THE CONVERGENCE OF RAYLEIGH-RITZ APPROXIMATIONS IN HYDRODYNAMICS

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1. Introduction

It is known that various cases of the steady isentropic irrotational motion of a compressible fluid are expressible as variational principles [1], [5]. In particular, the aerofoil problem i.e. the case of plane flow in which a uniform stream is locally deflected, without circulation, by a bounded obstacle, can be expressed in such a form. Thus we make stationary

$$J_1[\phi] = \lim_{R\to\infty} \iint_R \{p - p_\infty + \rho_\infty \nabla\phi_0 \cdot \nabla(\phi - \phi_\infty)\} dx\,dy$$

where the region R is that bounded internally by the obstacle (C_0) and externally by a circle (C_R) of radius R. In this expression ϕ_∞ is the velocity potential for a uniform stream, and ϕ_0 is the velocity potential for the corresponding incompressible flow. It is assumed that p is a function of the density ρ only, and we are to express p in terms of ϕ by use of Bernoulli's equation. The class of admissible functions is restricted to functions for which (i) $\partial\phi/\partial n = 0$ on C_0, and (ii) $\phi = u_\infty x + v_\infty y + \chi$ where for r large $|\chi| \leqq K' r^{-1}$, $|\nabla\chi| \leqq K' r^{-2}$, the constant K' being independent of the polar angle θ, and independent of the function considered.

The integral $J_1[\phi]$ may be used to obtain approximations to the velocity potential in the Rayleigh-Ritz manner [1]. I propose to show that in the case of a convex obstacle, if the flow is everywhere subsonic and if the approximations so obtained converge at any point Q of the fluid, they converge uniformly in any bounded subregion of the fluid containing Q.

2. An associated variational principle for the aerofoil problem

In his proof of the existence of subsonic flows past a prescribed obstacle, Shiffman [2] used the integral

$$J_2[\psi] = \lim_{R\to\infty} \iint_R \{p - p_\infty + \rho u(u - u_\infty) + \rho v(v - v_\infty)\} dx\,dy.$$

To any ψ of class C_2, ρ is defined by Bernoulli's equation with $p = p(\rho)$, and thence q, u, v by

$$q^2 = u^2 + v^2, \quad \rho u = \frac{\partial \psi}{\partial y}, \quad \rho v = -\frac{\partial \psi}{\partial x}.$$

If the class of admissible functions is restricted to those (single-valued) functions for which (i) ψ is constant on C_0, and (ii) $\psi = \rho_\infty(u_\infty y - v_\infty x) + \Psi'$ where for r large $|\Psi'| \leqq K'' r^{-1}$, $|\nabla \Psi'| \leqq K'' r^{-2}$, the constant K'' being independent of the angle θ and independent of the function considered. Any admissible ψ which makes $J_2[\psi]$ stationary specifies an irrotational flow.

If we put $\psi = \rho_\infty \psi_0 + \Psi$ where ψ_0 is the stream function for the corresponding incompressible flow, the last two terms of J_2 give

$$\lim_{R\to\infty} \iint_R \left\{ \rho_\infty \frac{\partial \psi_0}{\partial y}(u - u_\infty) - \rho_\infty \frac{\partial \psi_0}{\partial x}(v - v_\infty) - \Psi(u_y - v_x) \right\} dx\, dy$$
$$+ \lim_{R\to\infty} \int_{C_0, C_R} \Psi \boldsymbol{t} \cdot (\boldsymbol{q} - \boldsymbol{q}_\infty) ds.$$

As $\Psi = O(r^{-1})$, $\boldsymbol{q} - \boldsymbol{q} = O(r^{-2})$ the integral over C_R vanishes in the limit and, as Ψ is constant on C_0, the integral over C_0 vanishes for non-circulatory flow. For an extremal $u_y - v_x = 0$ and as ψ_0 is conjugate to ϕ_0

$$\begin{aligned} J_2[\psi_{\text{ext}}] &= \lim_{R\to\infty} \iint_R \{p - p_\infty + \rho_\infty \nabla\phi_0 \cdot (\boldsymbol{q} - \boldsymbol{q}_\infty)\} dx\, dy \\ &= J_1[\phi_{\text{ext}}]. \end{aligned}$$

For the subsonic case the extremal minimizes J_2 whereas it maximizes J_1 (Serrin [5] pp. 204—5), and it then follows that for any admissible ϕ and ψ

(2.1) $$J_1[\phi] \leqq J_1[\phi_{\text{ext}}] \leqq J_2[\psi].$$

3. Rayleigh-Ritz approximations

For a given case of the aerofoil problem we obtain Rayleigh-Ritz approximations to the velocity potential by setting

(3.1) $$\phi_\nu(x, y) = \phi_\infty(x, y) + \sum_1^\nu A_i f_i(x, y)$$

where, for a suitably chosen set of functions $f_1(x, y)$, $f_2(x, y)$, $\cdots$, we are to determine the constants A_i so as to make $J_1[\phi_\nu]$ stationary. The functions f_i are to be chosen so that (i) for all ν, ϕ_ν is an admissible function in the sense of § 1; and (ii) by proper choice of the A_i, any function of the type χ defined in § 1, together with its first derivatives, may be approximated to as closely as we please by $\Sigma A_i f_i$.

The (algebraic) equations for the determination of the A_i are

$$\frac{\partial J_1}{\partial A_i} = \iint_R (\rho_\infty \nabla\phi_0 - \rho_\nu \nabla\phi_\nu) \cdot \nabla f_i dx dy = 0 \qquad (i = 1, 2, \cdots, \nu)^*$$

where $\rho = \rho(\phi)$. We evaluate ϕ_ν for the values of A_i given by these equations, and we write the equations in the equivalent form

$$\iint_R (\rho_\infty \nabla\phi_0 - \rho_\nu \nabla\phi_\nu) \cdot \nabla\zeta_\nu dx dy = 0 \tag{3.2}$$

where $\zeta_\nu = \sum_1^\nu B_i f_i$ with B_i arbitrary.

For any two approximations ϕ_m, ϕ_{m+n} the difference $(\phi_m - \phi_{m+n})$ is a function of type ζ — call it ζ_{m+n} — and by Taylor's theorem we have

$$\begin{aligned} J_1[\phi_m] = J_1[\phi_{m+n}] &+ \iint_R (\rho_\infty \nabla\phi_0 - \rho_{m+n} \nabla\phi_{m+n}) \cdot \nabla\zeta_{m+n} dx dy \\ &- \tfrac{1}{2} \iint_R \overline{\frac{\rho}{c^2}} Q[\zeta_{m+n}] dx dy \end{aligned} \tag{3.3}$$

[1]. The bar in the last term indicates that the "velocities" determining ρ, c^2 are intermediate between $\nabla\phi_m$, $\nabla\phi_{m+n}$. We use Q for the quadratic expression

$$Q[\zeta] = (c^2 - u^2)\zeta_x^2 - 2uv\zeta_x\zeta_y + (c^2 - v^2)\zeta_y^2$$

which, for $q^2 = u^2 + v^2 \leqq q^{*2} < c^2$, is positive definite in its arguments; and so for some constants k, K

$$k(\nabla\zeta)^2 \leqq \tfrac{1}{2}\frac{\rho}{c^2} Q[\zeta] \leqq K(\nabla\zeta)^2. \tag{3.4}$$

The second term on the right hand side of (3.3) vanishes, and if we require that the ϕ_m, for all m, give rise to subsonic "velocities", the quadratic form Q is positive definite and the $J_1[\phi_m]$ form a monotonically increasing sequence.

Since the admissible functions give rise to subsonic "velocities", we may write

$$k_1 \iint_R \{\nabla(\psi - \psi_\infty)\}^2 dx dy \leqq J_2[\psi] \leqq K_1 \iint_R \{\nabla(\psi - \psi_\infty)\}^2 dx dy$$

where $\psi_\infty = \rho_\infty(u_\infty y - v_\infty x)$ [2]. Now $\rho_\infty\psi_0$ is an admissible ψ, and from (2.1) it follows that

$$J_1[\phi_m] \leqq K_1 \iint_R \{\nabla(\rho_\infty\psi_0 - \psi_\infty)\}^2 dx dy,$$

and thus the sequence $J_1[\phi_m]$ has a limit. Putting

$$|J_1[\phi_m] - J_1[\phi_{m+n}]| < \varepsilon$$

* We now use R for the infinite region exterior to C_0.

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there follows from (3.3), (3.4)

$$k \iint_R \{\nabla(\phi_m - \phi_{m+n})\}^2 dx\,dy < \varepsilon \tag{3.5}$$

for $m > M(\varepsilon)$.

We use (3.5) together with a theorem due to Morrey [4] to establish the convergence of the ϕ_m. Consider any two points P, Q such that a semicircle upon diameter PQ lies wholly within the fluid. Introduce Cartesian coordinates such that P, Q are given by $(\pm a, 0)$, and let T be the point $(0, a)$. Since $\nabla\phi_m$ is uniformly bounded

$$\iint_{C_r} \{\nabla(\phi_m - \phi_{m+n})\}^2 dx\,dy \leqq Lr^2$$

for any circle C_r of radius r, and Shiffman's summary [2] of Morrey's theorem then shows that

$$|\zeta_{m+n}(P) - \zeta_{m+n}(Q)| \leqq L_1 PQ^{\frac{1}{2}} \left\{\iint_R (\nabla\zeta_{m+n})^2 dx\,dy\right\}^{\frac{1}{2}}$$

where $\zeta_{m+n} = \phi_m - \phi_{m+n}$ and L_1 is a constant depending on L.

If the semicircle PTQ does not lie wholly within the fluid we may, if the surface of the obstacle is sufficiently regular, connect P to Q by a chain of non-overlapping semicircles lying within the fluid. For definiteness consider the case where C_0 is convex and, as it is bounded, enclose it within a square S. We can select a set of at most four points $P_1, \cdots, P_4$ such that, if we write $Q = P_0$, $P = P_5$, we can construct a set of required semicircles, one on each of the segments $P_{i-1}P_i$, $i = 1, \cdots 5$. It then follows that

$$|\zeta_{m+n}(P) - \zeta_{m+n}(Q)| \leqq L_1 \left\{\iint_R (\nabla\zeta_{m+n})^2 dx\,dy\right\}^{\frac{1}{2}} \{QP_1^{\frac{1}{2}} + \cdots + P_4P^{\frac{1}{2}}\}.$$

Let P and Q be any two points of a bounded subregion D of the fluid, and let D be enclosed within a circle of radius $d/2$, where d is sufficiently large for the circle to enclose the square S, then

$$|\zeta_{m+n}(P) - \zeta_{m+n}(Q)| \leqq 5L_1 d^{\frac{1}{2}} \left\{\iint_R (\nabla\zeta_{m+n})^2 dx\,dy\right\}^{\frac{1}{2}}. \tag{3.6}$$

From (3.5) we have finally

$$|\zeta_{m+n}(P) - \zeta_{m+n}(Q)| < L_2 k^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}} \tag{3.7}$$

for $m > M(\varepsilon)$.

If the approximations ϕ_m converge at Q

$$|\zeta_{m+n}(Q)| < \varepsilon^{\frac{1}{2}}$$

for $m > M'(\varepsilon)$, and as the contants L_2, k are independent of P, there follows from (3.7) the uniform convergence of the Rayleigh-Ritz approximations. For the case of the circular obstacle, all the ϕ_m are zero at $r = 1$, $\theta = \frac{1}{2}\pi$,

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and thus the ϕ_m converge in any bounded subregion of the fluid containing the point $r = 1, \theta = \frac{1}{2}\pi$.

Let ϕ be the velocity potential of the flow and set $\zeta_m = \phi_m - \phi$ then, since ϕ_m and ϕ are determined to an additive constant, we can adjust the constant so that the ζ_m are zero at some chosen point Q. We have from (3.6)

$$|\zeta_m(P)| < L_2 \left\{ \iint_R (\nabla \zeta_m)^2 dx\, dy \right\}^{\frac{1}{2}}$$

and using (3.4), (3.3) successively there follows

$$|\phi_m - \phi| \leqq L_2 k^{-\frac{1}{2}} \{J_1[\phi] - J_1[\phi_m]\}^{\frac{1}{2}}.$$

This makes definite the "criterion of mean error" of Lush and Cherry [1].

References

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