ON FREQUENCIES AND SEMICONTINUOUS FUNCTIONS

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THIS paper deals with a particular class of distributive[1] properties which appear to be important for Analysis and which I call *frequencies*. They can be defined for any kind of sets but it essential for proper application that a condition **L** (the statement of Lindelöf's lemma)[2] is satisfied. From this condition follows Theorem 1, which is characteristic for frequencies but does not hold for other distributive properties. For every frequency F of a space $\sum$, one can build up an Analysis mod F of $\sum$; the classical case is the Analysis mod F_0. It is convenient to introduce the words "nearly every" with such a meaning that "every" and "almost every" are the special cases which, when we use the notation of this paper, correspond to $F = F_0$ and $F = F_C$. These notations are applied to the semicontinuous functions which are obtained by the upper and lower limiting operations and their iteration. In this way an appropriate tool for investigating the discontinuities of a function is obtained. The iteration of the limiting process leads to interesting "pairs" of functions which are the upper (lower) limiting functions of a set of functions. The coordination into pairs is independent of the frequency F, a fact which proves to be important for the investigation of the pairs. The notion of frequency is also useful for other purposes, e.g. for a generalization of uniform convergence.

1. Consider a set $\sum$ (called space) of elements (called points) in which a family of subsets (called open sets) is distinguished which satisfy the following condition:

L. If A is the join of an aggregate of open sets O_ν, then there exists a countable subset of sets O_ν such that A is the join of them.

This condition is satisfied e.g. for locally compact metric spaces when "open" has the usual meaning (*Lindelöf's lemma*).

We use in this paper the symbols $\cap$ and $\cup$ for the set-theoretical "meet" and "join." In particular $\cup_n A_n$ will denote the join of a countable aggregate of sets $A_1, A_2, \ldots$.

A property which, for a subset of $\sum$, either holds or does not hold, is called a *frequency* F when it satisfies the following three conditions:

Condition 1. F holds in $A = \cup_n A_n$ if and only if F holds in at least one A_n.

Condition 2. F does not hold in the empty set.

Condition 3. F holds in $\sum$.

Received September 29, 1948.

[1]Regarding distributive properties, see [2] pp. 9-14 and the literature quoted on p. 48.

[2]See [1] p. 46, [2] p. 38.

Conditions 2 and 3 are introduced for convenience only to exclude properties which either hold for every subset of $\sum$ or for none. Every frequency is a distributive property, but the converse does not hold. If F holds in $S \subseteq A$, it holds also in $A = S \cup A$. The property "to be infinite" which plays an important role in Analysis, is distributive, but not a frequency. Note the following frequencies which may hold in $\sum$:

F_0 = the property "not to be empty",

F_1 = the property "not to be countable",

and—in a space admitting a regular measure function (Caratheodory)—

F_C = the property "to have a positive measure".

If S has the frequency F, then the property of A that $A \cap S$ has the frequency F, is a frequency $F(S)$. If A has the frequency $F(S)$ we say also: "S has the frequency F in A". If every open set which contains a point $x \in \sum$ has the frequency $F(S)$, we say: "S has the frequency F *at* x". That F_a *implies* F_b, is denoted by $F_a \subseteq F_b$; in particular,

$$F_a \subseteq F_0. \tag{1}$$

The frequencies form a partially ordered set, which is not a lattice. The property that a set has two given frequencies F_a and F_b, is not necessarily a frequency; however the property that it has F_a *or* F_b, is a frequency:

$$F_a \cup F_b \supseteq F_a. \tag{2}$$

If $S = S_1 \cup S_2$ has the frequency F, but S_2 has not the frequency F, then $F(S) = F(S_1)$.

If there exists a countable set T in which F holds, then F holds also for some one-point-set $\{x\}$, where $x \in T$. Denote the join of the one-point-sets which have the frequency F, by S_0. If $A \subseteq \sum - S_0$ has the frequency F, then A is non-countable. If $B \cap S_0$ is non-empty, then B has the frequency F. Thus if S_0 is non-empty,

$$F = F_0(S_0) \cup F(\textstyle\sum - S_0), \quad \text{where} \quad F(\textstyle\sum - S_0) \subseteq F_1(\textstyle\sum - S_0). \tag{3}$$

As **L** holds in $\sum$, the join of the open sets O_n which have not the frequency F, can be represented by

$$\textstyle\sum'' = \bigcup_n O_n,$$

and from condition 1 it follows that $\sum''$ has not the frequency F. Thus for $\sum' = \sum - \sum''$, $F = F(\sum) = F(\sum')$. Moreover, $x \in \sum'$ if and only if $\sum$ has the frequency F at x. Hence:

THEOREM 1. *The points at which the space $\sum$ has the frequency F, form a subset $\sum'$, such that $\sum'$ has the frequency F at each of its points. $\sum'$ is non-empty.*

By applying Theorem 1 to the frequency $F(S)$, we obtain the following corollary:

https://doi.org/10.4153/CJM-1950-004-6 Published online by Cambridge University Press

Corollary. If S has the frequency F, then the points of S at which S has the frequency F form a non-empty subset S' and S' has the frequency F at each of its points.

Theorem 1 is well known for topological spaces admitting **L** when $F = F_1(B)$. In this case $\sum'$ is the set of the points of condensation of B; moreover it is known for $F = F_C(B)$. That the theorem cannot be generalized to distributive properties which are not frequencies, is seen from the property "to contain an infinite subset of B". This property holds at every limiting point of B, but these may be finite in number.

Given a frequency F, the word *nearly* will mean: *except a set which has not the frequency F*. Thus for $F = F_C$, "nearly every" becomes synonymous with "almost every", whereas for $F = F_0$, it means "every".

2. The frequencies are closely connected with the theory of measure; they can even be considered as special cases of a generalized measure theory which includes both the measure (Lebesgue, Haar etc.) in a locally compact metric space as well as the frequencies.

A measure function μ^* is a set function which for every $\alpha \subseteq \sum$ takes only real non-negative numbers and $+\infty$ as values. It is supposed to satisfy the conditions:

I $\mu^*(\alpha) \neq 0$ for a suitable α,

II $\mu^*(\alpha) \leqslant \mu^*(\alpha \cup \beta)$,

III $\mu^*(\bigcup_n \alpha_n) \leqslant \sum \mu^*(\mu_n)$,

IV If ω is open and $\alpha \subseteq \omega$, $\beta \subseteq \sum - \omega$ are compact, then $\mu^*(\alpha \cup \beta) = \mu^*(\alpha) + \mu^*(\beta)$.

The theory can be generalized; the values of $\mu^*(\alpha)$ may belong to any linearly ordered system V provided an *addition* for every countable subset of V is defined and this addition satisfies the condition:

$$\sum_n a_n \geqslant a_n. \tag{4}$$

V may consist of two elements only, say 0 and 1, and a sum may be equal to 1 if and only if at least one of the terms is equal to 1; e.g., we may put $\mu^*(\alpha) = 1$ when α has the frequency F, otherwise 0. The theory of measurable sets can be developed without any reference to properties of real numbers other than those supposed to hold for V. From any generalized measure function we can deduce frequencies in the following way. We subdivide V into a well-ordered sequence of "sections", say

$$\ldots \subset S_\nu \subset S_{\nu+1} \subset \ldots,$$

such that every section is closed for addition and contains, with every element a, also the elements $\leqslant a$. The property $\mu^*(\alpha) > S_k$ is a frequency $F^{(k)}$ with $F^{(k)} \supseteq F^{(\lambda)}$ for $K \leqslant \lambda$. The values used in the classical theory form only two such sections and therefore give rise to a single frequency ($\mu^*(\alpha) > 0$).

The theory of frequencies admits a modification when we restrict the notion of "subset of $\sum$" to that of "*admissible* subset of $\sum$". Every family ϕ of subsets may be taken as admissible when:

(a) Every open set belongs to ϕ,

(b) If $A_1, A_2, \ldots$ belong to ϕ, then $\bigcup_n A_n$ belongs to ϕ.

A similar restriction has been applied successfully in the theory of the distributive properties.

3. Let $\sum$ be a topological space which satisfies the second axiom of countability; then **L** holds for open sets O_n. Every open set which contains a point x, will be called a *neighbourhood* of x. Let $C \subseteq \sum$ have the frequency F at each of its points. We consider functions $f(x)$, $g(x), \ldots$ whose domain is C and whose values are real numbers, $+\infty$ or $-\infty$.

We define $f_1(x)$ as the *upper limiting function* mod F of $f(x)$ and $f_2(x)$ *as the lower limiting function* mod F of $f(x)$ in the following way:

$f_1(x)$ is the l.u.b. of the values k which satisfy the condition that for every positive ϵ, the points x' for which

$$f(x') > k - \epsilon \tag{5}$$

have the frequency F at x; $f_2(x)$ is the g.l.b. of the values g which satisfy the condition that for every positive ϵ, the points x'' for which

$$f(x'') < g + \epsilon, \tag{5'}$$

have the frequency F at x. The (upper and lower) limiting functions of the classical theory[3] are those mod F_0. If $f_1(x_0) < h$, then the set of points x, for which $f(x) \geqslant h$, does not have the frequency F at x_0; therefore the set of points x'', for which $f(x'') < h$, has the frequency F, hence $f_2(x_0) < h$. As this holds for every $h > f_1(x_0)$, $x_0 \in A$, it follows that

$$f_2(x) \leqslant f_1(x), \; x \in C. \tag{6}$$

It is convenient to use the notation

$$g(x_0) \prec f(x_0) \tag{7}$$

when there exists a neighbourhood Ω of x_0 such that $g(x) \leqslant f(x)$ for nearly every $x \in \Omega$. The relation (7) is therefore not a relation between the values $g(x_0)$ and $f(x_0)$, but between the pairs $\{g, x_0\}$ and $\{f, x_0\}$. If (7) and $f(x_0) \prec g(x_0)$ hold, we write

$$g(x_0) \sim f(x_0); \tag{7'}$$

in this case there exists a neighbourhood Ω' of x_0 such that $g(x) = f(x)$ for nearly every $x \in \Omega'$; (7) implies that $g(x) \prec f(x)$ for every $x \in \Omega$ and (7′) implies that $g(x) \sim f(x)$ for every $x \subset \Omega'$. Moreover it follows from the definition of the upper and lower limiting functions mod F that (7) implies

[3] See [1], p. 122.

https://doi.org/10.4153/CJM-1950-004-6 Published online by Cambridge University Press

(8) $$g_1(x) \leqslant f_1(x),\ g_2(x) \leqslant f_2(x),\ x \in \Omega,$$

and that (7′) implies

(8′) $$g_1(x) = f_1(x),\ g_2(x) = f_2(x),\ x \in \Omega'.$$

THEOREM 2. *For nearly every* $x \in C$, *we have* $f(x) \leqslant f_1(x)$, $f_2(x) \leqslant f(x)$, *and therefore* $f_2(x) \prec f(x) \prec f_1(x)$ *for* $x \in C$.

Proof. By symmetry it suffices to prove the first statement of the theorem. If for some x, $f(x) > f_1(x)$, then $f(x) \neq -\infty$, $f_1(x) \neq +\infty$, and we can therefore represent C as the join of the (disjoint) sets:

$$C = \cup_n C_n \cup C_* \cup C^* \cup C_0,$$

where $x \in C^*$ if and only if $f_1(x) < f(x) = +\infty$,
$x \in C_*$ " " " " $-\infty = f_1(x) < f(x)$,
$x \in C_n$ " " " " $\dfrac{1}{n-1} > f(x) - f_1(x) \geqslant \dfrac{1}{n}$ for $n = 1,2,\ldots,$
$x \in C_0$ " " " " $f_1(x) \geqslant f(x)$.

We prove that none of these sets has the frequency F except C_0. Suppose C^* has the frequency F; then it follows from Theorem 1 that there exists an $x_0 \in C^*$ at which C^* has the frequency F, but as $f(x) = +\infty$ for $x \in C^*$, $f_1(x_0) = +\infty$, contrary to the supposition. If C_* or C_n has the frequency F, we partition the set $C_* = \cup_m C_{*,m}$, $C_n = \cup_m C_{n,m}$ where the second index indicates that $m - 1 < f(x) \leqslant m$ holds ($m = 0, +1 + 2, \ldots$). If C_* has the frequency F, then some $C_{*,m}$ has the frequency F and therefore there exists some $y \in C_{*,m}$ at which $C_{*,m}$ has the frequency F, but then $f_1(y) \geqslant m - 1 \neq -\infty$. Suppose now that $C_{n,m}$ has the frequency F and $z \in C_{n,m}$; then $f_1(z) \leqslant f(z) - 1/n$. Therefore, if U is any neighbourhood of z, the points $z_1 \in C' = U \cap C_{n,m}$ for which $f(z_1) \leqslant f(z) - 1/n$, have the frequency F. In every neighbourhood U' of z_1 again, the points $z_2 \in C'' = U' \cap C'$, for which $f(z_2) \leqslant f(z_1) - 1/n \leqslant f(z) - 2/n$, have the frequency F; after n steps we find a subset $C^{(n)} \subseteq C_{n,m}$ in which the points $z_n \in C^{(n)}, f(z_n) \leqslant f(z) - 1$ have the frequency F. Hence $C_{n,m} \cap C_{n,m+1}$ is non-empty, contrary to the definition of $C_{n,m}$. Therefore $C_{n,m}$ has not the frequency F. Thus $C - C_0$ has not the frequency F.

As $f(x) \prec f_1(x)$, it follows from (8) that $f_1(x) \leqslant f_{11}(x)$ throughout C. Suppose that for some $x \in C, f_1(x) = c, f_{11}(x) = c + 3k, k > 0$; then in every neighbourhood Ω of x, the set of points x' for which $f_1(x') \geqslant c + 2k$ has the frequency F, and therefore in every neighbourhood $\Omega' \subseteq \Omega$ of x', the set of points x'' for which $f(x'') > c + k$ has the frequency F. Thus the set of these points x'' has the frequency F at x and therefore $f_1(x) > c$, contrary to the supposition. Hence:

(9) $$f_{11}(x) = f_1(x),\ f_{22}(x) = f_2(x)$$

for every $x \in C$. If now $g_1(x_0) \prec f_1(x_0)$, then it follows from (8) that $g_{11}(x) \leqslant f_{11}(x)$, $x \in \Omega$ (neighbourhood of x_0), and from (9) that $g_1(x) \leqslant f_1(x)$. Hence:

THEOREM 3. *$g_i(x) < f_i(x_0)$ implies that there exists a neighbourhood Ω of x_0, such that $g_i(x) \leqslant f_i(x)$, $x \in \Omega$, $i = 1$ or 2.*

When we consider several frequencies $F_a, F_b, \ldots$ which C has at every $x \in C$, the limiting functions will be distinguished by an upper index. Suppose $F_a \subseteq F_b$; the set of points x'' for which $f(x'') < f^a{}_2(x) + \epsilon, \epsilon > 0$ has the frequency F_a and therefore also the frequency F_b at $x \in C$. Therefore $f^b{}_2(x) \leqslant f^a{}_2(x)$.

Hence

$$(10) \qquad F_a \subseteq F_b \text{ implies } f^b{}_2(x) \leqslant f^a{}_2(x) \leqslant f^a{}_1(x) \leqslant f^b{}_1(x),\ x \in C.$$

THEOREM 4. *$F_a \subseteq F_b$ implies $f^a{}_i(x) = (f^a{}_i)^b{}_i(x)$, for $x \in C$, $i = 1$ or 2.*

Proof. It suffices to consider $i = 1$. From (9) and (10) we deduce

$$f^a{}_1(x) = (f^a{}_1)^a{}_1(x) \leqslant (f^a{}_1)^b{}_1(x).$$

As in the proof of (9), we put $f^a{}_1(x) = c$, $(f^a{}_1)^b{}_1(x) = c + 3k$, $k > 0$. Every neighbourhood Ω of x contains points x' where $f^a{}_1(x') > c + 2k$ and therefore neighbourhoods Ω' of x' where the sets of points x'' satisfying $f(x'') > c + k$ has the frequency F_a. Therefore $f^a{}_1(x) > c$, contrary to the supposition. Hence the theorem.

Formula (9) is a special case of the theorem, as it corresponds to $F_a = F_b$. By putting $F_b = F_0$ we obtain:

COROLLARY 1. For every F, the functions $f_1(x)$ and $f_2(x)$ are semi-continuous above and below, respectively.

From a well known theorem[4] therefore follows:

COROLLARY 2. If $\sum$ is an n-dimensional Euclidean space, then $f_1(x)$ and $f_2(x)$ are measurable functions.

This corollary admits generalization to other spaces.

That $(f^a{}_1)^b{}_1(x)$ and $(f^b{}_1)^a{}_1(x)$ may be different functions, can be shown by the following example.

$$f(x) \begin{cases} =1, & x \text{ a rational number,} \\ =0, & x \text{ an irrational number.} \end{cases}$$

$$\left.\begin{aligned} (f^0{}_1)^1{}_1(x) &= 1 \\ (f^1{}_1)^0{}_1(x) &= f^1{}_1(x) = 0 \end{aligned}\right\} \text{ for every } x.$$

We call the functions $f(x)$ for which $f(x) = f_1(x)$ (or $f(x) = f_2(x)$), *semi-continuous* mod F *above* (or *below*), generalizing ordinary semicontinuity which corresponds to $F = F_0$. The semicontinuity mod F implies also the corresponding semicontinuity mod every weaker frequency. Addition of a continuous function and multiplication with a positive continuous function leave semi-continuity mod F invariant. Multiplication with -1 interchanges semi-continuity above and below (mod F).

[4][1] p. 403.

The obvious inequalities:

$$(f(x) + g(x))_1 \leqslant f_1(x) + g_1(x),$$
$$(f(x) + g(x))_2 \geqslant f_2(x) + g_2(x)$$

can be replaced by the corresponding equalities when $F = F_0$, but not in the general case (example $f(x) = 1$ for $0 \leqslant x \leqslant 1$, otherwise $f(x) = 0$; $g(x) = f(x+1)$). Semicontinuity above mod F_0 can be tested by the necessary and sufficient condition that $x_i \to x_0$ and $f(x_i) \to a$ imply $f(x_0) \geqslant a$; however an arbitrary frequency has no test involving convergence of sequences only.

4. To investigate the functions obtained by applying alternatively the upper and lower limiting operations mod a given F, a more general way of approach is convenient.

Given an arbitrary set A which contains a partially ordered subset A'; let L_1 and L_2 be two mappings of the elements $a, b, \ldots$ of A on elements of A'

$$L_1 : a \to a_1, \quad L_2 : a \to a_2 \tag{11}$$

which satisfy the following conditions, when i, j, k, stand for 1 and 2:

$$a_{ii} = a_i \qquad \text{(idempotent)}, \tag{12}$$

$$a_j \prec b_k \text{ implies } a_{ji} \prec b_{ki} \quad \text{(monotonic)} \tag{13}$$

$$a_2 \prec a_1. \tag{14}$$

Then $a_{i_1 \ldots i_m 2 k_1 \ldots k_n} \prec a_{i_1 \ldots i_m 1 k_1 \ldots k_n}$. In particular,

$$a_{1212} \prec a_{1112} = a_{12} = a_{1222} \prec a_{1212},$$

and therefore

$$a_{1212} = a_{12}, \quad a_{(121)(121)} = a_{12121} = a_{121}. \tag{15}$$

Hence the operations L_{12} mapping $a \to a_{12}$ and L_{121} mapping $a \to a_{121}$ are idempotent and obviously monotonic. The same statements hold for the mappings L_{21} and L_{212} which are defined in a corresponding manner. The six mappings $L_1, L_2, L_{12}, L_{21}, L_{121}, L_{212}$ form a semigroup of idempotents, and the four mappings $L_{12}, L_{121}, L_{21}, L_{212}$ form a subsemigroup in it. We have

$$\begin{aligned} &a_2 \prec a_{212} \prec a_{12} \prec a_{121} \prec a_1, \\ &a_2 \prec a_{212} \prec a_{21} \prec a_{121} \prec a_1. \end{aligned} \tag{16}$$

These formulae do not establish any order relation between a_{12} and a_{21}. By the mappings L_1, L_2 and L_{12}, and with the help of (16), we obtain easily:

$$a_{12} \prec a_{21} \text{ implies } a_{21} = a_{121} \text{ and } a_{212} = a_{12}, \tag{17}$$

$$a_{12} \prec b_{121} \prec a_{121} \text{ implies } a_{121} = b_{121} \text{ and } a_{12} = b_{12}. \tag{18}$$

5. We apply now the methods and results of **4** to the space $\sum$ considered in **3**. The system A consists of the pairs $\{f, x\}$ represented by $f(x)$, where x runs over

C and f over the functions with domain C; the system A' consists correspondingly of the $f_i(x)$; the order relation in A' is the relation $\prec$ introduced by (7), and the mappings L_i are defined by

(19) $$L_i : f(x) \to f_i(x), \quad i = 1,2.$$

From (16) it follows that for every $x \in C$,

(20) $$f_2(x) \prec f_{212}(x) \prec f_{21}(x) \prec f_1(x),$$

(21) $$f_2(x) \prec f_{12}(x) \prec f_{121}(x) \prec f_1(x).$$

It may be remembered that $g(x_0) \prec f(x_0)$ does not necessarily imply $g(x_0) \leqslant f(x_0)$, (nor does $g(x_0) \sim f(x_0)$ imply $g(x_0) = f(x_0)$), but the relation implies $g(x) \leqslant f(x)$ for nearly every x of a suitable neighbourhood Ω of x_0 (similarly for $\sim$). However, by Theorem 3, $g_i(x_0) \prec f_i(x_0)$ implies $g_i(x) \leqslant f_i(x)$ for every $x \in \Omega$. In the case of functions with several suffixes, it is the last one that matters. For $x \in \Omega$,

(22) $$\begin{array}{lll} f_{12}(x_0) \prec f_{21}(x_0) & \text{implies } f_{21}(x) = f_{121}(x) & \text{and } f_{212}(x) = f_{12}(x), \\ f_{12}(x_0) \prec g_{121}(x_0) \prec f_{121}(x_0) & \text{implies } f_{121}(x) = g_{121}(x) & \text{and } f_{12}(x) = g_{12}(x). \end{array}$$

If $f(x)$ is continuous at $x = x_0$, then $f_1(x_0) = f_2(x_0)$. Conversely this equation does not necessarily imply the continuity of $f(x)$ at x_0. If for $x \in \Omega$ (open), $f_1(x) = f(x) = f_2(x)$, then (by Corollary 1 of Theorem 4) $f(x)$ is semicontinuous above and below and therefore continuous. Furthermore we prove

THEOREM 5. *Let $f_1(x) \sim f_2(x)$ for $x \in \Omega$ (open), then there exists $K \subseteq \Omega$ such that K contains nearly all the points of Ω and $f(x)$ is continuous when considered as a function with the domain K.*

Proof. $f_2(x) = f_1(x)$, for nearly every $x \in \Omega$; moreover, by Theorem 2, $f_2(x) \leqslant f(x) \leqslant f_1(x)$ for nearly every $x \in C$. Hence the points of Ω which satisfy both these conditions form a set $K \leqslant \Omega$, where $f_2(x) = f(x) = f_1(x)$. Thus $f(x)$ is continuous when considered on K alone.

The theorem admits a converse statement, since the values of $f_1(x)$ and $f_2(x)$ do not depend on the values of $f(x)$ on the complement of K in Ω. Therefore the continuity of $f(x)$ on K implies the equivalence of $f_1(x)$ and $f_2(x)$, $x \in \Omega$. The integrability (C) of a function does not depend on its values on a set which has not the frequency F_C. Hence:

COROLLARY. When $F = F_C$ and for $x \in \Omega, f_1(x) \sim f_2(x)$ and $f(x)$ is bounded, then $f(x)$ is integrable (C) and $\int f(x)\,dx = \int f_1(x)\,dx$.

It should be noticed that we have here a sufficient condition for integrability (C) which depends on a property "im Kleinen" only. For a Euclidean space $\sum$ and Lebesgue integration, the class of functions satisfying the conditions of the corollary does not include all the bounded functions which are measurable (L), but is larger than the class of the functions which are bounded and integrable (R).

https://doi.org/10.4153/CJM-1950-004-6 Published online by Cambridge University Press

THEOREM 6. *Let $g(x)$ be continuous on the open set Ω, let A be the subset of Ω where $g(x) \geqslant f_{121}(x)$ and B the subset where $g(x) \leqslant f_{12}(x)$; then $A \cup B$ has the frequency F at every point of Ω.*

Proof. Suppose $A \cup B$ has not the frequency F at $x_0 \in \Omega$; then $f_{12}(x_0) \prec g(x_0) \prec f_{121}(x_0)$. As $g(x)$ is continuous, $g(x) = g_{12}(x) = g_{121}(x)$. Therefore it follows from (22) that $g(x)$ is equal and equivalent to $f_{12}(x)$ and to $f_{121}(x)$ at x_0. Hence $x_0 \in A \cap B \subseteq A \cup B$. At the points of the complement C of $A \cup B$ in Ω, therefore, $A \cup B$ has the frequency F. This leads to a contradiction, since when $A \cup B$ has not the frequency F at x_0, this point is a limiting point of C and therefore $A \cup B$ has the frequency F at x_0.

6. Consider now *pairs* of functions $g(x)$, $h(x)$ for which, for $x \in C$,

$$g(x) = g_1(x) = h_1(x) \text{ and } h(x) = g_2(x) = h_2(x); \tag{23}$$

then

$$\begin{aligned} g(x) &= g_{121}(x) = h_{121}(x) = g_{21}(x) = h_{21}(x); \\ h(x) &= g_{12}(x) = h_{12}(x) = g_{212}(x) = h_{212}(x). \end{aligned}$$

On the other hand, for every function $f(x)$, the pairs $f_{121}(x), f_{12}(x)$ and $f_{21}(x)$, $f_{212}(x)$ satisfy (23). Now suppose

$$h(x_0) \prec f(x_0) \prec g(x_0); \tag{24}$$

then $g(x_0) = h_1(x_0) \leqslant f_1(x_0) \leqslant g_1(x_0) = g(x_0)$; hence $f_1(x_0) = g(x_0)$. Similarly $f_2(x_0) = h(x_0)$. On the other hand, $f_1(x_0) = g(x_0)$ implies (see Theorem 2) $f(x_0) \prec g(x_0)$, and if $f_1(x) = g(x)$ for every $x \in \Omega$ (open set), then $f(x) \leqslant g(x)$ for nearly every $x \in \Omega$. Thus:

THEOREM 7. *Let the pair of functions $g(x)$, $h(x)$ satisfy* (23); *then the necessary and sufficient condition for $f_1(x_0) = g(x_0), f_2(x_0) = h(x_0)$ is* (24).

When Ω is an open set, the necessary and sufficient condition for $f_1(x) = g(x)$, $f_2(x) = h(x)$, $x \in \Omega$ is $h(x) \prec f(x) \prec g(x)$.

We consider now two frequencies $F_a \subseteq F_b$ which $\sum$ has at every point. To avoid clumsy formulas, we use the indices 1,2 mod F_b and correspondingly the indices α, β mod F_a. If $g(x)$ and $h(x)$ satisfy (23), then it follows from (10) that

$$h(x) = g_2(x) \leqslant g_\beta(x) \leqslant g_\alpha(x) \leqslant g_1(x) = g(x),$$

but from Theorems 4 and 7, $g_\alpha(x) = g_{\alpha 1}(x) = g(x)$,

$$g_\beta(x) = g_{\beta 2}(x) = h(x).$$

Now suppose that $r(x) = r_\alpha(x) = s_\alpha(x), s(x) = s_\beta(x) = r_\beta(x)$, then $r_1(x) = r(x)$, $s_2(x) = s(x)$ and therefore $s(x) = r_\beta(x) \geqslant r_2(x) = r_{\beta 2}(x) = s_2(x) = s(x)$. Hence $r_2(x) = s(x)$, and similarly $s_1(x) = r(x)$. Therefore if (23) holds for $F = F_b$, it holds also for F_a, and, conversely, if it holds for any F, it holds for F_0 and for every other F' (as it is necessarily $\subseteq F_0$). Hence

THEOREM 8. *If F and F' are frequencies which C has at every point, and the equations* (23) *are satisfied* mod F, *they are also satisfied* mod F'.

In the supposition of Theorem 8, C may be replaced by any subspace. Moreover it follows from this theorem, that when A and B are defined as in Theorem 6, $A \cup B$ has every frequency that $\Omega \cap C$ has at each of its points.

COROLLARY.[5] For every $\epsilon > 0$ and arbitrary x_0, the set of points x for which $g(x) \geqslant g(x_0) - \epsilon$ has all the frequencies at x_0 which C has at that point.

As the relation (23) between $g(x)$ and $h(x)$ does not depend on the selection of F, we may put $F = F_0$; therefore $g(x_0) = h(x_0)$ is the necessary and sufficient condition for $g(x)$ ($= h(x)$) to be continuous at x_0. The difference

$$\delta(x) = g(x) - h(x) \tag{25}$$

can therefore be used as a measure of the discontinuity of $g(x)$ and $h(x)$. $\delta(x)$ is non-negative and semicontinuous above mod F_0, but not every function with these properties is a $\delta(x)$. Select an arbitrary F which C has at every point, then $\delta(x) \geqslant \delta_1(x)$ (for (10) holds nearly everywhere). Suppose $\epsilon > 0$; then there exists a neighbourhood Ω of x_0, such that for $x \in \Omega \cap C$, $g(x) \leqslant g(x_0) + \epsilon$; moreover, there exists a subset $S \subseteq \Omega \cap C$ which has the frequency F at x_0, such that for $x' \in S$, $h(x') \geqslant h_1(x_0) - \epsilon = g(x_0) - \epsilon$. Hence $\delta(x') \leqslant 2\epsilon$ for $x' \in S$. As $\delta(x) \geqslant 0$ and S has the frequency F, it follows that $\delta_2(x_0) = 0$ for every x_0. Moreover $\delta_{12}(x) \leqslant \delta_2(x)$. Hence $0 = \delta_{12}(x) = \delta_{121}(x) = \delta_{21}(x) = \delta_{212}(x) = \delta_2(x)$. Therefore $\delta(x)$ is continuous only at those points where it vanishes, i.e., where $g(x) = h(x)$ is continuous.

The pairs $g(x)$, $h(x)$ which satisfy (23) are the same for every F; but for a given function $f(x)$, the pairs $f_{121}(x)$, $f_{12}(x)$ are in general different for different frequencies F.

7. A discontinuous function $f(x)$ can be characterized by the two semicontinuous functions $f_1(x)$ and $f_2(x)$, and furthermore by the two "pairs" $f_{121}(x)$, $f_{12}(x)$ and $f_{21}(x)$, $f_{212}(x)$. This characterization depends on the choice of the frequency F. It is therefore interesting to know all the frequencies which the domain of definition of $f(x)$ has at each of its points. We have mentioned frequencies which are derived from the powers of the subsets of $\sum$ and frequencies derived from measure functions. To know all the frequencies of the first kind would imply the solution of the "problem of the continuum"; the frequencies of the second kind include those generated by measures of lower dimension (e.g., Gillespie measure).[6] Moreover, if F is a frequency which $T \subseteq \sum$ has at every point of $\sum$, then $\sum$ has also the frequency $F(T)$ at every point. However there might exist frequencies of a different kind.

[5]If one tries to describe the domain of the values between $g(x)$ and $h(x)$, spread over C in a pictorial way, this corollary states that the domain is "soft" inside, whereas Theorem 6 means that it is "hard" outside.

[6]See [3].

The functions $f_1(x)$ and $f_2(x)$ satisfy $f_2(x) \leqslant f_{12}(x)$. To show that there is no other relation between those two semicontinuous functions, we select an arbitrary function which is semicontinuous above, say $r(x)$, and a function $s(x)$, semicontinuous below, such that $s(x) \leqslant r_2(x)$ for $x \in \sum$. Then we split $\sum$ into $\sum = A \cup B$, $A \cap B = 0$, such that A as well as B has the frequency F at every point of $\sum$. We define

$$f(x) \begin{cases} = r(x) \text{ for } x \in A, \\ = s(x) \text{ for } x \in B; \end{cases}$$

then $f_1(x) = r(x), f_2(x) = s(x), x \in \sum$.

That the splitting of $\sum$ into A and B is always possible for $F = F_0$ follows from the separability of $\sum$. To split the n-dimensional Euclidean space into two sets A and B which have a positive Lebesgue measure at each point of the space, one can construct a suitable sequence of disjoint perfect sets of positive measure $A_1, B_1, A_2, B_2, \ldots$ such that $A = \bigcup_n A_n$ and $B = \bigcup_n B_n$ have the required property.

Thus there is no relation other than $f_2(x) \leqslant f_{12}(x)$, between the upper and the lower limiting function, which holds for every frequency; and therefore a closer investigation of the nature of the discontinuities must be split into the "discontinuity above" (characterized by $f_1(x)$ and the pair $f_{121}(x), f_{12}(x)$) and the "discontinuity below" ($f_2(x)$ and $f_{21}(x), f_{212}(x)$).

It has been suggested[7] that we might modify the notion of upper (lower) limiting function by considering the functions

$$D_1(f,x_0) = \overline{\lim_{x \to x_0}}\ f(x); \quad D_2(f,x_0) = \underline{\lim_{x \to x_0}}\ f(x).$$

This would be an Analysis modulo the distributive property D: "To contain an infinite number of points". These limiting functions, however, do not lead to an idempotent operation, not even after infinite repetition, as is seen from the following example:

Represent the numbers $0 \leqslant x < 1$ by decimal fractions; then $x \in S_{m,n}$ ($n \leqslant m$) if and only if either $x = 0$, or x admits a finite decimal expansion which starts with *exactly* m zeros and has altogether $m - n$ non-zeros. Put

$$\bigcup_m \{S_{m,0} \ldots S_{m,k}\} = S^k, \quad \bigcup_k S^k = S,$$

and define $f(x) = 1$ for $x \in S$, $= 0$ otherwise; then $D_1(f, x) = 1$ if and only if $x \in S - S^0$, and when we indicate the iteration of the D_1-operation by an upper index, $D_1^{k+1}(f, x) = 1$ if and only if $x \in S - S^k$. All these operators are therefore non-idempotent, and the functions form a monotonic decreasing sequence. The lower limit is $D_\omega(f, x)$ which $= 1$ for $x = 0$ only; thus the operator D_ω is not idempotent either. The example can be modified in such a way that even some operators with higher transfinite indices are non-idempotent. For this purpose, one may admit several batches of non-zeros separated by sequences of consecutive zeros of prescribed length.

[7]See [4] p. 1003, footnote 452a.

8. Given a frequency F which $C \subseteq \sum$ has at every point of C, and a sequence $f(x,1), f(x,2), f(x,3), \ldots$ which converges to a function $f(x)$ defined on a subset of C in such a way that for every $\epsilon > 0$ there exists an $N(\epsilon)$, such that for $n > N(\epsilon)$,

$$(26) \qquad | f(x,n) - f(x) | < \epsilon \quad \text{for nearly every } x \in C,$$

then the sequence is said to converge *nearly uniformly* to $f(x)$ on C. Let $\epsilon_i \to 0$; if (26) holds, then the corresponding inequality holds also for every $\epsilon_i > \epsilon$. If $C_{n,i}$ is the set of points x for which

$$| f(x,n) - f(x) | \geqslant \epsilon_i,\ n > N(\epsilon_i)$$

then $\cup_{n,i} C_{n,i}$ has not the frequency F and therefore $B = C - \cup_{n,i} C_{n,i}$ has the frequency F at every point of C. Therefore the given sequence is uniformly convergent on B to $f(x)$; this function is defined at every point of B, and its limiting functions $f_1(x), f_2(x)$ are defined for every $x \in C$. We prove now:

THEOREM 9. *(26) implies that $f_i(x,n)$ converge to $f_i(x)$ uniformly on C for $i = 1,2$.*

Proof. Let U_n be a suitable neighbourhood of $x_0 \in C$. Then $f(x,n) < f_1(x_0,n) + \epsilon$ for nearly every $x \in U_n \cap B$, and therefore

$$f(x) < f_1(x_0,n) + 2\epsilon,\ \ n > N(\epsilon).$$

Moreover, U_n has a subset K_n with the frequency F, such that $f(x',n) > f_1(x_0,n) - \epsilon$ and therefore

$$f(x') > f_1(x_0,n) - 2\epsilon,\ \ n > N(\epsilon),\ \ x' \in K_n.$$

Now $f_1(x_0)$ exists for $x_0 \in C$, and it follows from the two inequalities that

$$f_1(x_0,n) - 2\epsilon \leqslant f_1(x_0) \leqslant f_1(x_0,n) + 2\epsilon,\ \ n > N(\epsilon).$$

Therefore $f_1(x_0,n) \to f_1(x)$ and similarly $f_2(x_0,n) \to f_2(x)$. Moreover,

$$| f_1(x_0) - f_1(x_0,n) | < 2\epsilon,\ \ n > N(\epsilon)\ \text{ independent of } x_0.$$

Thus the convergence is uniform on C.

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