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The late William Matthew Makeham.

BY the death of Mr. William Matthew Makeham, so feelingly alluded to by the President of the Institute at the opening of the present session, the scientific literature of life contingencies has lost one of its most brilliant names. A reference to the pages of this *Journal* will convey some impression of the importance and variety of the subjects he dealt with from time to time, but the productions of such a master mind cannot be fitly judged by their length or number. In some of his shorter contributions the highest level of scientific ability and invention is reached, and in all—with scarcely a single exception—are exhibited remarkable powers of insight and analysis, combined with a lucid and luminous mode of exposition, rarely found in association save in intellects of the highest calibre. The value of Makeham's writings will endure for all time, and his will always be a foremost place among those early exponents of actuarial science who guided it into the channels along which it has steadily progressed.

Although Makeham's name is so closely associated with his formula of graduation, or "Law of Mortality", as it is indifferently termed, there are other eminent achievements of his which will immediately present themselves to the minds of readers of this *Journal.* The concentration of thought and mathematical ingenuity, shown in his masterly contributions on Compound Survivorship Problems (*J.I.A.*, x, 241, and xii, 61), have called forth high praise, and led the way to greater simplification in the solution of these intricate and important matters. Another striking instance of Makeham's extraordinary ability to grapple with the most complicated problems, and reduce them to simple elements, is seen in his paper "On the Theory of Annuities-Certain" (*J.I.A.*, xiv, 189), than which nothing more elegant has, perhaps, found a place in the *Journal*. The easily applied formulas he there obtained for deriving the values of increasing annuities-certain are too well known to need further remark.

With equal success he dealt with the question of Extra Risks (*J.I.A.*, xiv, 159, 242, and xvii, 153), and it is interesting to reproduce the editorial note appended to the last of these papers, as evidence of his great practical foresight:

"*** The above paper will, no doubt, prove practically useful to the managers of the companies which adopt the method of charging for impaired health therein discussed,* but it appears to us to be even more valuable to the student of the theory of life contingencies, as suggesting the course which future investigations into the mortality among under-average lives should follow. Mr. Makeham's investigations clearly demonstrate that the point to be ascertained is whether the increased mortality among such lives is more apparent immediately after the grant of the policy or in later years; and any future investigation into such mortality must be considered incomplete that does not give special attention to this point."—ED. *J.I.A.*

Other contributions of great merit, too numerous to mention, are scattered throughout the pages of this *Journal*. The "continuous method" now so often employed, and destined, we may confidently predict, to play an even more important part in actuarial analysis, received its impetus from his early papers. His many purely mathematical essays—the last of which appeared shortly before his death was announced—are valuable additions to the scientific equipment of a modern actuary. They have been justly admired, both in this country and in America, where Makeham's writings are held in the highest esteem. The amount of instruction conveyed in a short space is often astonishing, and, to appraise Makeham's worth as an author, we have only to imagine what actuarial knowledge would now be could his writings be expunged from the records and obliterated from the memory.

* That is, by way of contingent debt on the policy.

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We have left till the last our reference to the papers dealing with the formula of graduation, or law of mortality, known as Makeham's hypothesis, to which a somewhat extended consideration will not be thought untimely. The first indication of his important modification of Gompertz's theoretical law of mortality was given to the world in January 1860 (*J.I.A.*, viii, 301). After premising that Gompertz's assumption could be defined "by "stating that the logarithms of the probabilities of living over "any given period proceed in geometrical progression", Makeham showed, by the aid of the Carlisle, Seventeen Offices', and Government Annuitants' Tables, that the facts did not conform to Gompertz's theory; and he proposed to make them obey the law assigned by the addition of a certain uniform quantity (x) to each term. The terms here referred to were the logarithms of ${}_np_x$; and, Gompertz's formula containing three constants, three values of the function were necessary for obtaining the values of such constants. Makeham operated upon l_{20}, l_{40}, and l_{60}, whence he passed to ${}_{20}p_{20}$, ${}_{20}p_{40}$, and ${}_{20}p_{60}$, and the logarithms of these quantities formed the terms which, by Gompertz's hypothesis, should be in geometrical progression. Denoting these terms by the letters (a), (b), and (c), the numerical value of (x) was easily found. Since $(a+x)$, $(b+x)$, and $(c+x)$ are in geometrical progression, we have

$$(a+x)(c+x)=(b+x)^2$$

whence

$$x=\frac{b^2-ac}{a+c-2b}.$$

By this means, the terms were brought into conformity with the hypothesis, and the law of mortality, as modified, was defined by stating that the "probabilities of living, *increased or diminished* "*in a certain constant ratio*, form a series whose logarithms are "in geometrical progression."

From this simple device flowed all the merits and advantages which are associated with Makeham's formula. The close agreement of the hypothetical law of mortality with that shown in the Seventeen Offices' Experience was first pointed out, and the unique facilities afforded for the computation of annuities on any given number of lives were then demonstrated. The next stage in the development of the theory (*J.I.A.*, xiii, 325) was to deduce the well-known formula for the force of mortality:

$$\mu_x=\text{A}+\text{B}q^x.$$

This formula, the author explains, "is derived from a modification "of Mr. Gompertz's simple and highly ingenious theory, that the "power to oppose destruction loses equal proportions in equal "times. The modification which I propose to introduce consists "in the limitation of the theory to a *portion only* of the partial "forces of mortality, and the assumption that the remaining "forces operate (in the aggregate) with equal intensity at all "ages." In the above formula, it is scarcely necessary to say, A is the sum of certain partial forces assumed to be of equal amount at all ages, and Bq^x the aggregate of several forces of a similar nature, but increasing obviously in constant relation to the age.

It is curious to look back, and observe how slow Makeham's valuable and ingenious suggestions were to gain acceptance. No comment on his exposition of the new hypothesis is to be found in our *Journal* until, four years after publicity had been given to it, the late Peter Gray, in a generous and highly appreciative letter, written in November 1863 (*J.I.A.*, xi, 236), enforced the value and importance of the new investigations. We cannot refrain from thinking that the time of their publication was, by a freak of chance, unfavourable to that early discussion which the merit and originality of Makeham's suggestions must be held to have deserved. The burning topic then, and for some time afterwards, was the claim—the unwarrantable claim, as we must now regard it—of Mr. T. R. Edmonds to have discovered an original and independent law of mortality; and the leading mathematical minds of that generation, De Morgan, Mr. Sprague, and Mr. Woolhouse, were engaged in proving that Edmonds' "law of mortality" was, to all intents and purposes, identical with that which Gompertz had given to the world some years previously. If this conjecture appear far-fetched, it is not easy to comprehend the neglect of Makeham's theory on any other ground.

After Gray's letter, no mention was made of the subject until Makeham resumed his investigations in November 1865 (*J.I.A.*, xii, 305). The physiological interpretation of his theory was now given, and the hypothesis may be considered to have been fully enunciated. An incidental feature of this paper was the discovery that in the second enumeration of the population of Carlisle (in December 1787), as used by Milne in conjunction with the earlier enumeration (in January 1780) in the construction of the Carlisle Table, the ages of the members living were not derived by actual enumeration of the inhabitants age by age, but by distributing the later population, 8,677 in total, according to the numbers living at each age in January 1780, 7,677 in total, by increasing each item in the proportion of 8,677 to 7,677. This removed one of the pillars on which the Carlisle Table rested, and by diminishing confidence in the original data, paved the way to its being superseded as a standard table for life assurance calculations by more exact experiences.

Makeham's modified law of mortality was next mentioned, cursorily, by Mr. M. N. Adler, in his "Memoir of the late Benjamin Gompertz" (*J.I.A.*, xiii, 14); and, passing by brief congratulatory references at the annual meetings of the Institute, we then find it examined and discussed, in July 1870, by Mr. Woolhouse (*J.I.A.*, xv, 403), who, after pointing out the great advantages to be derived from the formula, expressed the view that future investigations might reveal other important relations at present unknown. From this point it would be useless to trace the progress of Makeham's theory in the actuarial mind. It had at last received due recognition, and has since steadily increased its adherents and advocates throughout the scientific world.

Passing notice may be made of the later developments of Makeham's famous hypothesis. Mr. Emory McClintock, pursuing an elaborate investigation by Makeham, showed how the value of annuities on any number of lives, by any given constants of mortality, at any given rate of interest, may be found by means of the ordinary tables of the well-known gamma-function (*J.I.A.*, xviii, 242). In treating the formula as a means of adjusting tables of experience, Messrs. G. F. Hardy and G. King (*J.I.A.*, xxii, 191) introduced a most important modification by making the constants depend, not upon isolated values of $\log l_x$, but upon the summation of the values of $\log l_x$ in groups. This process (termed the Aggregate Method) gave a broader base to the resulting table, and it will no doubt be invariably adopted in future attempts to graduate tables by Makeham's formula.

Quite recently Makeham, returning to the subject of his early triumphs, has shown how Gompertz's law may be still further developed (*J.I.A.*, xxviii, 191). The successive developments may be exhibited as follows:

1. Gompertz's law l_x	$= dg^{q^x}$	($\Delta^1 \log l_x$	in geom$^{l.}$	progression)
2. Makeham's first modification	$= dg^{q^x} s^x$	(Δ^2	"	")
3. Makeham's second modification	$= dg^{q^x} s^x \omega^{x^2}$	(Δ^3	"	")

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By means of an elaborate analysis, Makeham then proves that the new term, ω^{x^2}, introduced into (3) would involve an increase in the rate of interest, a set of tables for equal ages at *successive rates* of *interest* being required for the computation of annuities on joint lives. The effect of this further extension of Gompertz's hypothesis is well summarized by Mr. Woolhouse (*J.I.A.*, xxviii, 481*): "The great value of Mr. Makeham's first development is the "beautiful maintenance of the law of seniority under a somewhat "different but equally convenient form to that belonging to "Gompertz's law. When we come to Mr. Makeham's second "development, there is found to exist a similar law of seniority, "but it is unfortunately accompanied by an inconvenient impo-"sition of a new rate of interest, having a determinate value "depending on the relative set of lives which may enter any "special calculation." At this time, it would appear that the second development, owing to its complexity, offers little scope or inducement for its practical application, though it is, of course, impossible to say to what use it may be put in the future.

Makeham's law is open to the great objection that, when employed to adjust mortality tables, it produces a curve of unbroken smoothness, and thus erases all irregularities in the series, which may be accidental or characteristic, according to the nature of the original observations. It thus becomes an unsuitable method of graduation under certain conditions, as when, for instance, the curve of mortality rapidly changes its form and progression. It would, therefore, be inapplicable to tables showing the influence of "selection" in the early years of insurance, where the mortality exhibits a totally different course for the first few years to that which it afterwards takes. But in the adjustment of an "average" table, required mainly as a basis for monetary values, the consistent regularity of the results ceases to be an objection, and, provided the graduated series reproduces the rough series with general fidelity, it becomes a positive advantage. Hence, Makeham's formula has been frequently employed in dealing with important tables of experience. It was adopted by the Thirty American Offices for their Male Life Table, by the Gotha Life Office in their Experience, and has been very skilfully applied to the H^M Mortality by Mr. G. King in the table appended to the Institute *Text-Book*, Part II. In many minor but important investigations it has also been used.

Experience in the use of Makeham's method of graduation

* On page 481, the 17th line from the bottom, s^s is a misprint for s^x.

has brought to light one very interesting fact. The values of the constants *d, g,* and *s,* are different for different tables of observations, and even for different selected points (or groups) in the same table of observations. But a singular coincidence has been found to exist in the values of *q,* independently obtained from various bodies of facts. It is strange that, after his first investigation of the subject, Makeham himself should nowhere have given the constants derived from any of the numerous bodies of observations to which he applied his formula; but the curious agreement in the values of *q* was pointed out by him in 1867 (*J.I.A.*, xiii, 347) as significant and worthy of consideration. We may here collate the results of some of the principal applications of his method:

Makeham, Seventeen Offices' Table .	$\log q =$	·0409075
Woolhouse, H^{MF} 1st Curve . . .	,,	·0402225
,, ,, 2nd ,, ...	,,	·0395573
,, ,, Mean Curve . .	,,	·0400008
,, Seventeen Offices' Mean Curve	,,	·03956
Thirty American Offices . . .	,,	·041280
Gotha Life Office	,,	·039625
King, H^{M} (*Text-Book*) Aggregate Curve	,,	·03965686

It was stated by Makeham that in the tables which he had calculated by means of his formula, an average value of *q* (or $\log q = {\cdot}04$) could be used without materially affecting the results. Returning to this theme in one of his latest papers (*J.I.A.*, xxviii, 319), he proceeded to argue that the practically identical agreement in the value of $\log q$ could only result from the rate of deterioration of the vital force being the same for each individual. And he concluded, "Thus extended, Gompertz's law may be stated as " follows:—The vital force or recuperative power of each individual " loses equal proportions in equal times; and the proportion of " vital force so lost by each is *universally the same,* being approximately represented by $\log q = {\cdot}04$." Readers of these pages will not think we have spent too much time in bringing together the facts which bear upon and justify this memorable inference. Few such instances of the manifestation of a law, influencing and threading its way through vital statistics, are to be met with in the records of actuarial science; and it is a singular result, both full of suggestion and likely to prove of wide and useful application.

A final remark may be made regarding an interesting illustration of the significance of Makeham's formula for the *force* of *mortality*, $\mu_x = A + Bq^x$, given by Messrs. G. F. Hardy and H. J. Rothery in their paper on Mortality in the West Indies (*J.I.A.*, xxvii, 179). By comparing the values of the constants A and B for various tables of normal and extra mortality, the nature of the

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incidence of the extra risk was shown in a simple and striking manner. Taking two examples only, the respective values of the constants are as follows:

	A	B
H^M Table	·0061	·000093
West Indies (Barbados)	·0099	·000113

Comparing the values of A and B in the H^M and the West Indies observations, the authors pointed out that the effect of exposure to tropical climate upon the rate of mortality could be represented by a constant addition to the force of mortality (to allow for the increase in A) combined with an addition to the age (to correspond with the increase in B). And seeing that a constant addition to the force of mortality affects the annuity-values in the same way as an equivalent change in the rate of interest, Messrs. Hardy and Rothery found they were able to derive their special Barbados annuity-values at 4 per-cent interest from the H^M annuity-values by simply taking the latter values at $4\frac{1}{2}$ per-cent interest for ages three years older. In other words, a_x (Barbados 4 per-cent) is approximately equal to a_{x+3} (H^M $4\frac{1}{2}$ per-cent). In considering the influence of special mortality upon policy reserves, it was also clearly brought out that an increase in A while B remains constant diminishes the values of policies, while an increase in B while A remains constant increases them. In the course of the discussion, it was remarked that a further investigation of the relative values of A and B seemed to afford the means of determining the vexed question "How does an increased mortality affect policy-values?" There can be little doubt that further examinations of the constituent elements of the force of mortality, on the lines of Messrs. Hardy and Rothery's suggestive analysis, will throw light upon many of the problems to be met with in regard to the incidence of increased mortality.

This is not the place, nor is this the occasion, to dwell upon Makeham's personal characteristics—his unobtrusiveness, his love of the peace of domesticity, his sound literary judgment, and his many accomplishments outside the arena in which his skill was chiefly exhibited. But in closing this very imperfect account of the work he so effectively performed in the service of his profession, it is at once a pleasure and a duty to pay a tribute of admiration and gratitude to his memory for the zeal with which he devoted his energies to the advancement of actuarial science, and for the rare ability and inexhaustible resourcefulness which have opened up such fruitful fields of speculation to all earnest enquirers.

R.