LOGICS FROM ULTRAFILTERS

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Abstract. Ultrafilters play a significant role in model theory to characterize logics having various compactness and interpolation properties. They also provide a general method to construct extensions of first-order logic having these properties. A main result of this paper is that every class Ω of uniform ultrafilters generates a Δ-closed logic $\mathcal{L}_\Omega$. $\mathcal{L}_\Omega$ is ω-relatively compact iff some $D \in \Omega$ fails to be ω_1-complete iff $\mathcal{L}_\Omega$ does not contain the quantifier "there are uncountably many." If Ω is a set, or if it contains a countably incomplete ultrafilter, then $\mathcal{L}_\Omega$ is not generated by Mostowski cardinality quantifiers. Assuming $\neg 0^\sharp$ or $\neg L^\mu$, if $D \in \Omega$ is a uniform ultrafilter over a regular cardinal ν, then every family Ψ of formulas in $\mathcal{L}_\Omega$ with $|\Phi| \leq \nu$ satisfies the compactness theorem. In particular, if Ω is a proper class of uniform ultrafilters over regular cardinals, $\mathcal{L}_\Omega$ is compact.

§1. Introduction. Lindström's 1969 theorem [14] (reprinted in [2, pp. 237–246]; also see [9, theorem 1.1.4]) characterizes first-order logic $\mathcal{L}_{\omega\omega}$ as the maximal compact logic satisfying the downward Löwenheim–Skolem theorem. After this fundamental result a variety of methods were developed for constructing extensions of $\mathcal{L}_{\omega\omega}$. The six Parts A–F of the book [1] may give an idea of the range of techniques available for such extensions, and their applications to algebra, probability, topology, set theory and game theory. In [1] one can also find a comprehensive study of extensions of $\mathcal{L}_{\omega\omega}$ satisfying many forms of interpolation, compactness, and Löwenheim–Skolem properties.

Ultrafilters and ultraproducts have a pervasive role in the literature, both for the *characterization* of all these properties in extensions of $\mathcal{L}_{\omega\omega}$ and for the *construction* of such extensions. For instance, in [15], [16, pp. 230–234] and [17, sec. 3] various compactness properties of a logic $\mathcal{L}$ are characterized in terms of ultrafilters. As another example, in [20] one can find a general method to construct logics on classes of models satisfying a maximality condition with respect to a suitable variant of the Łoś ultraproduct theorem. Likewise, the paper [21] is devoted to the construction of logics via the property of being preserved from the models to their ultraproduct. Specific examples are given involving cardinality quantifiers. From a fragment of second-order logic having this preservation property, in [21, p. 636] the author constructs a compact Δ-closed logic $\mathcal{L}$ (in the sense of [7, p. 18]).

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Pursuing this line of research, in the present paper *every* (nonempty) class Ω of uniform ultrafilters is shown to generate a Δ-closed logic $\mathcal{L}_\Omega$. Depending on Ω, $\mathcal{L}_\Omega$ may satisfy various forms of compactness.

All logics constructed in the present paper naturally arise from classes of ultrafilters and automatically satisfy the basic regularity/closure properties considered in abstract model theory [7, definitions 1.2.1–1.2.3].

1.1. Main results. With $\mathrm{Str}(\tau)$ denoting the class of all structures of type τ, and $\sigma \subseteq \tau$, let $\restriction^\tau_\sigma\colon \mathrm{Str}(\tau) \to \mathrm{Str}(\sigma)$ denote the reduct operation. For any class Ω of uniform ultrafilters, let $L(\tau) = L(\tau, \Omega)$ be the family of subclasses $\psi \subseteq \mathrm{Str}(\tau)$ such that there is a finite type $\sigma \subseteq \tau$ and a class $\chi \subseteq \mathrm{Str}(\sigma)$ having the following property:

> For every $D \in \Omega$, both χ and its complement $\mathrm{Str}(\sigma) \setminus \chi$ are closed under isomorphisms and ultraproducts modulo D, and ψ is the class of expansions to τ of all structures in χ.

In Sections 3 and 4 the following results are proved:

(i) There is a unique regular logic $\mathcal{L} = \mathcal{L}_\Omega$ whose elementary classes of any type τ coincide with the classes in $L(\tau)$. $\mathcal{L}$ is Δ-closed. If Ω contains a κ-descendingly incomplete ultrafilter,[1] $\mathcal{L}$ is κ-relatively compact.[2] $\mathcal{L}$ is compact only if Ω is a proper class. See Theorem 3.1(i).

(ii) (Assuming $\neg 0^\sharp$ or $\neg L^\mu$.[3]) Let $D \in \Omega$ be an ultrafilter over a regular cardinal ν. Then for a family Ψ of $\mathcal{L}$-sentences with $|\Psi| \leq \nu$ to have a model it is sufficient that every finite subfamily of Ψ has a model. In particular, if Ω is a proper class and each $D \in \Omega$ is an ultrafilter over a regular cardinal, then $\mathcal{L}$ is compact. See Theorem 3.1(ii).

(iii) If Ω is a set or Ω contains a countably incomplete ultrafilter, then $\mathcal{L}$ is not generated by Mostowski cardinality quantifiers. See Theorem 4.1.

(iv) Every ultrafilter $D \in \Omega$ is ω_1-complete iff $\mathcal{L}$ contains the quantifier "there are infinitely many" iff $\mathcal{L}$ contains the quantifier "there are uncountably many" iff $\mathcal{L}$ contains the well-ordering quantifier iff $\mathcal{L}$ is not ω-relatively compact. See Theorem 4.3 and Corollary 4.4.

(v) If in (ii) we assume, instead of $\neg L^\mu$, the existence of a proper class of measurable cardinals, then (ii) no longer holds. See Corollary 4.5.

Using Lindström's characterization theorem, in Theorem 4.7 and Corollary 4.8, certain specific assumptions on the map $\Omega \mapsto \mathcal{L}$ are shown to be related to the Chang–Keisler conjecture [4, p. 599, conjecture 18].

Throughout this paper we will work with *classes*, typically with subclasses of $\mathrm{Str}(\tau)$, and with families or collections of classes, such as the family of all elementary classes of type τ in a logic $\mathcal{L}$. All these mathematical entities can be handled by adding an extra stage of flexibility to the Gödel–Bernays–von Neumann set theory. For the sake of definiteness, throughout this paper we will adopt the Isbell–Mac Lane–Feferman approach, where our families are called "conglomerates." See [10, pp. 329–331] for details. Our syntax-free approach to abstract logics via their elementary classes agrees with Lindström's approach in his characterization theorem.

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[1] See [13].
[2] In the sense of [16, p. 230].
[3] For $\neg 0^\sharp$ see [11, p. 312]. $\neg L^\mu$ is shorthand for "there is no inner model with a measurable cardinal." See [6, p. 56].

We will refrain from introducing new logics. By letting Ω range over arbitrary classes of uniform ultrafilters, our readers will be able to manufacture the logics that best suit their needs.

Standard background references for model theory, abstract model theory and set theory are, respectively, given by [1, 4, 11], where the reader can find the definitions of all unexplained notions occurring in this paper.

§2. Basic notation and terminology. Following common usage we let α, β, γ denote ordinals and let $\kappa, \lambda, \mu, \nu$ denote cardinals. V is the class of all sets. A *symbol* is a pair $R = \langle \alpha, a \rangle$ where α is an ordinal and a is an integer: the absolute value $|a|$ is the *arity* (= number of places) of R. R is said to be a function, constant, or relation symbol according as $a < 0$, $a = 0$, or $a > 0$.

A *type* ("language" in [4, p. 18], "vocabulary" in [7, p. 26]) is a set of symbols. Following [4], in this paper we will only consider one-sorted types. We let σ, τ denote types. We say that τ is *relational* if it does not contain function or constant symbols. The empty type is known as the *pure identity language*. Following [4, p. 22], identity is understood as a *logical symbol*, denoted $=$.[4]

A *structure* ("model" in [4, p. 20]) of type τ is a function

$$\mathfrak{A}\colon \{\emptyset\} \cup \tau \to \mathsf{V},$$

where $A = \mathfrak{A}(\emptyset)$ is a nonempty set, called the *universe* of $\mathfrak{A}$, and for each constant symbol $c \in \tau$, $\mathfrak{A}(c)$ is an element of A, and for each $|a|$-ary relation (resp. function) symbol $R \in \tau$, $\mathfrak{A}(R)$ is an $|a|$-ary relation (resp. function) over A. We write $c^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$ instead of $\mathfrak{A}(c)$ and $\mathfrak{A}(R)$. $\mathrm{Str}(\tau)$ denotes the class of all structures of type τ. R and S will usually denote unary relation symbols, E a binary relation symbol, and c a constant symbol. Further, M, N, A, B are the universes of $\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \mathfrak{B}$. Structures will be displayed as in [4, p. 20]: thus, e.g., by writing $\mathfrak{A} = \langle A, E, c, S_\alpha \rangle_{\alpha<\kappa}$ we mean that $\mathfrak{A}$ is a structure of type $\{E, c, S_\alpha\}_{\alpha<\kappa}$ and by a traditional abuse of notation, $E = E^{\mathfrak{A}}, c = c^{\mathfrak{A}}, S_\alpha = S_\alpha^{\mathfrak{A}}$ for each $\alpha < \kappa$.

If $\mathfrak{M} \in \mathrm{Str}(\tau)$ and $\sigma \subseteq \tau$ then the *reduct* of $\mathfrak{M}$ to σ (in symbols, $\mathfrak{M}\restriction\sigma$, or $\mathfrak{M}\restriction^\tau_\sigma$ for greater definiteness) is the structure of type σ obtained by restricting to $\sigma \cup \{\emptyset\}$ the domain $\tau \cup \{\emptyset\}$ of $\mathfrak{M}$. We say that $\mathfrak{B} \in \mathrm{Str}(\tau)$ is an *expansion* (to τ) of $\mathfrak{A} \in \mathrm{Str}(\sigma)$ if $\tau \supseteq \sigma$ and $\mathfrak{A} = \mathfrak{B}\restriction\sigma$. If σ is a relational type, $\mathfrak{M} \in \mathrm{Str}(\sigma)$ and M' is a nonempty subset of M, then

$$\mathfrak{M}|M' \text{ is the substructure ("submodel" in [4, p.21]) of } \mathfrak{M} \text{ with universe } M'. \tag{1}$$

For Θ a set of *first-order* sentences, we let $\mathfrak{A} \models \Theta$ mean that $\mathfrak{A}$ satisfies every $\theta \in \Theta$. For D an ultrafilter over an infinite set I, we let $\Pi_D \mathfrak{A}$ denote the *ultrapower of* $\mathfrak{A}$ modulo D. As in [4, p. 215], for any map $g\colon I \to A$ we let $g_D \in \Pi_D A$ denote the $=_D$-equivalence class of g. Given a family $\{\mathfrak{A}_i\}_{i\in I}$ of structures of type τ, we let $\Pi_D \langle \mathfrak{A}_i \mid i \in I \rangle$ denote their *ultraproduct modulo D*. A class $\psi \subseteq \mathrm{Str}(\tau)$ is *closed under ultraproducts modulo D* if $\{\mathfrak{A}_i\}_{i\in I} \subseteq \psi$ implies $\Pi_D \langle \mathfrak{A}_i \mid i \in I \rangle \in \psi$. We say that ψ is *closed under isomorphisms* if $\mathfrak{B} \cong \mathfrak{A} \in \psi$ implies $\mathfrak{B} \in \psi$.

[4] The identity symbol is denoted $\equiv$ in [4].

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2.1. Preparatory results. The next four lemmas are routine exercises for readers familiar with Łoś ultraproduct theorem and its consequences ([12, theorem 3.1], [4, sec. 4]). As a warm up we only prove the first one.

LEMMA 2.1. *Let D be a κ-*d. i. *ultrafilter over a set I. Let $\mathfrak{A} = \langle A, E, S_\alpha\rangle_{\alpha<\kappa}$ be a structure of type $\tau = \{E, S_\alpha\}_{\alpha<\kappa}$, where E is an equivalence relation having exactly κ distinct equivalence classes, denoted $\{A_\beta \mid \beta < \kappa\}$, and $S_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for each $\alpha < \kappa$ (with $S_0 = \emptyset$).*

Then for every expansion $\mathfrak{M}$ of $\mathfrak{A}$ to any type $\tau^+ \supseteq \tau$ and every constant symbol $c \notin \tau^+$, the ultrapower $\Pi_D\mathfrak{M}$ has an expansion $\mathfrak{B}$ of type $\tau^+ \cup \{c\}$ such that

$$\mathfrak{B} \models \{\neg S_\alpha c \mid \alpha < \kappa\}.$$

Proof. We first consider the particular case $\tau^+ = \tau$, i.e., $\mathfrak{M} = \mathfrak{A}$. Let

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_\alpha \supseteq \cdots \quad (\alpha < \kappa)$$

be a sequence of elements of D with empty intersection. Let the function $t \colon I \to \kappa$ map each $i \in I$ to

$$t(i) = \text{ least } \alpha < \kappa \text{ such that } i \notin X_\alpha.$$

For each $\alpha < \kappa$ pick an element $x_\alpha \in A_\alpha$. For each $i \in I$ let $\mathfrak{A}_i$ be the expansion of $\mathfrak{A}$ to the type $\tau \cup \{c\}$ such that $c^{\mathfrak{A}_i} = x_{t(i)}$. Let $\mathfrak{B} = \Pi_D\langle \mathfrak{A}_i \mid i \in I\rangle \in \mathrm{Str}(\tau \cup \{c\})$. By Łoś theorem,

$$\{i \in I \mid \mathfrak{A}_i \models \neg S_\alpha c\} = \{i \in I \mid x_{t(i)} \notin S_\alpha\} = \{i \in I \mid t(i) \geq \alpha\} \supseteq X_\alpha \in D.$$

Thus $\mathfrak{B} \models \{\neg S_\alpha c \mid \alpha < \kappa\}$. By the expansion theorem ([4, theorem 4.1.8], [12, proposition 4.1]),

$$\mathfrak{B}\restriction\tau = \Pi_D\langle \mathfrak{A}_i\restriction\tau \mid i \in I\rangle = \Pi_D\mathfrak{A},$$

which shows that $\mathfrak{B}$ is the desired expansion of $\Pi_D\mathfrak{M}$ $(= \Pi_D\mathfrak{A})$.

In the general case when $\mathfrak{M}$ is an expansion of $\mathfrak{A}$ in a type τ^+ strictly containing τ, again from the expansion theorem it follows that

$$(\Pi_D\mathfrak{M})\restriction\tau = \Pi_D(\mathfrak{M}\restriction\tau) = \Pi_D\mathfrak{A}.$$

Arguing as in case $\tau^+ = \tau$, it is not hard to see that the universe of the structure $(\Pi_D\mathfrak{M})\restriction\tau$ has an element c with

$$\langle(\Pi_D\mathfrak{M})\restriction\tau, c\rangle \models \{\neg S_\alpha c \mid \alpha < \kappa\}.$$

As a consequence, $\langle \Pi_D\mathfrak{M}, c\rangle \models \{\neg S_\alpha c \mid \alpha < \kappa\}$. $\square$

LEMMA 2.2. *Let D be an ultrafilter over a set I, τ a* relational *type, and $\mathfrak{M} \in \mathrm{Str}(\tau)$. Assume $E \in \tau$ and $E^{\mathfrak{M}}$ is an equivalence relation. For any map $g : I \to M$ let $g_D \in \Pi_D\mathfrak{M}$ denote the $=_D$-equivalence class of* g. *Let further*

$$Eg_D = \{h_D \in \Pi_D M \mid \Pi_D\mathfrak{M} \models h_D E g_D\},$$

and for every $i \in I$, $Eg(i) = \{x \in M \mid \mathfrak{M} \models xEg(i)\}$.

Then, with "$\restriction$" as defined in (1), $(\Pi_D\mathfrak{M})\restriction Eg_D \;\cong\; \Pi_D\langle(\mathfrak{M}\restriction Eg(i)) \mid i \in I\rangle$.

DEFINITION 2.3. *For τ a relational type, let R, E be relation symbols not in τ, with R unary and E binary. For each class $\psi \subseteq \mathrm{Str}(\tau)$ the class $\psi^{RE} \in \mathrm{Str}(\tau \cup \{R, E\})$ is defined by stipulating that for any $\mathfrak{M} \in \mathrm{Str}(\tau \cup \{R, E\})$,*

$$\mathfrak{M} \in \psi^{RE} \textit{ iff } E^{\mathfrak{M}} \textit{ is an equivalence relation and } R^{\mathfrak{M}} = \{x \in M \mid (\mathfrak{M}|Ex)\restriction\tau \in \psi\},$$

where $Ex = \{y \in M \mid \mathfrak{M} \models yEx\}$.

Intuitively, $R^{\mathfrak{M}}$ is the union of the equivalence classes Ex such that the substructure of $\mathfrak{M}$ with universe Ex satisfies ψ.

LEMMA 2.4. *Let D be an ultrafilter over I, and τ a relational type not containing the relation symbols R (unary) and E (binary). Let us assume that the class $\psi \subseteq \mathrm{Str}(\tau)$ and its complementary class $\mathrm{Str}(\tau) \setminus \psi$ are both closed under isomorphisms and ultraproducts modulo D.*

Then $\Pi_D\mathfrak{M}$ belongs to ψ^{RE} for all $\mathfrak{M} \in \mathrm{Str}(\tau \cup \{R, E\})$ such that $\mathfrak{M} \in \psi^{RE}$.

Combining [4, theorem 4.1.8 and exercise 4.1.11] with [8, theorem 4.2 and p. 114] we obtain:

LEMMA 2.5. *Let D be an ultrafilter over I, and τ a relational type not containing the unary relation symbol R. Assume the class $\psi \subseteq \mathrm{Str}(\tau)$ and its complementary class $\mathrm{Str}(\tau) \setminus \psi$ are both closed under isomorphisms and ultraproducts modulo D. Define $\psi^R \subseteq \mathrm{Str}(\tau \cup \{R\})$ by the following stipulation:*

$$\textit{For any } \mathfrak{M} \in \mathrm{Str}(\tau \cup \{R\}), \quad \mathfrak{M} \in \psi^R \quad \textit{iff} \quad (R^{\mathfrak{M}} \neq \emptyset \textit{ and } (\mathfrak{M}\restriction\tau)|R^{\mathfrak{M}} \in \psi).$$

Then both ψ^R and its complement are closed under isomorphisms and ultraproducts modulo D.

Following [22, p. 251], a topological space X is said to be $[\nu, \mu]$-*compact* if every open cover of cardinality $\leq \mu$ has a subcover of cardinality $< \nu$.

PROPOSITION 2.6. *Let D be a κ-*d. i. *ultrafilter over a set I. Let τ be a relational type, and J be a set having the property that for each $j \in J$ there is a class $\psi_j \subseteq \mathrm{Str}(\tau)$ such that both ψ_j and $\neg\psi_j = \mathrm{Str}(\tau) \setminus \psi_j$ are closed under isomorphisms and ultraproducts modulo D.*

Then $\mathrm{Str}(\tau)$ is $[\kappa, \kappa]$-compact for the topology generated by the subbase $\{\neg\psi_j\}_{j \in J}$.

Proof. Arguing by way of contradiction, let $\{F_\alpha \mid \alpha < \kappa\}$ be a family of closed subspaces of $\mathrm{Str}(\tau)$ with empty intersection, such that for no $W \subseteq \kappa$ with $|W| < \kappa$ we have $\bigcap_{\alpha \in W} F_\alpha = \emptyset$. For each $\beta < \kappa$ let $\hat{F}_\beta = \bigcap_{\alpha < \beta} F_\alpha$ (with $\hat{F}_0 = \mathrm{Str}(\tau)$). Then

$$\hat{F}_0 \supseteq \hat{F}_1 \supseteq \cdots \supseteq \hat{F}_\beta \supseteq \cdots, \ (\beta < \kappa), \quad \hat{F}_\beta \neq \emptyset, \quad \bigcap_{\beta < \kappa} \hat{F}_\beta = \emptyset. \tag{2}$$

We may safely assume that the subbase $\{\neg\psi_j\}_{j \in J}$ is closed under finite intersections. In this way, every closed set is a (possibly infinite) intersection of closed subsets of $\mathrm{Str}(\tau)$ taken from the family $\{\psi_j\}_{j \in J}$. By (2) there is a sequence $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_\beta \subseteq \cdots \ (\beta < \kappa)$ of subsets of J satisfying

$$\hat{F}_\beta = \bigcap\{\psi_j \mid j \in J_\beta\}, \text{ for each } \beta < \kappa. \tag{3}$$

Since each ψ_j is closed under isomorphisms, we also have a sequence of structures $\mathfrak{A}_\beta \in \mathrm{Str}(\tau)$ satisfying

$$\mathfrak{A}_\beta \in \hat{F}_\beta, \ \ A_\alpha \cap A_\beta = \emptyset, \ \ (\alpha < \beta < \kappa). \tag{4}$$

Let

$$\tau^+ = \tau \cup \{E\} \cup \{S_\alpha\}_{\alpha<\kappa} \cup \{R_j\}_{j\in I}\,,$$

where each S_α and R_j is a unary relation symbol, and E is binary. Since J is a set, τ^+ is a type. Define $\mathfrak{M} \in \mathrm{Str}(\tau^+)$ by the following stipulations:

(i) $M = \bigcup_{\alpha<\kappa} A_\alpha$, $(M{\restriction}A_\alpha)\restriction\tau = \mathfrak{A}_\alpha$, and for every $T \in \tau$, $T^{\mathfrak{M}} = \bigcup_{\alpha<\kappa} T^{\mathfrak{A}_\alpha}$.
(ii) $E^{\mathfrak{M}}$ is an equivalence relation with κ components, and for every $x, y \in M$, $xE^{\mathfrak{M}}y$ iff there is $\alpha < \kappa$ with $x, y \in A_\alpha$.
(iii) $S_0^{\mathfrak{M}} = \emptyset$, $S_\alpha^{\mathfrak{M}} = \bigcup_{\beta<\alpha} A_\beta$ for every $\alpha < \kappa$.
(iv) $R_j^{\mathfrak{M}} = \bigcup\{A_\beta \mid \mathfrak{A}_\beta \in \psi_j\}$ for every $j \in J$.

The existence of $\mathfrak{M}$ satisfying condition (i) is ensured by our assumption that τ is relational and the A_α's are pairwise disjoint. With the notation of Definition 2.3, by (i), (ii), and (iv) we can write

$$R_j^{\mathfrak{M}} = \{x \in M \mid (\mathfrak{M}{\restriction}Ex)\restriction\tau \in \psi_j\},$$

i.e.,

$$\mathfrak{M}\restriction(\tau \cup \{R_j, E\}) \in \psi_j^{R_jE} \quad \text{for every } j \in J. \tag{5}$$

If $x \in M \setminus S_{\alpha+1}^{\mathfrak{M}}$ then by (iii), $Ex = A_\beta$ for a unique β with $\alpha < \beta < \kappa$. Then by (2)–(4), conditions (i) and (ii) yield

$$(\mathfrak{M}{\restriction}Ex)\restriction\tau = \mathfrak{A}_\beta \in \bigcap\{\psi_j \mid j \in J_\beta\}$$

whence, by (iv), $x \in R_j^{\mathfrak{M}}$ for each $j \in J_\beta$. A fortiori, $x \in R_j^{\mathfrak{M}}$ for each $j \in J_\alpha$, and we may write

$$\mathfrak{M} \models \forall x(\neg S_{\alpha+1}x \to R_jx) \quad \text{for all } \alpha < \kappa \text{ and } j \in J_\alpha. \tag{6}$$

By Lemma 2.1, the ultrapower $\Pi_D\mathfrak{M}$ has an expansion $\mathfrak{B} = \langle \Pi_D\mathfrak{M}, c\rangle$ such that $\mathfrak{B} \models \neg\{S_\alpha c \mid \alpha < \kappa\}$. Combining Łoś theorem with (6) we can write

$$\mathfrak{B} \models \forall x(\neg S_{\alpha+1}x \to R_jx), \quad \text{for all } \alpha < \kappa \text{ and } j \in J_\alpha.$$

As a consequence,

$$\langle \Pi_D\mathfrak{M}, c\rangle \models R_jc \quad \text{for all} \quad j \in \bigcup_{\alpha<\kappa} J_\alpha. \tag{7}$$

For every $j \in J$ let

$$\tau_j = \tau \cup \{R_j, E\} \text{ and } \mathfrak{M}_j = \mathfrak{M}\restriction\tau_j.$$

From (5) we have $\mathfrak{M}_j \in \psi_j^{R_jE}$. Lemma 2.4 now yields

$$\Pi_D\mathfrak{M}_j \in \psi_j^{R_jE}, \text{ whence by expansion, } (\Pi_D\mathfrak{M})\restriction\tau_j \in \psi_j^{R_jE}.$$

Setting now $\mathfrak{N} = \Pi_D\mathfrak{M}$, by Definition 2.3 we have

$$R_j^{\mathfrak{N}} = \{x \in N \mid (\mathfrak{N}{\restriction}Ex)\restriction\tau \in \psi_j\} \text{ for each } j \in J.$$

By (7), $(\mathfrak{N}{\restriction}Ec)\restriction\tau \in \psi_j$ for all $j \in \bigcup_{\alpha<\kappa} J_\alpha$. Finally, by (3), $(\mathfrak{N}{\restriction}Ec)\restriction\tau$ belongs to $\hat{F}_\beta$ for all $\beta < \kappa$, in contradiction with (2). The proof is complete. □

§3. From classes of ultrafilters to logics.

3.1. Logics, elementary classes, Δ-closure. Statement of the main theorem. Following [7, definition 1.1.1], by a *logic* $\mathcal{L}$ we mean a pair $(\mathrm{Stc}_{\mathcal{L}}, \models_{\mathcal{L}})$, where $\mathrm{Stc}_{\mathcal{L}}$ is a map assigning to every type τ a family $\mathrm{Stc}_{\mathcal{L}}(\tau)$, called the family of *sentences in $\mathcal{L}$ of type τ*, and $\models_{\mathcal{L}}$ (called *$\mathcal{L}$-satisfaction*) is a relation between structures and sentences satisfying conditions (i)–(v) in [7, p. 28]. Via $\models_{\mathcal{L}}$, each sentence ψ of type τ is identified with a class, also denoted ψ, of structures of type τ. We write $\mathcal{L}(\tau)$ instead of $\mathrm{Stc}_{\mathcal{L}}(\tau)$. Given $\Phi \subseteq \mathcal{L}(\tau)$ we say that Φ *has a model* if there is $\mathfrak{N} \in \mathrm{Str}(\tau)$ such that $\mathfrak{N} \models_{\mathcal{L}} \phi$ for all $\phi \in \Phi$.

As already remarked, $\models$ denotes the satisfaction relation of first-order logic.

Following [16, p. 230] logic $\mathcal{L}$ is *κ-relatively compact* (κ-r. c.) if for any two sets Ψ, Ξ of sentences in $\mathcal{L}$ with $|\Psi| = \kappa$, if $\Psi_0 \cup \Xi$ has a model for every $\Psi_0 \subseteq \Psi$ with $|\Psi_0| < \kappa$ and arbitrary Ξ, then so does $\Psi \cup \Xi$.

A class $\chi \subseteq \mathrm{Str}(\tau)$ is *elementary of type τ in $\mathcal{L}$* if there is $\psi \in \mathcal{L}(\tau)$ such that $\chi = \{\mathfrak{M} \in \mathrm{Str}(\tau) \mid \mathfrak{M} \models_{\mathcal{L}} \psi\}$. $EC^{\tau}_{\mathcal{L}}$ denotes the family of all such classes. Two logics are *equivalent* if they have the same elementary classes. For types $\sigma \subseteq \tau$ and $\chi \subseteq \mathrm{Str}(\sigma)$ we let

$$(\restriction^{\tau}_{\sigma})^{-1}\chi = \{\mathfrak{M} \in \mathrm{Str}(\tau) \mid \mathfrak{M}\restriction^{\tau}_{\sigma} \in \chi\},$$

where $\restriction^{\tau}_{\sigma} \colon \mathrm{Str}(\tau) \to \mathrm{Str}(\sigma)$ is the reduct operation.

Following [7, p. 18] we say that $\mathcal{L}$ is *Δ-closed* if it has the following property: Let $\tau \subseteq \tau' \cap \tau''$, $\psi' \in EC^{\tau'}_{\mathcal{L}}$, $\psi'' \in EC^{\tau''}_{\mathcal{L}}$. Suppose $(\restriction^{\tau'}_{\tau})\psi' = \mathrm{Str}(\tau) \setminus (\restriction^{\tau''}_{\tau})\psi''$. Then $(\restriction^{\tau'}_{\tau})\psi' \in EC^{\tau}_{\mathcal{L}}$.

The following theorem is our first main result in this paper:

THEOREM 3.1. *Let Ω be a nonempty class of uniform ultrafilters. Let the map $L = L_{\Omega}$ assign to every type τ the family $L(\tau)$ specified as follows*:

> *$\psi \in L(\tau)$ iff $\psi \subseteq \mathrm{Str}(\tau)$ and there is a finite type $\sigma \subseteq \tau$ and a class $\chi \subseteq \mathrm{Str}(\sigma)$ such that $\psi = (\restriction^{\tau}_{\sigma})^{-1}\chi$, with both χ and $\mathrm{Str}(\sigma) \setminus \chi$ closed under isomorphisms and ultraproducts modulo D for all $D \in \Omega$.*

We then have:

(i) *Up to equivalence, there is a unique regular logic $\mathcal{L} = \mathcal{L}_{\Omega}$ such that for every type τ, $L(\tau) = EC^{\tau}_{\mathcal{L}}$. $\mathcal{L}$ is Δ-closed. If Ω contains a κ-descendingly incomplete ultrafilter, $\mathcal{L}$ is κ-relatively compact. $\mathcal{L}$ is compact only if Ω is a proper class.*

(ii) ($\neg 0^{\sharp}$ *or* $\neg L^{\mu}$) *If $D \in \Omega$ is an ultrafilter over a regular cardinal ν then for a family Ψ of $\mathcal{L}$-sentences with $|\Psi| \leq \nu$ to have a model it is sufficient that every finite subfamily of Ψ has a model. In particular, if Ω is a proper class and each $D \in \Omega$ is an ultrafilter over a regular cardinal, then $\mathcal{L}$ is compact.*

3.2. First part of the proof of Theorem 3.1 (i): Basic properties of $L = L_{\Omega}$.

PROPOSITION 3.2. *With the above notation and terminology concerning Ω and L, we have*:

(1) *Both $\emptyset$ and $\mathrm{Str}(\tau)$ belong to $L(\tau)$.*

(2) *If ψ belongs to $L(\tau)$ then so does its complement $\neg\psi = \mathrm{Str}(\tau) \setminus \psi$.*

(3) *Every $\psi \in L(\tau)$ (as well as $\neg\psi$) is closed under isomorphisms and ultraproducts modulo D for all $D \in \Omega$.*

(4) *For every finite type τ, if both $\psi \subseteq \mathrm{Str}(\tau)$ and its complement are closed under isomorphisms and ultraproducts modulo D for all $D \in \Omega$, then ψ belongs to $L(\tau)$.*
(5) *If $\psi \in L(\tau)$ and $\zeta \supseteq \tau$ then $(\upharpoonright^{\zeta}_{\tau})^{-1}\psi \in L(\zeta)$.*
(6) *Let $\rho\colon \tau \to \sigma$ be an arity preserving bijection of τ onto σ sending relations, constants and functions to relations, constants and functions. (For short, ρ is a* renaming.) *For each $\mathfrak{M} \in \mathrm{Str}(\tau)$ let $\mathfrak{M}^{\rho} \in \mathrm{Str}(\sigma)$ be the structure with the same universe as $\mathfrak{M}$, where each symbol $S \in \sigma$ is interpreted precisely as the symbol ρ^{-1} is interpreted in $\mathfrak{M}$. Let $\psi^{\rho} = \{\mathfrak{M}^{\rho} \mid \mathfrak{M} \in \psi\}$. Then for all $\psi \in L(\tau)$, $\psi^{\rho} \in L(\sigma)$.*
(7) *For every first-order sentence θ of a finite type τ, the class*

$$\psi = \{\mathfrak{N} \in \mathrm{Str}(\tau) \mid \mathfrak{N} \models \theta\}$$

belongs to $L(\tau)$.
(8) *If $\psi, \phi \in L(\tau)$ then $\psi \cap \phi \in L(\tau)$.*
(9) *For any $\psi \in L(\tau)$ and constant symbol c, let the class $\exists c\psi \subseteq \mathrm{Str}(\tau \setminus \{c\})$ be defined by*

$$\exists c\psi = \left(\upharpoonright\tau^{\tau}_{\tau\setminus\{c\}}\right)\ \psi.$$

If $c \notin \tau$ then $\exists c\psi = \psi$. If $c \in \tau$ then for every $\mathfrak{M} \in \mathrm{Str}(\tau \setminus \{c\})$ we have: $\mathfrak{M} \in \exists c\psi$ iff there is $m \in M$ such that $\langle \mathfrak{M}, c/m\rangle \in \psi$. Here $\langle \mathfrak{M}, c/m\rangle$ denotes the expansion of $\mathfrak{M}$ to τ where c is interpreted by m.
(10) *For every $\psi \in L(\tau)$ and constant symbol c, $\exists c\psi$ belongs to $L(\tau \setminus \{c\})$.*
(11) *Let τ, τ', τ'' be types, with τ finite and coinciding with $\tau' \cap \tau''$. Let $\psi' \in L(\tau')$ and $\psi'' \in L(\tau'')$. Suppose further $(\upharpoonright^{\tau'}_{\tau})\psi' = \mathrm{Str}(\tau) \setminus (\upharpoonright^{\tau''}_{\tau})\psi''$. Then $(\upharpoonright^{\tau'}_{\tau})\psi'$ belongs to $L(\tau)$.*
(12) *For μ a cardinal and τ a fixed but otherwise arbitrary type, suppose we are given classes of structures $\{\chi_\alpha \mid \alpha < \mu\} \subseteq L(\tau)$. Let $\mathcal{T}$ be the topology on $\mathrm{Str}(\tau)$ generated by the subbase $\{\neg\chi_\alpha \mid \alpha < \mu\}$. Then $\mathcal{T}$ is a $[\kappa, \kappa]$-compact topology for each cardinal κ such that there is a κ-*d. i. *ultrafilter $D \in \Omega$.*

Proof. (1)–(7) are immediate consequences of the definition of $L(\tau)$. (8) routinely follows from (2), (3) and (5). (9) is trivial.

To prove (10), skipping all trivialities, assume $c \in \tau$. For some finite type $\sigma_0 \subseteq \tau$ and class $\chi_0 \subseteq \mathrm{Str}(\sigma_0)$ we can write $\psi = (\upharpoonright^{\tau}_{\sigma_0})^{-1}\chi_0$. Let $\sigma = \sigma_0 \cup \{c\}$ and $\chi = (\upharpoonright^{\sigma}_{\sigma_0})^{-1}\chi_0$. By (3) and (5), both χ and $\neg\chi$ are in $L(\sigma)$ and are closed under isomorphisms and ultraproducts modulo D for all $D \in \Omega$. Furthermore, $\psi = (\upharpoonright^{\tau}_{\sigma})^{-1}\chi$ (and $c \in \sigma$). Let $\sigma' = \sigma \setminus \{c\}$ and $\tau' = \tau \setminus \{c\}$. By making repeated use of (9) we get $\exists c\psi = (\upharpoonright^{\tau'}_{\sigma'})^{-1}\exists c\chi$. From the expansion theorem [4, Theorem 4.1.8] it follows that for all $D \in \Omega$ both classes $\exists c\chi$ and $\neg\exists c\chi$ are closed under ultraproducts modulo D. Both classes are also closed under isomorphisms. Since σ' is finite we conclude that $\exists c\psi$ belongs to $L(\tau') = L(\tau \setminus \{c\})$, which settles (10).

Next, (11) is easily proved using the expansion theorem and (3) and (4).
The special case of (12) when τ is relational is taken care of by (3) and Proposition 2.6. The general case for an arbitrary type τ is a tedious but routine variant of the special case, again using Proposition 2.6 together with (10). ☐

3.3. Second part of the proof of Theorem 3.1 (i): The construction of $\mathcal{L} = \mathcal{L}_\Omega$. For every structure $\mathfrak{M}$ there is exactly one type $\tau = \tau_{\mathfrak{M}}$ such that $\mathfrak{M} \in \mathrm{Str}(\tau)$. For every nonempty class ψ of structures of the same type there is exactly one $\tau = \tau_\psi$ such that $\psi \subseteq \mathrm{Str}(\tau)$.

DEFINITION 3.3. *Adopt the above notation and terminology concerning Ω and $L = L_\Omega$. Given any type τ, the family $\mathcal{L}(\tau)$ of* sentences of type τ in $\mathcal{L} = \mathcal{L}_\Omega$ *and its* satisfaction relation $\models_{\mathcal{L}}$ *are defined by*:

- $(*)$ *$\psi \in \mathcal{L}(\tau)$ if either ψ is empty or ψ is a nonempty class of structures of the same type τ_ψ, with τ_ψ finite, $\tau_\psi \subseteq \tau$, and $\psi \in L(\tau_\psi)$.*
- $(**)$ $\mathfrak{M}$ satisfies ψ in $\mathcal{L}$, *in symbols, $\mathfrak{M} \models_{\mathcal{L}} \psi$, if $\tau_\psi \subseteq \tau_{\mathfrak{M}}$ and $\mathfrak{M} \restriction \tau_\psi \in \psi$.*

Since ψ is a sentence of $\mathcal{L}(\tau)$ for some τ, the type τ_ψ is well defined in $(**)$. Moreover, τ_ψ is finite and ψ is a member of $\mathcal{L}(\tau_\psi)$. We use the notation

$$\mathrm{Mod}^\tau_{\mathcal{L}}\, \psi = \{\mathfrak{M} \in \mathrm{Str}(\tau) \mid \mathfrak{M} \models_{\mathcal{L}} \psi\}.$$

Recalling the definition at the outset of Section 3, we say that $\chi \subseteq \mathrm{Str}(\tau)$ is an elementary class of typeτ in $\mathcal{L}$, *in symbols, $\chi \in EC^\tau_{\mathcal{L}}$, if $\chi = \mathrm{Mod}^\tau_{\mathcal{L}}\, \psi$ for some $\psi \in \mathcal{L}(\tau)$.*

The proof of the following proposition is a main prerequisite to prove (in the next subsection) that $\mathcal{L} = \mathcal{L}_\Omega$ is a regular logic having all the properties stated in Theorem 3.1.

PROPOSITION 3.4. *With the above notation we have*:

(1) *For every τ, $EC^\tau_{\mathcal{L}} = L(\tau)$.*
(2) *(Monotonicity) If $\sigma \subseteq \tau$ then $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\tau)$.*
(3) *("Type" property) If $\mathfrak{M} \models_{\mathcal{L}} \psi$ then $\psi \in \mathcal{L}(\tau_{\mathfrak{M}})$.*
(4) *(Isomorphism) If $\mathfrak{M} \models_{\mathcal{L}} \psi$ and $\mathfrak{N} \cong \mathfrak{M}$ then $\mathfrak{N} \models_{\mathcal{L}} \psi$.*
(5) *(Reduct) Let $\mathfrak{M} \in \mathrm{Str}(\sigma)$, $\psi \in \mathcal{L}(\tau)$, and $\tau \subseteq \sigma$. Then $\mathfrak{M} \models_{\mathcal{L}} \psi$ iff $\mathfrak{M} \restriction \tau \models_{\mathcal{L}} \psi$.*
(6) *(Renaming) Let ρ be a renaming of τ onto σ. (See Proposition 3.2(6).) Then for every $\psi \in \mathcal{L}(\tau)$ there is $\chi \in \mathcal{L}(\sigma)$ such that for all $\mathfrak{M} \in \mathrm{Str}(\tau)$, $\mathfrak{M} \models_{\mathcal{L}} \psi$ iff $\mathfrak{M}^\rho \models_{\mathcal{L}} \chi$.*
(7) *(Atoms) For every τ and atomic first-order sentence ξ of type τ there is $\psi \in \mathcal{L}(\tau)$ with $\mathrm{Mod}^\tau_{\mathcal{L}}\, \psi = \{\mathfrak{N} \in \mathrm{Str}(\tau) \mid \mathfrak{N} \models \xi\}$.*
(8) *(Negation) For all τ and $\psi \in \mathcal{L}(\tau)$ there is $\chi \in \mathcal{L}(\tau)$ with $\mathrm{Mod}^\tau_{\mathcal{L}}\, \chi = \mathrm{Str}(\tau) \setminus \mathrm{Mod}^\tau_{\mathcal{L}}\, \psi$.*
(9) *(Conjunction) For every type τ and classes $\psi, \phi \in \mathcal{L}(\tau)$ there is $\chi \in \mathcal{L}(\tau)$ with $\mathrm{Mod}^\tau_{\mathcal{L}}\, \chi = \mathrm{Mod}^\tau_{\mathcal{L}}\, \psi \cap \mathrm{Mod}^\tau_{\mathcal{L}}\, \phi$.*
(10) *(Quantification) For every τ, constant symbol* c, *and $\psi \in \mathcal{L}(\tau)$ there is $\chi \in \mathcal{L}(\tau \setminus \{c\})$ such that for all $\mathfrak{M} \in \mathrm{Str}(\tau \setminus \{c\})$,*

$$\mathfrak{M} \models_{\mathcal{L}} \chi \text{ iff } \langle \mathfrak{M}, c/m\rangle \models_{\mathcal{L}} \psi \text{ for some } m \in M,$$

with $\langle \mathfrak{M}, c/m\rangle$ as in Proposition 3.2(10).
(11) *(Δ-closure) For all $\tau \subseteq \tau' \cap \tau''$, $\psi' \in EC^{\tau'}_{\mathcal{L}}$, and $\psi'' \in EC^{\tau''}_{\mathcal{L}}$, if $(\restriction^{\tau'}_{\tau})\psi' = \mathrm{Str}(\tau) \setminus (\restriction^{\tau''}_{\tau})\psi''$, then $(\restriction^{\tau'}_{\tau})\psi' \in EC^\tau_{\mathcal{L}}$.*
(12) *If Ω contains a κ-descendingly incomplete ultrafilter, $\mathcal{L}$ is κ-relatively compact.*

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(13) (*Finite Occurrence Property*) *For every sentence $\psi \in \mathcal{L}$ there is a smallest type τ such that $\psi \in \mathcal{L}(\tau)$. Furthermore, such τ is finite.*

(14) *Let τ be a finite relational type, $R \notin \tau$ a unary relation symbol, $\phi \in \mathcal{L}(\tau)$ and $\tau^+ = \tau \cup \{R\}$. Then there exists $\psi \in \mathcal{L}(\tau^+)$ such that for all $\mathfrak{M} \in \mathrm{Str}(\tau^+)$,*

$$\mathfrak{M} \models_{\mathcal{L}} \psi \textit{ iff } (R^{\mathfrak{M}} \neq \emptyset \textit{ and } (\mathfrak{M}\restriction\tau)|R^{\mathfrak{M}} \models_{\mathcal{L}} \phi).$$

Proof. (1) follows by definition from Proposition 3.2(1)–(5). Properties (2), (3) and (5) are immediate consequences of Definition 3.3. (4) Follows from Proposition 3.2(3). (6) follows from (1) and Proposition 3.2(6). By (1), conditions (7)–(10), respectively, follow from Propositions 3.2(7), 3.2(2), 3.2(8) and 3.2(10) and (11).

To prove (11) we argue as follows: By the renaming property (6) we can safely assume $\tau = \tau' \cap \tau''$. By (1), $\psi' \in L(\tau')$ and $\psi'' \in L(\tau'')$. By Proposition 3.2(4), for some finite $\sigma' \subseteq \tau'$ and $\sigma'' \subseteq \tau''$ there are $\chi' \in L(\sigma')$ and $\chi'' \in L(\sigma'')$ such that

$$\psi' = (\restriction^{\tau'}_{\sigma'})\chi' \text{ and } \psi'' = (\restriction^{\tau''}_{\sigma''})\chi''.$$

The finite type $\sigma = \sigma' \cap \sigma''$ is contained in τ. Let

$$\phi = (\restriction^{\tau'}_{\tau})\psi' \text{ and } \neg\phi = \mathrm{Str}(\tau) \setminus \phi = (\restriction^{\tau''}_{\tau})\psi''.$$

Then

$$(\restriction^{\sigma'}_{\sigma})\chi' \cap (\restriction^{\sigma''}_{\sigma})\chi'' = \emptyset.$$

A routine application of Proposition 3.2(3), (4), and (12) yields

$$(\restriction^{\sigma'}_{\sigma})\chi' \cup (\restriction^{\sigma''}_{\sigma})\chi'' = \mathrm{Str}(\sigma).$$

From Proposition 3.2(2) and (11) we obtain

$$\phi = (\restriction^{\tau}_{\sigma})^{-1}((\restriction^{\sigma'}_{\sigma})\chi').$$

By (1), we finally obtain $\phi = (\restriction^{\tau'}_{\tau})\psi' \in EC^{\tau}_{\mathcal{L}}$, which settles (11).

A proof of (12) can be obtained from (1) in combination with Proposition 3.2(12). (13) is trivial. (14) follows from (1) together with Proposition 3.2(3) and (4), in combination with Lemma 2.5. □

3.4. Last part of the proof of Theorem 3.1(i): $\mathcal{L}$ has the desired properties. By Proposition 3.4(1), $\mathcal{L}$ is uniquely determined by Ω up to equivalence.

We next prove that $\mathcal{L}$ is a regular logic.

By Proposition 3.4(2)–(10), $\mathcal{L}$ has the monotony, type, isomorphism, reduct, renaming properties, and is closed under first-order atomic sentences, negation, conjunction and existential quantification. The finite occurrence property holds by Proposition 3.4(13). By Proposition 3.4(11), $\mathcal{L}$ is Δ-closed. These properties, together with the finite occurrence property, imply the substitution property [7, definition 1.2.3]. As a consequence, $\mathcal{L}$ has elimination of function and constant symbols [7, definition 1.2.3 and p. 31]. To prove that $\mathcal{L}$ has the relativization property [7, definition 1.2.2], we argue as follows: Given sentences ϕ, χ, a relativization of ϕ to $\{c \mid \chi(c)\}$ is obtained by assuming $\phi \in \mathcal{L}(\tau'), \chi \in \mathcal{L}(\tau'')$ for finite types τ', τ'', with τ' relational. This assumption is made without loss of generality, because $\mathcal{L}$ has the finite occurrence property and allows elimination of function and constant symbols. We next let $R \notin \tau' \cup \tau''$ be a unary relation symbol. Then Proposition 3.4(14) yields a sentence

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$\tilde{\psi} \in \mathcal{L}(\tau' \cup \{R\})$ providing the desired relativization. Replacing now Rc in $\tilde{\psi}$ by $\chi(c)$ (which is allowed by the substitution property) we finally obtain a sentence $\psi \in \mathcal{L}(\tau' \cup (\tau'' \setminus \{c\}))$ yielding a relativization of ϕ to $\{c \mid \chi(c)\}$. Thus $\mathcal{L}$ is closed under relativization.

The proof that $\mathcal{L}$ is a regular logic is now complete.

By Proposition 3.4(12), $\mathcal{L}$ is κ-r. c. whenever Ω contains a κ-d. i. ultrafilter.

To conclude the proof of Theorem 3.1(i), assume Ω to be a *set*, with the intent of proving that its associated logic $\mathcal{L} = \mathcal{L}_\Omega$ is not compact. Without loss of generality, each $D \in \Omega$ is a uniform ultrafilter over some cardinal $\mu_D \geq \omega$. Let $\{\mu_D \mid D \in \Omega\}$ be the set of cardinals which are the index set of some ultrafilter in Ω. Let

$$\mu = \sup\{\mu_D \mid D \in \Omega\} \text{ and } \mu^* = (2^\mu)^+.$$

We then have

$$|\Pi_{i \in I} \kappa_i| \leq (2^\mu)^{\mu_D} \leq (2^\mu)^\mu = 2^\mu < (2^\mu)^+ = \mu^*.$$

As a consequence, each μ_D is *small* for μ^*, in the sense that for any family $\langle \kappa_i \mid i \in I\rangle$, if $|I| \leq \mu_d$ and $\kappa_i < \mu^*$ for all $i \in I$, then the cardinality of the cartesian product $\Pi_{i \in I} \kappa_i$ is $< \mu^*$. Now let $\psi^* \subseteq \mathrm{Str}(\emptyset)$ be defined by $\psi^* = \{\mathfrak{M} \in \mathrm{Str}(\emptyset) \mid |M| \geq \mu^*\}$. Thus ψ^* is the class of all structures in the pure identity language, whose universe has at least μ^* elements. For every ultrafilter $D \in \Omega$, say D over μ_D, and for every set $\{\mathfrak{M}_\alpha \mid \alpha < \mu_D\} \subseteq \neg\psi^*$, the ultraproduct $\Pi_D \langle \mathfrak{M}_\alpha \mid \alpha < \mu_D\rangle$ belongs to $\neg\psi^*$. As a matter of fact, since μ_D is small for μ^*,

$$|\Pi_D \langle M_\alpha \mid \alpha < \mu_d\rangle| \leq |\Pi_{\alpha < \mu_d} M_\alpha| < \mu^*.$$

Likewise, by [4, proposition 4.3.6(ii) and (iv)], for every $\{\mathfrak{N}_\alpha \mid \alpha < \mu_D\} \subseteq \psi^*$ we have $\Pi_D \langle \mathfrak{N}_\alpha \mid \alpha < \mu_D\rangle \in \psi^*$.

Thus for every $D \in \Omega$ both classes ψ^* and $\neg\psi^*$ are closed under ultraproducts modulo D. Trivially, both classes are also closed under isomorphisms. By Propositions 3.2(4) and 3.4(1),

$$\psi^* \text{ is a sentence in the pure identity language of } \mathcal{L}. \tag{8}$$

One now routinely checks that $\mathcal{L}$ is not compact. Let $\tau = \{c_\beta \mid \beta < \mu^*\}$. Let $\Upsilon \subseteq \mathcal{L}(\tau)$ be defined by

$$\Upsilon = \{\neg\psi^*\} \cup \{c_\alpha \neq c_\beta \mid \alpha < \beta < \mu^*\},$$

where $c_\alpha \neq c_\beta$ is an abbreviation of the class $\{\mathfrak{N} \in \mathrm{Str}(\tau) \mid \mathfrak{N} \models c_\alpha \neq c_\beta\}$. By Proposition 3.2(5) and (7), $\Upsilon \subseteq \mathcal{L}(\tau)$. Furthermore, every finite subfamily of Υ has a model, but Υ has none. This proves that $\mathcal{L}$ is not compact. The proof of Theorem 3.1(i) is now complete.

Proof of Theorem 3.1(ii). We prepare:

PROPOSITION 3.5. *Let κ be an infinite cardinal. Then a topological (not necessarily Hausdorff) space is $[\lambda, \lambda]$-compact for all regular λ with $\omega \leq \lambda \leq \kappa$ iff every open cover of cardinality $\leq \kappa$ has a finite subcover.* □

Proof [22, theorem 2A]. □

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PROPOSITION 3.6. *$\mathcal{L}$ is κ-r. c. iff it has the following property:*
For every set J, type τ and family $\{\psi_j \mid j \in J\} \subseteq L(\tau)$ closed under complement (i.e., negation), closed under finite intersections, and containing $EC^\tau_{\mathcal{L}_{\omega\omega}}$, the class $\mathrm{Str}(\tau)$ *is $[\kappa, \kappa]$-compact with respect to the topology generated by the subbase $\{\neg\psi_j \mid j \in J\}$.*

Proof. [18, theorem 1] which uses a different notation. □

Let $D \in \Omega$ be a uniform ultrafilter over a regular (infinite) cardinal ν. By [6, theorem 4.3 and remarks on pages 89–91], D is λ-d. i. for all regular λ with $\omega \leq \lambda \leq \nu$. (Here the authors of [6] make use of the hypotheses $\neg L^\mu$ or $0^\sharp$.) By Proposition 3.4(12), $\mathcal{L}$ is λ-r. c. for all regular λ with $\omega \leq \lambda \leq \nu$. Let Ψ be a family of sentences of $\mathcal{L}(\tau)$ with $|\Psi| \leq \nu$. For each $\psi \in \Psi$ let $\dot{\psi} = \mathrm{Mod}^\tau_{\mathcal{L}}\, \psi$. Next let us equip $\mathrm{Str}(\tau)$ with the topology $\mathcal{T}$ generated by the subbase

$$EC^\tau_{\mathcal{L}_{\omega\omega}} \cup \{\dot{\psi} \mid \psi \in \Psi\} \cup \{\neg\dot{\psi} \mid \psi \in \Psi\}.$$

Without loss of generality this subbase is closed under finite intersections. By Proposition 3.5, every open cover of $\mathrm{Str}(\tau)$ of cardinality $\leq \nu$ has a finite subcover. By Proposition 3.6, $\mathrm{Str}(\tau)$ is $[\lambda, \lambda]$-compact for all regular λ with $\omega \leq \lambda \leq \nu$. Thus, if every finite subfamily of Ψ has a model, Ψ has a model. This proves the first statement in Theorem 3.1(ii).

To prove the last statement we argue as follows: From the assumption that Ω be a proper class with each $D \in \Omega$ a uniform ultrafilter over a regular cardinal it follows that the class

$$\{\nu \mid \text{ there is } D \in \Omega \text{ with } D \text{ an ultrafilter over } \nu\}$$

is proper. The same argument in the first part of the proof now shows that every family $\Psi \subseteq \mathcal{L}(\tau)$ with $|\Psi| \leq \nu$ satisfies the compactness theorem. Since ν is an arbitrarily large regular cardinal, $\mathcal{L}$ is compact.

Having thus proved Theorem 3.1(ii), the proof of Theorem 3.1 is complete.

COROLLARY 3.7. *The logic $\mathcal{L}$ generated by the class Ω^* of all uniform ultrafilters over all infinite cardinals is (equivalent to) $\mathcal{L}_{\omega\omega}$.*

Proof. By Theorem 3.1(i), $\mathcal{L}$ is κ-r.c for every regular cardinal κ. By Propositions 3.5 and 3.6, $\mathcal{L}$ is compact. By Proposition 3.4(4)–(10), every elementary class of $\mathcal{L}_{\omega\omega}$ is an elementary class of $\mathcal{L}$. Thus $\mathcal{L}$-equivalence $\equiv_{\mathcal{L}}$ is finer than $\mathcal{L}_{\omega\omega}$-equivalence $\equiv$. Conversely, let us assume $\mathfrak{A} \equiv \mathfrak{B}$. The proof of the Keisler–Shelah theorem [4, Theorem 6.1.15] yields $\lambda \geq \omega$ and a uniform ultrafilter D over λ such that $\Pi_D\mathfrak{A} \cong \Pi_D\mathfrak{B}$. Since $D \in \Omega^*$ and the elementary classes of $\mathcal{L}$ are closed under ultrapowers and isomorphisms, Propositions 3.2(3) and 3.4(1) yield $\mathfrak{A} \equiv_{\mathcal{L}} \Pi_D\mathfrak{A}$ and $\mathfrak{B} \equiv_{\mathcal{L}} \Pi_D\mathfrak{B}$, whence $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$. Therefore, $\equiv$ is finer than (whence it coincides with) $\equiv_{\mathcal{L}}$. The compactness of $\mathcal{L}$ and $\mathcal{L}_{\omega\omega}$ now routinely yields the desired conclusion. □

§4. Applications and examples.

Q is a *cardinality quantifier* if for some ordinal α

$$Q = Q_\alpha = \text{ the quantifier “there exist at least } \aleph_\alpha \text{ many.”}$$

We say that $\mathcal{L}$ is *generated by cardinality quantifiers* if $\mathcal{L} = \mathcal{L}(Q_\alpha)_{\alpha \in W}$ for some nonempty class W of ordinals. We also say that $\mathcal{L}$ *contains the quantifier Q_α* if $EC^\tau_{\mathcal{L}} \supseteq EC^\tau_{\mathcal{L}(Q)}$ for all τ.

THEOREM 4.1. *Let Ω be a class of uniform ultrafilters over infinite cardinals. Assume that either Ω is a set or Ω contains a countably incomplete ultrafilter. Then the logic $\mathcal{L}$ obtained from Ω via Theorem 3.1(i) is not generated by cardinality quantifiers.*

Proof. The case when Ω is a set.

The final part of the proof of Theorem 3.1(i) yields a cardinal $\mu^* > \omega$ such that the class $\psi^* = \{\mathfrak{M} \in \mathrm{Str}(\tau) \mid |M| \geq \mu^*\}$ belongs to $EC^{\emptyset}_{\mathcal{L}}$. See (8). Since $\mathcal{L}$ is a Δ-closed regular logic with the finite occurrence property then, by [7, theorem 4.1.3], $\mathcal{L}$ contains the quantifier Q^* "there are at least μ^* many."

By way of contradiction, assume $\mathcal{L}$ is generated by cardinality quantifiers, say, $\mathcal{L} = \mathcal{L}(Q^*, Q_\alpha)_{\alpha \in W}$ for some class W of ordinals. Let On be the class of all ordinals, and $\mathcal{L}^+ = \mathcal{L}(Q_\alpha)_{\alpha \in \mathsf{On}}$. For any $\kappa, \lambda \geq \omega$ let $\mathfrak{A}^\kappa_\lambda = \langle A, E\rangle$, where E is an equivalence relation over A having exactly λ distinct equivalence classes, each one of cardinality κ. The back and forth argument of [3, proof of theorem 4.4, p. 93], combined the proof of $\mathfrak{A}^{\omega_1}_{\omega_1} \equiv_{\mathcal{L}(Q_1)} \mathfrak{A}^{\omega_1}_{\omega}$ in [7, corollary 4.2.8] yields

$$\mathfrak{A}^\kappa_\kappa \equiv_{\mathcal{L}^+} \mathfrak{A}^\kappa_\omega, \text{ whence } \mathfrak{A}^{\mu^*}_{\mu^*} \equiv_{\mathcal{L}} \mathfrak{A}^{\mu^*}_{\omega}. \tag{9}$$

Let $\mathfrak{B} = \langle \mathfrak{A}^{\mu^*}_{\mu^*}, f\rangle$, with the unary function f picking one element from every equivalence class. Then $\mathcal{L}(E, f)$ contains a sentence ψ stating "the range of f has $\geq \mu^*$ elements" with $\mathfrak{B} \models_{\mathcal{L}} \psi$. Similarly, let $\mathfrak{D} = \langle \mathfrak{A}^{\mu^*}_{\omega}, g\rangle$, where g chooses a member of every equivalence class. Then some sentence $\chi \in \mathcal{L}(E, g)$ stating "the range of g has $< \mu^*$ elements" is satisfied by $\mathfrak{D}$ in $\mathcal{L}$. Since $\mathcal{L}$ contains all first-order sentences, $\mathcal{L}(E)$ contains a sentence stating "E is an equivalence relation." Therefore we can safely assume that for every structure $\mathfrak{M}$ of type $\{E\}$ exactly one of the following alternatives holds:

(i) either $\mathfrak{M}$ can be expanded to a model of ψ of type $\{f, E\}$,

(ii) or $\mathfrak{M}$ can be expanded to a model of χ of type $\{g, E\}$.

Upon setting

$$\dot\psi = \mathrm{Mod}^{\{E,f\}}_{\mathcal{L}}(\psi), \quad \text{and} \quad \dot\chi = \mathrm{Mod}^{\{E,g\}}_{\mathcal{L}}(\chi),$$

by Proposition 3.4(1) we obtain

$$\left(\restriction^{\{E,f\}}_{\{E\}}\right)\dot\psi = \mathrm{Str}(E) \setminus \left(\restriction^{\{E,g\}}_{\{E\}}\right)\dot\chi.$$

Since $\mathcal{L}$ is Δ-closed, Proposition 3.4(1) yields a sentence $\phi \in \mathcal{L}(E)$ such that

$$\mathfrak{A}^{\mu^*}_{\mu^*} \models_{\mathcal{L}} \phi \text{ and } \mathfrak{A}^{\mu^*}_{\omega} \models_{\mathcal{L}} \neg\phi.$$

This contradicts (9), and settles the case when Ω is a set.

The case when Ω contains a countably incomplete ultrafilter D.

Proceeding as in the first case, by way of contradiction assume $\mathcal{L}$ is generated by cardinality quantifiers, say, $\mathcal{L} = \mathcal{L}(Q^*, Q_\alpha)_{\alpha \in W}$ for some nonempty class W of ordinals. Assume D is an ultrafilter over μ.

CLAIM. *The quantifier Q_0 saying "there are infinitely many" is not contained in $\mathcal{L}$.*

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As a matter of fact, by [4, lemma 4.2.3], our standing hypothesis about D amounts to assuming the existence of a partition $\tilde{X}$ of μ into $<\omega_1$ components, none of which belongs to D. Now $\tilde{X}$ has exactly ω many components, say,

$$\tilde{X} = \{X_n \mid n \in \omega\} \text{ with } X_n \neq \emptyset, X_n \notin D, X_n \cap X_m = \emptyset \text{ for all } n < m < \omega.$$

For each $n < \omega$ let $\mathfrak{M}_n \in \mathrm{Str}(\emptyset)$ be such that M_n has exactly $n+1$ elements. For each $\alpha < \mu$ let $n(\alpha)$ be the only $n < \omega$ such that $\alpha \in X_n$. Let

$$\mathfrak{N} = \Pi_D \langle \mathfrak{M}_{n(\alpha)} \mid \alpha < \mu \rangle.$$

From Łoś theorem it follows that the universe N of $\mathfrak{N}$ is infinite. Letting now

$$\psi = \{\mathfrak{M} \in \mathrm{Str}(\emptyset) \mid M \text{ is infinite}\},$$

we have proved that $\neg\psi$ is not closed under ultraproducts modulo D. From Propositions 3.2(3) and 3.4(1) it follows that $\psi \notin EC^{\emptyset}_{\mathcal{L}}$, whence, a fortiori, $\mathcal{L}$ does not contain the quantifier Q_0. Our claim is thus settled.

Since $0 \notin W \neq \emptyset$, there exists an ordinal $\beta > 0$ such that $\mathcal{L}$ contains the quantifier "there are at least $\aleph_\beta$ many." The argument in the proof of Case 1 yields a contradiction also in the present case. □

Remark 4.2. *Theorem 4.1 shows that for any set Ω of uniform ultrafilters the expressive power of the logic $\mathcal{L}$ generated by Ω goes beyond the crude cardinality properties of universes of structures. Specifically, the proof of the first case gives examples of quantifiers, such as equivalence quantifiers, contained in $\mathcal{L}$, but not contained in any logic generated by cardinality quantifiers. As an application, from the Δ-closure property of $\mathcal{L}$ it follows that in the language $\tau = (\cdot, ^{-1}, e)$ of groups one can express in $\mathcal{L}$ the fact that the center of a group has $\geq\mu^*$ elements, where $\mu^* = (2^\mu)^+$ and $\mu = \sup\{\mu_D \mid D \in \Omega\}$. As another application, $\mathcal{L}$ can express the fact that a boolean algebra has $\geq\mu^*$ atoms.*

The expressibility of properties increasingly outside the scope of cardinality quantifiers is the subject matter of the following result. We refer to [7, definition 4.1.2] and [19, chapters 1.1 and 2.1] for the well ordering quantifier Q^{WO}, the Chang quantifier Q^C, the cofinality ω quantifier $Q^{\mathrm{cf}\omega}$, the Härtig quantifier I, and the Henkin quantifier Q^H.

Theorem 4.3. *Let Ω be a class of uniform ultrafilters over infinite cardinals. Let $\mathcal{L}$ be the logic generated by Ω via Theorem 3.1(i). We then have:*

(i) *If every $D \in \Omega$ is ω_1-complete*[5] *then $\mathcal{L}$ contains all quantifiers Q_0, Q_1, Q^{WO} and $Q^{\mathrm{cf}\omega}$.*
(ii) *If some $D \in \Omega$ is countably incomplete,*[6] *then $\mathcal{L}$ does not contain the following quantifiers: Q^{WO}, I, Q^H, Q^C, and the quantifier "there are at least λ many" for every λ satisfying $\omega \leq \lambda \leq 2^\omega$.*

Proof. (i) Following [4, p. 231], let $\mathcal{L}_{\omega_1}$ be the logic with countable conjunctions and countable iterations of the universal quantifier, as specified in [4, definitions 4.1.9 and 4.1.10, p. 230]. By [4, theorem 4.2.11], for every $D \in \Omega$ and sentence $\psi \in \mathcal{L}_{\omega_1}$ the class $\dot{\psi}$ of models of ψ of type τ is closed under ultraproducts modulo D, and so

[5] In the sense of [4, sec. 4.2, p. 227].
[6] In the sense of [4, definition 4.3.1].

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is its complementary class $\neg\dot{\psi} = \mathrm{Str}(\tau) \setminus \dot{\psi}$. When τ is finite, by Propositions 3.2(4) and 3.4(1) we may identify $\dot{\psi}$ with a sentence of $\mathcal{L}$ of type τ. Now, $Q_0, Q_1, Q^{WO}, Q^{\mathrm{cf}\omega}$ are quantifiers of finite type and are contained in the logic $\mathcal{L}_{\omega_1}$. Furthermore, $\mathcal{L}$ is Δ-closed. Then by [7, theorem 4.1.3] all these quantifiers are also contained in $\mathcal{L}$.

(ii) We first show that for every $\omega \leq \lambda \leq 2^\omega$, $\mathcal{L}$ does not contain the quantifier "there are at least λ many."

The proof of the second case in Theorem 4.1 shows that $\mathcal{L}$ does not contain Q_0. Next let Q be the quantifier "there are at least ξ many," with $\omega_1 \leq \xi \leq 2^\omega$. By [4, propositions 4.3.4 and 4.3.9], $|\Pi_D \omega| \geq 2^\omega$ whenever D is countably incomplete. Since $\omega < \xi$ and $|\Pi_D \omega| \geq \xi$, letting

$$\psi = \{\mathfrak{M} \in \mathrm{Str}(\emptyset) \mid |M| \geq \xi\},$$

it follows that the class $\neg\psi$ is not closed under ultraproducts modulo D, whence, by Propositions 3.2(3) and 3.4(1), $\psi \notin EC^{\emptyset}_{\mathcal{L}}$, and Q is not contained in $\mathcal{L}$.

It is not hard to see that $\mathcal{L}$ does not contain Q^{WO}. For otherwise (absurdum hypothesis), let $\mathcal{L}' = \Delta\mathcal{L}(Q^{WO})$ be the smallest Δ-closed logic containing $\mathcal{L}(Q^{WO})$. Then all $\mathcal{L}'$-elementary classes are in $\mathcal{L}$, because also $\mathcal{L}$ is Δ-closed. Consequently, $\mathcal{L}'$ contains Q_0, because a set is finite iff it can be endowed with a well ordering R such that the reverse ordering $R^{\leftarrow}$ is also a well ordering, and a set is infinite iff it can be equipped with a non-well-ordered linear ordering. It follows that $\mathcal{L}$, too, contains Q_0, a contradiction.

One similarly shows that $\mathcal{L}$ does not contain Q^C, because $\Delta\mathcal{L}(Q^C)$ contains Q_0. Finally, $\mathcal{L}$ does not contain I and Q^H because $\Delta\mathcal{L}(I)$ contains Q^{WO} and $\Delta\mathcal{L}(Q^H)$ contains Q_0. (See [7, sections 2.3 and 2.5] and [19].) $\square$

Corollary 4.4. *Let Ω be a nonempty class of uniform ultrafilters over infinite cardinals. Let $\mathcal{L}$ be the logic generated by Ω via Theorem 3.1(i). Then the following conditions are equivalent*:

(i) *Each $D \in \Omega$ is ω_1-complete.*
(ii) *$\mathcal{L}$ contains Q_0.*
(iii) *$\mathcal{L}$ contains Q_1.*
(iv) *$\mathcal{L}$ contains Q^{WO}.*
(v) *$\mathcal{L}$ is not ω-relatively compact.*

Proof. Theorem 4.3 immediately yields the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). One easily proves the implication (ii) $\Rightarrow$ (v). Finally, to prove (v) $\Rightarrow$ (i), let us suppose that $D \in \Omega$ is not ω_1-complete. In other words, D is *countably incomplete*, in the sense that there is a countable set $E \subseteq D$ such that $\bigcap E \notin D$. (See [4, definition 4.3.1, p. 248].) Equivalently, D is ω-d.i. To see this, recall [13, definition 0.1] and [4, exercise 4.3.10, p. 258]. By Theorem 3.1(i), $\mathcal{L}$ is ω-r.c., whence (v) fails. $\square$

The following proposition shows that Theorem 3.1(ii) no longer holds without the special hypotheses $\neg 0^\sharp$ or $\neg L^\mu$. It also shows that, assuming the existence of a proper class of measurable cardinals, not all proper classes of uniform ultrafilters over regular cardinals generate $\mathcal{L}_{\omega\omega}$. By Theorem 3.1(i), no non-proper class generates $\mathcal{L}_{\omega\omega}$.

Corollary 4.5. *If there is a proper class of measurable cardinals then some proper class of uniform ultrafilters over regular cardinals generates a noncompact logic.*

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Proof. Let $\{\mu_i \mid j \in W\}$ be a proper class of measurable cardinals. Let $\Omega = \{D_j \mid j \in W\}$ be a class where each D_j is a μ_j-complete nonprincipal ultrafilter over μ_j. By [4, theorem 4.2.14(i)], for each $j \in W$ the ultrafilter D_j is uniform and ω_1-complete, and μ_j is regular. By Corollary 4.4(i)⇒(ii), the logic $\mathcal{L}$ generated by Ω is not compact. □

Remark 4.6. *By Corollary 3.7, $\mathcal{L}_{\omega\omega}$ is generated (via Theorem 3.1(i)) by* some *proper class of all uniform ultrafilters over regular cardinals. Let* A *be short hand for the following statement*:

"*Every* proper class of uniform ultrafilters over regular cardinals generates $\mathcal{L}_{\omega\omega}$."

Next let B *stand for "Every uniform ultrafilter is regular." The existence of a measurable cardinal entails* ¬B.[7] *On the other hand, Donder* [5] *proved that* B *is consistent relative to ZFC. Finally, let* C *be an abbreviation of Conjecture* 18 *in* [4, *p.* 599], *which reads*:

"Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ with $\mathfrak{A} \equiv \mathfrak{B}$, and $|A|, |B|, |\tau|, \omega \le \kappa$.
Then $\Pi_D\mathfrak{A} \cong \Pi_D\mathfrak{B}$, for every regular ultrafilter D over κ."

Theorem 4.7. B&C ⇒ A ⇒ *"there is no proper class of measurable cardinals."*

Proof. The second implication follows from Corollary 4.5. For the first implication, assume B and C to hold. For Ω an arbitrary proper class of uniform ultrafilters over regular cardinals, let $\mathcal{L}$ be the logic generated by Ω via Theorem 3.1(i). Since all elementary classes of $\mathcal{L}_{\omega\omega}$ are in $\mathcal{L}$, then for all structures $\mathfrak{A}, \mathfrak{B}$ of type τ,

$$\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B} \;\Rightarrow\; \mathfrak{A} \equiv \mathfrak{B}, \;\text{ i.e., } \equiv_{\mathcal{L}} \text{ is finer than elementary equivalence } \equiv .$$

Conversely, we *claim* that $\equiv$ is finer than $\equiv_{\mathcal{L}}$. As a matter of fact, assume $\mathfrak{A} \equiv \mathfrak{B}$. Since Ω is a proper class there is $D \in \Omega$ and $\kappa \ge \omega$ such that D is a uniform ultrafilter over κ, and $|A|, |B|, |\tau| \le \kappa$. Assumption B ensures that D is regular. Assumption C yields $\Pi_D\mathfrak{A} \cong \Pi_D\mathfrak{B}$. By Propositions 3.2(3) and 3.4(1), every elementary class of $\mathcal{L}$ is closed under isomorphisms and ultrapowers. Therefore,

$$\mathfrak{A} \equiv_{\mathcal{L}} \Pi_D\mathfrak{A} \equiv_{\mathcal{L}} \Pi_D\mathfrak{B} \equiv_{\mathcal{L}} \mathfrak{B},$$

whence $\equiv$ is finer than $\equiv_{\mathcal{L}}$, as required to settle our claim. Having thus proved

$$\equiv \,=\, \equiv_{\mathcal{L}}, \tag{10}$$

there remains to prove that $\mathcal{L}$ is equivalent to $\mathcal{L}_{\omega\omega}$. Trivially, $\mathcal{L}$ cannot contain the quantifier Q_0. For otherwise, some sentence in the pure identity language of $\mathcal{L}$ could distinguish between finite and infinite sets, against (10). By Corollary 4.4(v)⇒(ii), $\mathcal{L}$ is ω-relatively compact. As a consequence, every countable set of sentences of $\mathcal{L}$ has the compactness property. In other words, $\mathcal{L}$ is $[\omega, \omega]$-compact. From (10) it follows that $\mathcal{L}$ has the Löwenheim property [7, 1.2.4(vii)], stating that every sentence ψ of $\mathcal{L}$ having an infinite model has a countable model. By Lindström's characterization theorem ([14] or [9, theorem 2.1.4]), first-order logic is the only $[\omega, \omega]$-compact logic having the Löwenheim property, in the sense that every sentence ψ of $\mathcal{L}$ having an infinite model has a countable model. Thus $\mathcal{L}$ is equivalent to $\mathcal{L}_{\omega\omega}$, as desired. □

Corollary 4.8. *In the core model,* C *implies* A.

[7] See, e.g., [4, exercise 6.5.9, and remarks on page 601 about conjectures 14 and 15].

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Proof. In the core model, every uniform ultrafilter is regular [5, sec. 4]. Now apply Theorem 4.7. □

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