BASIC SEQUENCES IN F-SPACES AND THEIR APPLICATIONS

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1. Introduction

The aim of this paper is to establish a conjecture of Shapiro (**10**) that an F-space (complete metric linear space) with the Hahn-Banach Extension Property is locally convex. This result was proved by Shapiro for F-spaces with Schauder bases; other similar results have been obtained by Ribe (**8**). The method used in this paper is to establish the existence of basic sequences in most F-spaces.

It was originally stated by Banach that every B-space contains a basic sequence, and proofs have been given by Bessaga and Pelczynski (**1**), (**2**), Gelbaum (**4**) and Day (**3**). In (**1**) Bessaga and Pelczynski give a general method of construction in locally convex F-spaces, but we shall show in Section 3 that this construction can be modified to apply in any F-space (X, τ) on which there is a weaker vector topology ρ such that τ has a base of ρ-closed neighbourhoods. The basic result of the paper is Theorem 3.2, and this is a natural generalisation of a locally convex version due to Bessaga and Mazur and given (essentially) in Pelczynski (**6**), (**7**).

In Section 4 we study the problem of existence of a basic sequence in an arbitrary F-space, and show that in fact repeated applications of Theorem 3.2 give a basic sequence in any F-space with a non-minimal topology. Since the only example we know of a minimal F-space is the space ω of all sequences (which has a basis) it seems likely that every F-space contains a basic sequence.

The results of Section 5 do not depend on Section 4; in this section are gathered together the applications of the existence theory of Section 3. We show that if (X, τ) is an F-space and $\rho \leqq \tau$ is a topology defining the same closed linear subspaces as τ, then ρ and τ define the same bounded sets—a result familiar in locally convex theory. The Shapiro conjecture follows immediately. The final theorem is a generalisation of the Eberlein-Smulian theorem employing techniques developed by Pelczynski (**7**).

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2. Preliminary results

An F-semi-norm η on a vector space X is a non-negative real-valued function defined on X such that

(i) $\eta(x+y) \leqq \eta(x)+\eta(y)$.

(ii) $\eta(tx) \leqq \eta(x) \quad |t| \leqq 1$,

(iii) $\lim_{t\to 0} \eta(tx) = 0 \quad x \in X$.

If in addition $\eta(x) = 0$ implies that $x = 0$ then we call η an F-norm. Any vector topology on X may be defined by a collection of F-semi-norms; any metrisable topology may be defined by one F-norm. From this point, unless specifically stated, all vector topologies are assumed to be Hausdorff.

Now suppose (X, ρ) is a topological vector space and τ is a vector topology on X; we shall say that τ is *ρ-polar* if τ has a base of neighbourhoods which are ρ-closed.

Proposition 2.1. *If τ is ρ-polar then τ may be defined by a collection of F-semi-norms $(\eta_\alpha\colon \alpha \in A)$ of the form*

$$\eta_\alpha(x) = \sup \{\lambda(x)\colon \lambda \in \Lambda_\alpha\}$$

where each Λ_α is a collection of ρ-continuous F-semi-norms. If τ is metrisable then τ may be defined by one such F-norm.

Proof. Let $(\gamma_\alpha\colon \alpha \in A)$ be a collection of F-semi-norms defining τ such that every τ-neighbourhood of 0 contains a set $\{x\colon \gamma_\alpha(x) \leqq \varepsilon\}$ for some $\alpha \in A$ and $\varepsilon > 0$; let Δ be the collection of all ρ-continuous F-semi-norms. We define Λ_α to be the collection of F-semi-norms of the form

$$\lambda_\delta^\alpha(x) = \inf(\delta(y)+\gamma_\alpha(z)\colon y+z = x).$$

(Thus $\Lambda_\alpha = \{\lambda_\delta^\alpha\colon \delta \in \Delta\}$.) As $\lambda_\delta^\alpha \leqq \delta$ each λ_δ^α is ρ-continuous and an F-semi-norm ($\lambda_\delta^\alpha \leqq \delta$ implies condition (iii) in particular). Now define

$$\eta_\alpha(x) = \sup(\lambda_\delta^\alpha(x)\colon \delta \in \Delta).$$

Clearly $\eta_\alpha \leqq \gamma_\alpha$ and so is an F-semi-norm. Now if U is a τ-neighbourhood of 0 we may find α_1 and $\varepsilon > 0$ such that if $x_0 \in \overline{\{x\colon \gamma_{\alpha_1}(x) \leqq \varepsilon\}}$ (closure in ρ) then $x_0 \in U$. Suppose now $x_0 \in \{x\colon \eta_{\alpha_1}(x) < \varepsilon\}$; then it is easy to show that $x_0 \in \overline{\{x\colon \gamma_{\alpha_1}(x) \leqq \varepsilon\}}$ and so $(\eta_\alpha\colon \alpha \in A)$ defines τ.

If τ is metrisable, A may be taken to be a singleton and therefore τ may be defined by a single F-norm of the required type.

Proposition 2.2. *Suppose (X, τ) is an F-space (complete metric linear space) and suppose $\rho < \tau$ is a vector topology on X. Then*

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(i) *If the net* $x_a \to 0(\rho)$ *but* $x_a \not\to 0(\tau)$, *then there are vector topologies* α, β *such that*

(*a*) $\rho \leqq \alpha < \beta \leqq \tau$;

(*b*) β *is metrisable and* α-*polar*;

(*c*) $x_a \to 0(\alpha)$ *but* $x_a \not\to 0(\beta)$.

(ii) *If U is a* τ-*neighbourhood of* 0 *but not a* ρ-*neighbourhood then there are vector topologies* α, β *satisfying* (*a*), (*b*) *and* (*c*)′ *U is a* β-*neighbourhood of* 0 *but not an* α-*neighbourhood of* 0.

(iii) *If* τ *is locally bounded then there is a topology* α *such that* $\alpha < \tau$ *but* τ *is* α-*polar*.

Proof. (i) Let α be the largest vector topology such that $\rho \leqq \alpha \leqq \tau$ and $x_a \to 0(\alpha)$ (it is easy to see that there is such a topology). Let β be the vector topology with a base of neighbourhoods consisting of the α-closures of τ-neighbourhoods of 0. Since $\alpha \leqq \tau$ it follows that $\alpha \leqq \beta \leqq \tau$. If $\alpha = \beta$ then the identity map i: $(X, \alpha) \to (X, \tau)$ is almost continuous and so by the Closed Graph Theorem (cf. Kelley (5), p. 213) $\alpha = \tau$ contrary to hypothesis on the net (x_a). Therefore $\alpha < \beta$; clearly also since τ is metrisable so is β, and $x_a \not\to 0(\beta)$.

(ii) (We are grateful to J. H. Shapiro for the following simplification of the original proof.) By an application of Zorn's Lemma it may be shown that there is a maximal vector topology α such that $\rho \leqq \alpha \leqq \tau$ and U is not an α-neighbourhood (we do not assert that α is the largest such topology). Then proceed as in (i).

(iii) Follows from (ii) by considering a single bounded neighbourhood ($\beta = \tau$).

Two vector topologies on X will be called *compatible* if they define the same closed subspaces.

Proposition 2.3. *Let* τ *and* ρ *be compatible topologies on X; they define the same continuous linear functionals.*

Proof. f is τ- or ρ-continuous according as its null space is τ- or ρ-closed.

A sequence (x_n) in a topological vector space X is called a *basis* if every $x \in X$ has a unique expansion in the form

$$x = \sum_{i=1}^{\infty} t_i x_i.$$

In this case we may define linear functionals f_n such that

$$f_n(x) = t_n$$

and linear operators S_n by

$$S_n(x) = \sum_{i=1}^{n} t_i x_i = \sum_{i=1}^{n} f_i(x) x_i.$$

If X is an F-space then it is well known (cf. (**10**), (**12**)) that each f_n is necessarily continuous and the family $\{S_n\}$ is equicontinuous.

Suppose now that X is metrisable but not necessarily complete; we shall call a sequence (x_n) in X a *basic sequence* if it is a basis for its closed linear span *in the completion of X*. We shall call (x_n) a *semi-basic sequence* if we simply have $x_n \notin \overline{\text{lin}}\,\{x_{n+1}, x_{n+2}, \ldots\}$ for every n.

We now give a useful and elementary criterion for a sequence (x_n) to be basic or semi-basic. Let (x_n) be linearly independent and let E be the linear span of (x_n); then for $x \in E$

$$x = \sum_{i=1}^{\infty} t_i x_i$$

uniquely where (t_i) is finitely non-zero. Define

$$f_n(x) = t_n$$

and

$$S_n x = \sum_{i=1}^{n} f_i(x) x_i,$$

where S_n: $E \to E$ is linear.

Lemma 2.4. (i) *(x_n) is semi-basic if and only if each S_n is continuous or equivalently each f_n is continuous.*

(ii) *(x_n) is basic if and only if the family $\{S_n\}$ is equicontinuous.*

Proof. (i) If $\{x_n\}$ is semi-basic, let N_k be the null space of f_k; then N_k is a maximal linear subspace of E. Then $N_1 = \text{lin}\,\{x_i: i \geqq 2\}$ and since $x_1 \notin \overline{N}_1$, N_1 is closed and f_1 is continuous; while if $k \geqq 2$,

$$N_k = \text{lin}\,\{x_i: i \neq k\} = \text{lin}\,\{x_i: i<k\} + \text{lin}\,\{x_i: i>k\}.$$

Hence

$$\overline{N}_k = \text{lin}\,\{x_i: i<k\} + \overline{\text{lin}}\,\{x_i: i>k\},$$

since the former space is finite-dimensional. Suppose $x_k \in \overline{N}_k$; then

$$x_k = \sum_{i=1}^{k-1} t_i x_i + y,$$

where $y \in \overline{\text{lin}}\,\{x_i: i>k\}$. Since $x_k \notin \overline{\text{lin}}\,\{x_i: i>k\}$ we conclude that there is a first index l such that $t_l \neq 0$. Then we obtain $x_l \in \overline{\text{lin}}\,\{x_{l+1}, x_{l+2}, \ldots\}$ and a contradiction. Hence $x_k \notin \overline{N}_k$ and by the maximality of N_k, N_k is closed and f_k is continuous.

The converse is trivial.

(ii) (Cf. Shapiro (**12**), Proposition C.)

It follows from the definition of basic sequence that if (x_n) is basic then the family $\{S_n\}$ is equicontinuous (consider (x_n) as a basis of its closed linear span in the completion of X). Conversely, $S_n(x) \to x$ for $x \in E$ and if the family is equicontinuous $S_n(x)\to x$ for $x\in \bar{E}$ (closure in the completion of X), and it easily follows that (x_n) is a basis for $\bar{E}$.

3. Construction of basic sequences

Lemma 3.1. *Let E be a finite-dimensional space and suppose V is a closed balanced subset of E. If V intersects every one-dimensional subspace of E in a bounded set then V is bounded.*

Proof. We may suppose E is normed; suppose $x_n\in V$ and $\| x_n \|\to\infty$. Then by selecting a subsequence we may suppose $\| x_n \|^{-1}x_n\to z$ where $\| z \| = 1$. Then for any N there is an m such that for $n \geqq m$, $\| x_n \| \geqq N$ and

$$\| x_n \|^{-1}x_n\in \| x_n \|^{-1}V\subset N^{-1}V.$$

Therefore $z\in N^{-1}V$ for all N and hence $V\supset \text{lin}\,\{z\}$.

Theorem 3.2. *Suppose (X, τ) is a metric linear space and ρ is a vector topology on X such that τ is ρ-polar. Suppose (x_a) is a net such that $x_a\to 0(\rho)$ but $x_a\not\to 0(\tau)$; suppose $z_1 \neq 0\in X$. Then there is a sequence $(a(k)\colon k \geqq 2)$ such that*

$$a(k+1)>a(k)$$

for all $k \geqq 2$ and the sequence $(z_n)_{n=1}^{\infty}$ is a basic sequence where $z_n = x_{a(n)}$ $n \geqq 2$.

Proof. We may suppose (Proposition 2.1) that (X, τ) is normed by an F-norm $\|.\|$ such that

$$\| x \| = \sup\,(\lambda(x)\colon \lambda\in\Lambda),$$

where Λ is a collection of ρ-continuous F-norms. Let $\theta>0$ be chosen such that

(i) $\| z_1 \| \geqq 4\theta$.

(ii) For all a, $\exists a' \geqq a$ such that $\| x_{a'} \| \geqq 4\theta$.

Let $V = \{x\colon \| x \| \leqq \theta\}$; then $V\cap\text{lin}\,\{z_1\}$ is compact (since $\| z_1 \| \geqq 4\theta$). We shall construct the sequence $(a(n)\colon n \geqq 2)$ by induction so that if

$$E_n = \text{lin}\,(z_1, x_{a(2)}, \ldots, x_{a(n)})$$

then $E_n\cap V$ is compact.

Suppose $\{a(2), \ldots, a(n)\}$ have been chosen (this set can be empty at the first step, the selection of $a(2)$) and let $E_n = \text{lin}\,(z_1, x_{a(2)}, \ldots, x_{a(n)})$. By the inductive hypothesis $V\cap E_n$ is compact.

For $1 \leqq k \leqq 2^{n+3}$ let

$$W_k^n = \{x\colon \| x \| = k.2^{-(n+3)}\theta\}\cap E_n.$$

Each W_k^n is compact and so we may choose finite subsets U_k^n so that for $w\in W_k^n$ there exists $u\in U_k^n$ with

$$\| w-u \| \leqq 2^{-(n+3)}\theta.$$

Let $U^n = \bigcup_{k=1}^{2^{n+3}} U_k^n$, and for $u\in U^n$ choose $\lambda_u\in\Lambda$ so that

$$\lambda_u(u) \geqq \| u \| -2^{-(n+3)}\theta. \tag{1}$$

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Then choose $b > a(n)$ so that if $c \geqq b$ then

$$\lambda_u(x_c) \leqq 2^{-(n+3)}\theta \tag{2}$$

for $u \in U^n$ (possible since U^n is finite and $x_a \to 0(\rho)$).

Choose a subnet $(x_d\colon d \in D)$ of $(x_c\colon c \geqq b)$ such that $\| x_d \| \geqq 4\theta$, and suppose for every such x_d the set $V \cap \text{lin}(E_n, x_d)$ is unbounded. By Lemma 3.1, for every d there exists $t_d x_d + u_d \neq 0$ where $u_d \in E_n$ such that the linear span of $(t_d x_d + u_d)$ is contained in V. Clearly $u_d \neq 0$ and so we may normalize in such a way that $\| u_d \| = \theta$ (since $V \cap E_n$ is compact). Then

$$\| t_d x_d \| \leqq \| t_d x_d + u_d \| + \| u_d \|$$
$$\leqq 2\theta$$

so that $| t_d | \leqq 1$. Hence since $x_d \to 0(\rho)$, $t_d x_d \to 0$ in (ρ). By selection again of a subnet we may suppose $u_d \to u$ in E_n (since $V \cap E_n$ is compact) and $\| u \| = \theta$. Then for any $t \in \mathbf{R}$

$$\| tu \| \leqq \liminf_{d \to \infty} \| t(t_d x_d + u_d) \|$$
$$\leqq \theta$$

so that $\text{lin}\{u\} \subset V \cap E_n$, a contradiction.

Hence we may choose $a(n+1) \geqq b$ such that $\| x_{a(n+1)} \| \geqq 4\theta$ and $V \cap E_{n+1}$ is compact. This completes the construction of $a(n)$; now let $z_n = x_{a(n)}$ $n \geqq 2$. It remains to establish that by using (1) and (2) (z_n) is a basic sequence.

For convenience we shall replace $\| . \|$ by an equivalent F-norm $\| . \|^*$ given by

$$\| x \|^* = \min(\| x \|, \theta).$$

We next show that if $t_1, \ldots, t_{n+1}$ is a scalar sequence

$$\left\| \sum_{i=1}^{n+1} t_i z_i \right\|^* \geqq \left\| \sum_{i=1}^{n} t_i z_i \right\|^* - 2^{-(n+1)}\theta. \tag{3}$$

Choose the greatest integer k such that

$$\left\| \sum_{i=1}^{n} t_i z_i \right\|^* \geqq k . 2^{-(n+3)}\theta.$$

Then $0 \leqq k \leqq 2^{n+3}$; if $k = 0$ there is nothing to prove. If $k \geqq 1$ then we may choose a scalar s with $| s | \leqq 1$ such that

$$\left\| \sum_{i=1}^{n} s t_i z_i \right\| = k . 2^{-(n+3)}\theta.$$

Then choose $u \in U_k^n$ so that

$$\left\| u - \sum_{i=1}^{n} s t_i z_i \right\| \leqq 2^{-(n+3)}\theta.$$

If $|st_{n+1}|\leqq 1$ then

$$\| u+st_{n+1}z_{n+1} \| \geqq \lambda_u(u)-\lambda_u(z_{n+1}) \\ \geqq (k-2).2^{-(n+3)}\theta$$

by (1) and (2). If $|st_{n+1}|\geqq 1$ then

$$\| u+st_{n+1}z_{n+1} \| \geqq \| z_{n+1} \| - \| u \| \\ \geqq 3\theta \geqq (k-2)2^{-(n+3)}\theta.$$

Hence

$$\left\| s\sum_{i=1}^{n+1} t_i z_i \right\| \geqq (k-2)2^{-(n+3)}\theta - 2^{-(n+3)}\theta$$

$$= (k-3)2^{-(n+3)}\theta$$

$$\geqq \left\| \sum_{i=1}^{n} t_i z_i \right\|^* - 2^{-(n+1)}\theta.$$

Hence since $|s|\leqq 1$

$$\left\| \sum_{i=1}^{n+1} t_i z_i \right\| \geqq \left\| \sum_{i=1}^{n} t_i z_i \right\|^* - 2^{-(n+1)}\theta$$

and (3) follows.

From (3) it is clear that (z_n) is linearly independent for if $\left\| \sum_{i=1}^{n} t_i z_i \right\| \geqq \theta$ then $\left\| \sum_{i=1}^{n+1} t_i z_i \right\| \geqq \frac{1}{2}\theta$; thus if $\sum_{i=1}^{n+1} t_i z_i = 0$, then for every s, $\left\| s\sum_{i=1}^{n} t_i z_i \right\| \leqq \theta$ and so since $V\cap E_n$ is compact, $\sum_{i=1}^{n} t_i z_i = 0$. Let E be the linear span of $\{z_n\}$ and define S_k by

$$S_k\left(\sum_{i=1}^{\infty} t_i z_i\right) = \sum_{i=1}^{k} t_i z_i$$

where (t_i) is finitely non-zero. Then by (3)

$$\| S_{n+k}x \|^* \geqq \| S_n x \|^* - 2^{-n}\theta \quad (k \geqq 0)$$

and therefore for $x\in E$ and $n\geqq 1$

$$\| x \|^* \geqq \| S_n x \|^* - 2^{-n}\theta.$$

Suppose $\| x_m \| \to 0$ but $\| S_k x_m \| \not\to 0$; then since $V\cap E_k$ is compact we may, by selecting a subsequence and multiplying by a bounded sequence of scalars, suppose that $\| S_k x_m \| = \theta$. Thus $\| x_m \| \geqq \frac{1}{2}\theta > 0$, and we have a contradiction. Thus each S_k is continuous.

To establish equicontinuity of $\{S_m \colon m \geqq 1\}$ we must show that if $p(m)$ is any sequence and $x_m \to 0$ then $S_{p(m)}x_m \to 0$. Suppose not; then we may suppose

$$\| S_{p(m)}x_m \|^* \geqq \gamma > 0$$

for all m. Then

$$\| x_m \|^* \geqq \gamma - 2^{-p(m)}\theta$$

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and as $\| x_m \|^* \to 0$ we conclude that $p(m)$ is bounded. But then we may select a constant subsequence and this contradicts the continuity of each S_n. Thus by Lemma 2.4 we have established the theorem.

Corollary 3.3. *Under the assumptions of Theorem* 3.2 *suppose* μ *is a pseudo-metrisable topology on* X *such that* $\mu \leqq \rho$. *Then* (z_n) *may be chosen so that* $z_n \to 0(\mu)$.

An examination of the proof of Theorem 3.2 reveals that we can insist that $\eta(z_n) \to 0$ for any single ρ-continuous F-semi-norm.

Corollary 3.4. *Suppose that* (X, τ) *is an F-space and that* ρ *is a vector topology on* X *with* $\rho < \tau$. *Suppose* $x_a \to 0(\rho)$ *but* $x_a \nrightarrow 0(\tau)$, *and that* $z_1 \in X$. *Then there is a sequence* $a(k)$ *so that* $a(k+1) > a(k)$ $k \geqq 2$ *and such that the sequence* (z_n) *is a semi-basic sequence where* $z_n = x_{a(n)} n \geqq 2$.

Proof. Proposition 2.2 combined with Theorem 3.2 establishes that we may choose (z_n) to be a basic sequence in a weaker topology than τ. This clearly implies that (z_n) is at least a semi-basic sequence in (X, τ).

4. Existence of basic sequences

In this section we consider the question of whether an F-space need possess a basic sequence. The results we obtain will not be used in Section 5, and this section may be omitted. We shall call a topological vector space (E, τ) *minimal* if for every Hausdorff vector topology $\rho \leqq \tau$ we have $\rho = \tau$. It is well known that ω is minimal if we restrict to locally convex topologies.

Proposition 4.1. ω *is a minimal F-space.*

Proof. Suppose ρ is a weaker vector topology on ω and $x_a \to 0(\rho)$ but $\| x_a \| \geqq \theta$ (where $\| . \|$ is an F-norm determining the topology of ω). Then there is a sequence (z_n), with $\| z_n \| \geqq \theta$, which is a basic sequence for some weaker Hausdorff vector topology on ω (Proof of 3.4). Let E be the closed linear span of (z_n) in the original topology, then $E \cong \omega$. However, the dual functionals of (z_n) induce on E a weaker Hausdorff locally convex topology. It follows that $z_n \to 0$ contrary to assumption.

We do not know any other examples of minimal F-spaces; their existence is crucial to the problem of basic sequences in view of the following theorem.

Theorem 4.2. *Every non-minimal F-space contains a basic sequence.*

Before proceeding to the proof of Theorem 4.2 we first prove a stability theorem for basic sequences similar to a locally convex version given by Weill (**13**) (cf. also Shapiro (**11**), p. 1085). A sequence in a topological vector space is *regular* if it is bounded away from zero.

Lemma 4.3. *Suppose X is an F-space and (x_n) is a regular basic sequence. Suppose $\Sigma\| u_n \| < \infty$, and let $y_n = x_n + u_n$. If whenever*

$$\sum_{n=1}^{\infty} t_n y_n = 0$$

then $t_n = 0$, then (y_n) is also a basic sequence.

Proof. Define a map S: $l_\infty \to X$ by

$$S(t) = \sum_{n=1}^{\infty} t_n u_n.$$

Since $\Sigma\| u_n \| < \infty$, S is well defined and S is continuous by the Banach-Steinhaus Theorem. Now suppose $(t^{(n)})$ is a sequence in l_∞ such that

$$\sup \| t^{(n)} \|_\infty < \infty$$

and

$$\lim_{n\to\infty} t_k^{(n)} = 0 \quad \text{for each } k.$$

Then it is easy to verify that $\| S(t^{(n)}) \| \to 0$.

Let E be the closed linear span of $\{x_n\}$ and suppose $f_n \in E'$ is the bi-orthogonal sequence. For $x \in E$, $\lim_{n\to\infty} f_n(x) = 0$, since (x_n) is regular. We define R: $E \to c_0$ by $R(x) = (f_n(x))$; R is continuous by the Closed Graph Theorem. Hence the map T: $E \to X$ defined by $T = I + SR$ is also continuous. Since T takes the form

$$T(x) = \sum_{n=1}^{\infty} f_n(x) y_n.$$

T is injective. Now suppose $(z_n) \subset E$ is a sequence such that $\| T(z_n) \| \to 0$; suppose $\| z_n \| > \varepsilon > 0$. We suppose at first

$$\sup_n \| R(z_n) \|_\infty < \infty.$$

Then by selecting a subsequence we may suppose $R(z_n) \to t$ co-ordinatewise in l_∞ and hence

$$S(R(z_n)) \to S(t) \text{ in } X.$$

Now

$$z_n = T(z_n) - S(R(z_n)) \to -S(t).$$

Therefore $S(t) \in E$ and

$$R(z_n) + RS(t) \to 0 \text{ in } l_\infty.$$

i.e.

$$t + RS(t) = 0$$

$$S(t) + SRS(t) = 0$$

$$T(S(t)) = 0$$

$$S(t) = 0$$

and so

$$\lim_{n\to\infty} z_n = 0$$

contrary to assumption. It follows that no subsequence of ($\| Rz_n \|_\infty$) is bounded.

If, on the contrary, $\| Rz_n \|_\infty \to \infty$, then we may consider ($\| Rz_n \|_\infty^{-1} z_n$) and obtain a similar contradiction. We establish that for such a sequence $\| Rz_n \|_\infty^{-1} z_n \to 0$ and hence $\| Rz_n \|_\infty^{-1} Rz_n \to 0$ in l_∞ which is a contradiction. Hence T is an isomorphism on to its image, and as $Tx_n = y_n$, (y_n) is a basic sequence.

Proof of Theorem 4.2. Let U_n be a base of neighbourhoods of 0 in (X, τ); We may assume, without loss of generality, that U_1 is not a neighbourhood of 0 in some weaker vector topology. By Proposition 2.2 there are vector topologies α, β in X such that $\alpha < \beta \leqq \tau$, β is metrisable and α-polar and U_1 is a β-neighbourhood. Then by Theorem 3.2 there is a basic sequence $(w_k^{(1)})$ in (X, β). Then let E_1 be the τ-closed linear hull of the sequence $(w_k^{(1)})$ and let F_1 be the linear span; let $\gamma_1 = \beta$. Then by induction we construct sequences $(h_k^{(n)})$, E_n, F_n, γ_n such that $F_n = \text{lin}\,\{w_k^{(n)}: k = 1, 2, \ldots\}$, E_n is the τ-closure of F_n and γ_n is a metrisable vector topology on E_n such that $(w_k^{(n)}: k = 1, 2, \ldots)$ is a basis of (E_n, γ_n). Furthermore

(i) $(w_k^{(n)})$ is block basic with respect to $(w_k^{(n-1)})$ for $n \geqq 2$, i.e. $w_k^{(n)}$ takes the form

$$w_k^{(n)} = \sum_{p_{k-1}+1}^{p_k} c_i w_i^{(n-1)},$$

where $p_0 = 0 < p_1 < p_2 \ldots$. Thus $F_n \subset F_{n-1}$ for $n \geqq 2$ and $E_n \subset E_{n-1}$ $n \geqq 2$.

(ii) The topology γ_n on E_n is finer than γ_{n-1} restricted to E_n for $n \geqq 2$, and coarser than τ.

(iii) $U_n \cap E_n$ is a γ_n-neighbourhood of 0.

We now describe the inductive construction; suppose $(w_k^{(n)})$, E_n, F_n and γ_n have been chosen. If $U_{n+1} \cap E_n$ is a γ_n-neighbourhood of 0 then let $\gamma_{n+1} = \gamma_n$ and $w_k^{(n+1)} = w_k^{(n)}$ for all k. Otherwise by Proposition 2.2 we may find topologies α and γ_{n+1} on E_n such that $\gamma_n \leqq \alpha < \gamma_{n+1} \leqq \tau$, γ_{n+1} is α-polar and metrisable and $U_{n+1} \cap E_n$ is a γ_{n+1}-neighbourhood of 0 but not an α-neighbourhood.

Since F_n is τ-dense in E_n, F_n is also γ_{n+1}-dense and hence $\alpha < \gamma_{n+1}$ on F_n. Thus by Corollary 3.3 we may determine a γ_{n+1}-regular basic sequence (z_k) in F_n such that $z_k \to 0(\gamma_n)$. Thus

$$z_k = \sum_{i=1}^{q(k)} c_{k,i} w_i^{(n)},$$

where $\lim_{k\to\infty} c_{k,i} = 0$ for each i (since the co-ordinate functionals for $(w_i^{(n)})$ are γ_n-continuous). It follows easily that we may find a subsequence (y_k) and a block basic sequence $(w_k^{(n+1)})$ such that $\sum_k \| y_k - w_k^{(n+1)} \|_{n+1} < \infty$ where $\| . \|_{n+1}$ is an F-norm determining γ_{n+1}. If

$$\sum_{k=1}^{\infty} t_k w_k^{(n+1)} = 0 \quad (\gamma_{n+1})$$

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then

$$\sum_{k=1}^{\infty} t_k w_k^{(n+1)} = 0 \quad (\gamma_n)$$

and thus since the co-ordinate functionals for $w_i^{(n)}$ are γ_n-continuous $t_k = 0$ for all k. Thus $(w_k^{(n+1)})$ is a γ_{n+1}-basic sequence, and we proceed by letting $F_{n+1} = \text{lin}\,\{w_k^{(n)}\}$, $E_{n+1} = \bar{F}_{n+1}$ (in τ). This completes the inductive construction.

Finally take the " diagonal sequence "

$$v_n = w_n^{(n)}.$$

Then for each n, $(v_k\colon k \geqq n)$ is block basic with respect to $(w_k^{(n)})$. In particular (v_k) is block basic with respect to $(w_k^{(1)})$ and hence there are γ_1-continuous linear functionals (f_k) defined on $\text{lin}\,\{v_k\}$ such that $f_i(v_j) = \delta_{ij}$. These are then also τ-continuous and extend to the closed linear span H of $\{v_k\}$. Now suppose $x \in H$; we show

$$\sum_{i=1}^{\infty} f_i(x)v_i = x.$$

For any n, $(v_k\colon k \geqq n)$ is a basic sequence in (E_n, γ_n); let

$$R_n(x) = x - \sum_{i=1}^{n-1} f_i(x)v_i.$$

Then $R_n(x)$ is in the τ-closure of $\text{lin}\,\{v_k\colon k \geqq n\}$, as this space is easily seen to be $\bigcap_{i=1}^{n-1} f_i^{-1}(0)$. Thus $R_n(x)$ is in E_n and in the γ_n-closure of $\text{lin}\,\{v_k\colon k \geqq n\}$. Therefore

$$R_n(x) = \sum_{i=n}^{\infty} f_i(x)v_i \quad (\gamma_n)$$

and so for some N and all $m \geqq N$,

$$R_n(x) - \sum_{i=n}^{m} f_i(x)v_i \in U_n,$$

and

$$x - \sum_{i=1}^{m} f_i(x)v_i \in U_n.$$

Thus $x = \sum_{i=1}^{\infty} f_i(x)v_i$ for $x \in H$, and (v_i) is a basic sequence.

If E is a minimal F-space, then E may still possess a basic sequence (see Proposition 4.1). The author does not know if every F-space must possess a basic sequence.

Theorem 4.4. *Let (X, τ) be an F-space; the following are equivalent:*

(i) *X contains no basic sequence.*

(ii) *Every closed subspace of X with a separating dual is finite-dimensional.*

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Proof. Clearly (ii)⇒(i) so we have to show (i)⇒(ii). If E is a subspace of X with a separating dual, then the weak topology σ on E is weaker than τ. If E is infinite-dimensional, then by Theorem 4.2 $\sigma = \tau$. But in this case $E \cong \omega$, and so has a basis. Therefore, E is finite-dimensional.

5. Applications

We now can apply basic sequences or rather semi-basic sequences to derive many results familiar in locally convex theory.

Theorem 5.1.

(i) *Let (X, τ) be an F-space and suppose $\rho \leqq \tau$ is a vector topology on X compatible with τ. Then every ρ-bounded set is τ-bounded.*

(ii) *Suppose X is a vector space and $\rho \leqq \tau$ are two vector topologies on X such that ρ and τ are compatible and τ is ρ-polar. Then any ρ-bounded set is τ-bounded.*

Proof. (i) It is enough to show that if $x_n \to 0(\rho)$ and c_n is a sequence of scalars such that $c_n \to 0$ then $c_n x_n \to 0$ (τ). Suppose $x_n \to 0$ (ρ); then choose $x_0 \neq 0$. For $c_n \to 0$, $c_n \neq 0$,

$$c_n(x_n + x_0) \to 0\ (\rho).$$

Suppose $c_n(x_n + x_0) \not\to 0\ (\tau)$; then by Corollary 3.4, there is a semi-basic sequence (z_n) with $z_1 = x_0$ and

$$z_n = c_{m_n}(x_{m_n} + x_0) \quad (n \geqq 2),$$

where (m_n) is an increasing sequence of integers. Then

$$c_{m_n}^{-1} z_n \to x_0\ (\rho)$$

and hence x_0 is in the ρ-closure of $\text{lin}\{z_n \colon n \geqq 2\}$. Thus x_0 is also in the τ-closure of $\text{lin}\{z_n \colon n \geqq 2\}$, contradicting the fact that (z_n) is a semi-basic sequence. Thus since $c_n x_0 \to 0$, $c_n x_n \to 0$ (τ).

The proof of (ii) is somewhat similar; let η be a ρ-lower-semi-continuous τ-continuous F-semi-norm and let $N = \{x \colon \eta(x) = 0\}$. Then X/N is metrisable under η and may be given the quotient topology $\hat{\rho}$ of ρ (N is ρ-closed). Every η-closed subspace of X/N is $\hat{\rho}$-closed and so an argument similar to (i) may be employed.

Corollary 5.2. *Suppose (X, τ) is an F-space and $\rho \leqq \tau$ is a metrisable vector topology compatible with τ. Then $\rho = \tau$.*

Corollary 5.3. *Let (X, τ) be an F-space with the Hahn-Banach Extension Property. Then X is locally convex.*

Proof. Let σ be the weak topology on N; then $\sigma \leqq \tau$ and σ and τ are compatible by the HBEP. For suppose Y is a τ-closed subspace and $x \notin Y$; then by HBEP there is a continuous linear functional ϕ such that $\phi(Y) = 0$ and $\phi(x) = 1$. Let μ be the associated Mackey topology; then (see Shapiro **(10)**, Proposition 3) $\sigma \leqq \mu \leqq \tau$ and μ is metrisable. Hence by Corollary 5.2 $\mu = \tau$ and τ is locally convex.

Corollary 5.4. *Suppose (X, τ) is an F-space and $\rho \leqq \tau$ is a vector topology compatible with τ. Then τ is ρ-polar.*

Proof. Let γ be the topology induced by the ρ-closures of τ-neighbourhoods of 0; then $\rho \leqq \gamma \leqq \tau$ and γ is metrisable. Hence by 5.2, $\gamma = \tau$.

Theorem 5.5. *Let (X, τ) be an F-space and let (x_n) be a basis of X in a compatible topology $\rho \leqq \tau$. Then (x_n) is a basis of X.*

Proof. By the previous corollary we may assume that τ is defined by a ρ-lower-semi-continuous F-norm $\|.\|$ (see Proposition 2.1). Each $x \in X$ may be expanded in the form

$$x = \sum_{i=1}^{\infty} f_i(x)x_i \, (\rho)$$

(the linear functionals f_n are not necessarily ρ-continuous). Now for each $x \in X$, the sequence $\left(\sum_{i=1}^{n} f_i(x)x_i\right)$ is ρ- and therefore τ-bounded (Theorem 5.1) and so we may define

$$\| x \|^* = \sup_n \left\| \sum_{i=1}^{n} f_i(x)x_i \right\|.$$

Then $\lim_{t\to 0} \| tx \|^* = 0$ since $\lim_{t\to 0} ty = 0$ uniformly for y in a bounded set; hence $\|.\|^*$ is an F-norm on X. Clearly also $\| x \|^* \geqq \| x \|$ by the ρ-lower-semi-continuity of $\|.\|$.

It remains to establish that $(X, \|.\|^*)$ is complete and then by the Closed Graph Theorem it will follow that $\|.\|^*$ and $\|.\|$ are equivalent. Let (y_n) be a $\|.\|^*$-Cauchy sequence; then since $\| y_n - y_m \| \leqq \| y_n - y_m \|^*$ for all m, n, (y_n) is τ-convergent to y say. Furthermore, it can be seen that the sequences

$$\left(\sum_{i=1}^{m} f_i(y_n)x_i\right)$$

are τ-convergent uniformly in m; clearly $\lim_{n\to\infty} f_i(y_n) = t_i$ exists and

$$\lim_{n\to\infty} \sum_{i=1}^{m} f_i(y_n)x_i = \sum_{i=1}^{m} t_i x_i$$

uniformly in m for the topology τ. Thus working in the weaker topology ρ

$$\lim_{m\to\infty} \sum_{i=1}^{m} t_i x_i = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{m} f_i(y_n)x_i = y.$$

(The limits are interchangeable by uniform convergence.) Therefore it follows that

$$\lim_{n\to\infty} \sum_{i=1}^{m} f_i(y_n)x_i = \sum_{i=1}^{m} f_i(y)x_i \ (\tau)$$

uniformly in m and that $\| y-y_n \|^* \to 0$. Hence $\| . \|$ and $\| . \|^*$ are equivalent, and by an application of Lemma 2.4, (x_n) is a basic sequence in $(X, \| . \|)$. By the compatibility of ρ, (x_n) is a basis of X.

Shapiro (**12**) proves that the Weak Basis Theorem fails in any non-locally convex locally bounded F-space. With regard to this theorem we establish that a weaker version of the Weak Basis Theorem holds always.

Proposition 5.6. *Let (x_n) be a weak basis of (X, τ), where (X, τ) is an F-space with a separating dual. Then the associated linear functionals $\{f_n\}$ are continuous.*

Proof. Let σ be the weak topology and μ the (metrisable) Mackey topology. Then (X, μ) is barrelled, for if C is a μ-barrel then C is τ-closed and by the Baire Category Theorem we may show C has τ-interior. It follows easily that C is a τ-neighbourhood of 0 and thus a μ-neighbourhood ((**10**), Proposition 3).

Now let $\| . \|_n$ be a sequence of semi-norms defining μ and let

$$\| x \|_n^* = \sup_m \left\| \sum_{i=1}^{m} f_i(x)x_i \right\|_n$$

(finite, since μ and σ have the same bounded sets). Let μ^* be the topology induced by the sequence $\| . \|_n^*$ and let $\hat{X}$ be the μ^*-completion of X. Consider the identity map i: $(X, \mu) \to (\hat{X}, \mu^*)$. Suppose $z_n \in X$, $z_n \to z$ (μ) and $z_n \to z'$ (μ^*). Then $\left\{ \sum_{i=1}^{m} f_i(z_n)x_i \right\}_{n=1}^{\infty}$ is uniformly μ-Cauchy for $m = 1, 2, \ldots$; thus in the topology $\sigma \leqq \mu$

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{m} f_i(z_n)x_i = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} f_i(z_n)x_i$$

and we conclude

$$\lim_{n\to\infty} f_i(z_n) = t_i \text{ exists for each } i$$

and

$$\lim_{n\to\infty} z_n = z = \sum_{i=1}^{\infty} t_i x_i \text{ in } \sigma.$$

Thus $f_i(z) = t_i$ and therefore

$$\lim_{n\to\infty} \sum_{i=1}^{m} f_i(z_n - z)x_i = 0 \ \mu\text{-uniformly in } m.$$

Hence $z_n \to z$ in (X, μ^*) and i has Closed Graph. By the Closed Graph Theorem ((**9**), p. 116), since $(\hat{X}, \mu^*)$ is complete and metric, $\mu \geqq \mu^*$ and it follows easily that each f_n is μ and hence τ-continuous.

The idea of the next theorem is due to Pelczynski (**7**).

Theorem 5.7. *Let (X, τ) be an F-space and suppose $\rho \leqq \tau$ is a compatible vector topology. Let K be a subset of X; then the following are equivalent*

(i) *K is ρ-compact,*

(ii) *K is ρ-sequentially compact,*

(iii) *K is ρ-countably compact.*

Proof. (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii) are well known. Let $\|.\|$ be an F-norm determining τ; by Corollary 5.4 we may suppose $\|.\|$ is ρ-lower-semi-continuous.

(iii)$\Rightarrow$(i). It is easy to see that K is ρ-precompact; we show that K is also ρ-complete. Let $(\hat{X}, \hat{\rho})$ be the ρ-completion of X and let $Y \subset \hat{X}$ be the vector space of all $y \in \hat{X}$ such that there is a ρ-bounded net $x_\alpha \in X$ such that $x_\alpha \to y$. By Theorem 5.1 a ρ-bounded net is τ-bounded. Let $B_\lambda = \{x \in X \colon \| x \| \geqq \lambda\}$; then for $y \in Y$ we define

$$\| y \|^* = \inf \{\lambda\colon\ y \in \bar{B}_\lambda, \text{ closure in } \hat{\rho}\}.$$

Let $y \in Y$ and suppose x_α is a τ-bounded net converging to y in $\hat{\rho}$; then

$$\| y \|^* \leqq \sup_\alpha \| x_\alpha \| < \infty$$

and

$$\lim_{t \to 0} \| ty \|^* \leqq \lim_{t \to 0} \sup_\alpha \| tx_\alpha \| = 0$$

since the net $\{x_\alpha\}$ is bounded (cf. Theorem 5.5). It follows without difficulty that $\|.\|^*$ is an F-semi-norm on Y, and that $\|.\|^*$ is $\hat{\rho}$-lower-semi-continuous; also from the definition, $\| x \| = \| x \|^*$ for $x \in X$, since each B_λ is ρ-closed. Next if $y \in Y$ and $\| y \|^* = 0$ then for each $\lambda > 0$ and V a neighbourhood of 0 in $(\hat{X}, \hat{\rho})$ we may find $x_{\lambda,V} \in X$ such that $x_{\lambda,V} - y \in V$ and $\| x_{\lambda,V} \| \leqq \lambda$. The set $\{(\lambda, V)\colon \lambda > 0,\ V \text{ a } \hat{\rho}\text{-neighbourhood of } 0\}$ is directed in the obvious way $[(\lambda, V) \geqq (\lambda', V')$ if and only if $\lambda \leqq \lambda'$ and $V \subset V']$; then the net $x_{\lambda,V}$ converges to 0 in (X, τ) and $x_{\lambda,V} \to 0$ in (X, ρ). However $x_{\lambda,V} \to y$ in $(\hat{X}, \hat{\rho})$ and so $y = 0$ Thus Y is a metrisable vector space under $\|.\|^*$ and $\|.\|^*$ is $\hat{\rho}$-lower-semi-continuous.

Now suppose $x_\alpha \in K$ is a ρ-Cauchy net; then $x_\alpha \to y$ in $(\hat{X}, \hat{\rho})$ and $y \in Y$. Suppose at first $\| x_\alpha - y \|^* \to 0$; then by the completeness of (X, τ) $y \in X$, and there is a sequence $(\alpha(n))$ such that $x_{\alpha(n)} \to y(\tau)$. Thus y is the sole ρ-cluster point of $\{x_{\alpha(n)}\}$ in X; since K is countably compact, $y \in K$, and $x_\alpha \to y$ in (K, ρ).

Now suppose $\| x_\alpha - y \|^* \not\to 0$ and that $y \notin X$; since $y \neq 0$ we may suppose $x_\alpha \notin V$ for all α, where V is a ρ-neighbourhood of 0. Then by Theorem 3.2 there is a basic sequence (z_n) in $(Y, \|.\|^*)$ such that:

(i) $z_1 = y$.

(ii) $z_n = w_n - y$, $n \geqq 2$ where $w_n = x_{\alpha(n)}$ for some increasing sequence.

(iii) $\inf \| z_n \|^* > 0$.

Let Z be the closed linear span of $\{z_n\}_{n=1}^{\infty}$ and let W be the closed linear span of $\{w_n\}_{n=2}^{\infty}$. Since $z_1 \notin X$ and $W \subset X$, W is a closed subspace of co-dimension one in Z. Let ϕ be the continuous linear functional on $(Z, \|.\|^*)$ such that $\phi(z_1) = 1$ and $\phi(W) = 0$; we define $A: Z \to Z$ by $Az = z - \phi(z)z_1$. Then for $n \geqq 2$

$$\begin{aligned} Az_n &= Aw_n - Az_1 \\ &= w_n. \end{aligned}$$

Similarly define $B: Z \to Z$ by

$$B\left(\sum_{i=1}^{\infty} t_i z_i\right) = \sum_{i=2}^{\infty} t_i z_i.$$

Then

$$\begin{aligned} Bw_n &= B(z_1 + z_n) \\ &= z_n. \end{aligned}$$

It follows that $BAz_n = z_n$, $n \geqq 2$ and hence that A is an isomorphism of $\overline{\text{lin}}\,\{z_n: n \geqq 2\}$ on to its image. In particular $(w_n: n \geqq 2)$ is a basic sequence in $(X, \|.\|)$. However $w_n \in K$ for $n \geqq 2$, and so (w_n) possesses a ρ-cluster point. Now suppose w_0 is a ρ-cluster point; then w_0 is in the τ-closed linear span of (w_n) by compatibility. It follows that

$$w_0 = \sum_{i=2}^{\infty} \psi_i(w_0) w_i,$$

where ψ_i is the dual sequence of τ-continuous linear functionals on W. Each ψ_i is also ρ-continuous by compatibility and hence

$$\psi_i(w_0) = 0 \quad i \geqq 2.$$

Therefore $w_0 = 0$. This contradicts the original choice of $x_\alpha \notin V$, where V is a ρ-neighbourhood of 0. Thus we have a contradiction.

Finally suppose $\| x_\alpha - y \|^* \nrightarrow 0$ and $y \in X$; determine the basic sequence $(z_n: n \geqq 2)$ satisfying (ii)-(iii). In this case if w_0 is a ρ-cluster point of $(w_n: n \geqq 2)$ then $w_0 - y$ is a ρ-cluster point of $(z_n: n \geqq 2)$. Since $w_0 - y \in X$ and $z_n \in X$ we conclude that $w_0 - y$ is in the τ-closed linear span of $\{z_n: n \geqq 2\}$ by compatibility and it follows as usual that $w_0 - y = 0$. Hence $y \in K$. We conclude that any ρ-Cauchy net converges in K and so K is complete and therefore compact.

(iii)$\Rightarrow$(ii). Let (x_n) be a sequence in K and let x_0 be a ρ-cluster point. Then there is a net (z_α) in K such that each z_α is some x_n and $z_\alpha \to x_0$ (ρ). If $z_\alpha \to x_0$ in τ then there is nothing to prove, as it will follow that some subsequence of (x_n) converges to x_0. Otherwise we may find a basic sequence (u_n) of the form $u_n = z_{\alpha(n)} - x_0$. Let w be a ρ-cluster point of $(z_{\alpha(n)})$ in K; then clearly $w - x_0 \in \overline{\text{lin}}\,\{u_n\}$ and since τ and ρ are compatible it follows as in (iii)$\Rightarrow$(i) that $w - x_0 = 0$. Hence x_0 is the sole cluster point of $(z_{\alpha(n)})$ and so $z_{\alpha(n)} \to x_0$. However $z_{\alpha(n)}$ is simply a subsequence of (x_n) ($\alpha(n) \to \infty$ since the $z_{\alpha(n)}$ are distinct).

[ADDED IN PROOF: The problem of determining conditions under which the Hahn-Banach Extension Property is equivalent to local convexity was originally posed by Duren, Romberg and Shields **(14)** p.59.]

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