1)$	24	$(\pm(1+\sqrt{2}), \pm 1, \pm 1)'$	Rhombicuboctahedron	$t_{0,2}\gamma_3$
$\langle 1, 1, 0 \rangle$	$(1+\sqrt{2}, 1+\sqrt{2}, 1)$	24	$(\pm(1+\sqrt{2}), \pm(1+\sqrt{2}), \pm 1)'$	Truncated cube	$t_{0,1}\gamma_3$
$\langle 1, 1, 1 \rangle$	$(1+2\sqrt{2}, 1+\sqrt{2}, 1)$	48	$(\pm(1+2\sqrt{2}), \pm(1+\sqrt{2}), \pm 1)'$	Truncated cuboctahedron	$t_{0,1,2}\gamma_3$

[2]A prime (′) implies that the coordinates are to be permuted in every possible way.

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reflection if the corresponding ϵ_i is zero, or is transformed into a point at unit distance if the corresponding ϵ_i is unity.

By allowing ϵ_i to take other values, the vertices of polytopes may be derived whose bounding figures are parallel to the corresponding bounding figures of a uniform polytope but whose edges are equal in length to the values of the non-zero ϵ_i. (See for example **(4**, p. 336**)**. Here a ringed node marked $\sqrt{2}$ is taken to indicate that ϵ_i is to be given the value $\sqrt{2}$.)

The basic vectors, and their reflections in the primes, are precisely the translations to be effected on the bounding figures in the "expansions" and "contractions" of Mrs. Stott's method **(14)**.

3. Infinite groups. A similar method may be used for the coordinates of the vertices of a degenerate polytope (honeycomb) in n dimensions. In this case the fundamental region consists of a Euclidean simplex $P_1 P_2 \ldots P_{n+1}$ of which the vertex P_{n+1} is chosen as the origin O of the coordinate system. The method differs from that for finite groups on account of the fact that if all the Wythoffian polytopes associated with a given group have edges of unit length, the "scale" of the group (i.e., the size of the fundamental simplex) may be different in each case. We proceed as follows:

The primes $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are defined to be the faces of the fundamental simplex that pass through O, and $\mathbf{p}_{n+1}(c)$ to be the prime

$$\sum_{i=1}^{n} a_i x_i = c$$

parallel to the face $P_1 P_2 \ldots P_n$ of the simplex, and normalized so that $\Sigma a_i^2 = 1$. The size of the simplex is therefore altered by varying the value of the constant c. The vectors $\mathbf{r}_i$ are defined as before, that is, $\mathbf{r}_i$ lies along the line of intersection of all the primes except $\mathbf{p}_i$ and $\mathbf{p}_{n+1}$ and its end point is at distance $\frac{1}{2}$ from $\mathbf{p}_i$. Also $\mathbf{r}_{n+1}$ is the zero vector. Define also the $n + 1$ constants by the relations

$$\sum_{j=1}^{n} a_j r_{ij} = c_i \qquad (i = 1, 2, \ldots, n),$$

$$\tfrac{1}{2} = c_{n+1},$$

where $\mathbf{r}_i$ is taken in the form 2.1.

The first vertex of the honeycomb has coordinate vector

$$\epsilon_1 \mathbf{r}_1 + \epsilon_2 \mathbf{r}_2 + \ldots + \epsilon_{n+1} \mathbf{r}_{n+1}$$

where $\epsilon_i = 1$ if the ith node of the graph is ringed, $\epsilon_i = 0$ if the ith node of the graph is not ringed, and the other vertices are given by repeated reflections of the first vertex in $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ and

$$\sum a_j x_j = \epsilon_1 c_1 + \epsilon_2 c_2 + \ldots + \epsilon_{n+1} c_{n+1}.$$

3.2 *Table of honeycombs associated with R_4.*

(All the coordinates in this table have been multiplied by 2, and so relate to honeycombs of edge length 2.)

$\langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle$	First vertex	Coordinates of vertices	Honey-comb	Polyhedra in honeycomb
$\langle 1, 0, 0, 0\rangle$	$(1, 1, 1)$	$(\pm 1, \pm 1, \pm 1)$ (mod 2)	$t_0\delta_4 = \delta_4$	Cubes
$\langle 0, 1, 0, 0\rangle$	$(\sqrt{2}, \sqrt{2}, 0)$	$(\pm\sqrt{2}, \pm\sqrt{2}, 0)'$ (mod $2\sqrt{2}$)	$t_1\delta_4$	Octahedra and cuboctahedra
$\langle 1, 1, 0, 0\rangle$	$(1+\sqrt{2}, 1+\sqrt{2}, 1)$	$(\pm(1+\sqrt{2}), \pm(1+\sqrt{2}), \pm 1)'$ (mod $2(1+\sqrt{2})$)	$t_{0,1}\delta_4$	Truncated cubes and octahedra
$\langle 1, 0, 1, 0\rangle$	$(1+\sqrt{2}, 1, 1)$	$(\pm(1+\sqrt{2}), \pm 1, \pm 1)'$ (mod $2(1+\sqrt{2})$)	$t_{0,2}\delta_4$	Rhombicuboctahedra, cubes and octahedra
$\langle 1, 0, 0, 1\rangle$	$(1, 1, 1)$	$(\pm 1, \pm 1, \pm 1)$ (mod 4)	$t_{0,3}\delta_4 = \delta_4$	Cubes
$\langle 0, 1, 1, 0\rangle$	$(2\sqrt{2}, \sqrt{2}, 0)$	$(\pm 2\sqrt{2}, \pm\sqrt{2}, 0)'$ (mod $4\sqrt{2}$)	$t_{1,2}\delta_4$	Truncated octahedra
$\langle 1, 1, 1, 0\rangle$	$(1+2\sqrt{2}, 1+\sqrt{2}, 1)$	$(\pm(1+2\sqrt{2}), \pm(1+\sqrt{2}), \pm 1)'$ (mod $2(1+2\sqrt{2})$)	$t_{0,1,2}\delta_4$	Truncated cuboctahedra, cubes and truncated octahedra
$\langle 1, 1, 0, 1\rangle$	$(1+\sqrt{2}, 1+\sqrt{2}, 1)$	$(\pm(1+\sqrt{2}), \pm(1+\sqrt{2}), \pm 1)'$ (mod $2(2+\sqrt{2})$)	$t_{0,1,3}\delta_4$	Truncated cubes, cubes, octagonal prisms and rhombicuboctahedra
$\langle 1, 1, 1, 1\rangle$	$(1+2\sqrt{2}, 1+\sqrt{2}, 1)$	$(\frac{1}{2}(1+2\sqrt{2}), \pm(1+\sqrt{2}), \pm 1)'$ (mod $4(1+\sqrt{2})$)	$t_{0,1,2,3}\delta_4$	Truncated cuboctahedra and octagonal prisms

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3.1 *Example*: $\mathfrak{G} = R_4$, the symmetry group of uniformly packed cubes.

Graph	Number of node	Basic vector	C_i	Reflecting plane
	1	$\mathbf{r}_1 = \frac{1}{2}(1, 1, 1)$	$\frac{1}{2}$	$x_3 = 0$
	2	$\mathbf{r}_2 = \frac{1}{2}(\sqrt{2}, \sqrt{2}, 0)$	$\frac{1}{2}\sqrt{2}$	$x_2 - x_3 = 0$
	3	$\mathbf{r}_3 = \frac{1}{2}(\sqrt{2}, 0, 0)$	$\frac{1}{2}\sqrt{2}$	$x_1 - x_2 = 0$
	4	$\mathbf{r}_4 = (0, 0, 0)$	$\frac{1}{2}$	$x_1 = c$

Reflections in $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are equivalent to permuting the coordinates in every way, and altering the sign of any coordinate. Reflections in all four planes include the operation of increasing any coordinate by a multiple of $2c$. Hence the coordinates of the vertices may be written

$$(x_1, x_2, x_3) \qquad \pmod{2c},$$

though this may not be the simplest or most elegant form. These points evidently form a number of lattices.

Owing to the fact that the graph is symmetrical, only nine of the fifteen Wythoffian derivatives (Table 3.2) are distinct (**5**, pp. 402–403).[3]

[3]On page 403, the symbols for $h_3\delta_4$ and $h_{2,3}\delta_4$ have been accidentally transposed.

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