ON $(\underline{L}^1)^*$ FOR GENERAL MEASURE SPACES

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1. Introduction. It is well known that certain results such as the Radon-Nikodym Theorem, which are valid in totally σ-finite measure spaces, do not extend to measure spaces in which μ is not totally σ-finite. (See § 2 for notation.) Given an arbitrary measure space (X, S, μ) and a signed measure ν on (X, S), then if $\nu << \mu$ for X, $\nu << \mu$ when restricted to any $e \in \mathbf{S}_f$ and the classical finite Radon-Nikodym theorem produces a measurable function $g_e(x)$, vanishing outside e, with

$$\nu(e') = \int_{e'} g_e(x) d\mu ,$$

for every measurable $e' \subset e$. When μ is totally σ-finite there exist disjoint measurable sets e_i with $X = \bigcup_1^\infty e_i$ and, defining $g(x) = \Sigma_{i=1}^\infty g_{e_i}(x)$ extends the Radon-Nikodym theorem to X. Standard arguments then show that every continuous linear functional on L^p, $1 \leq p < \infty$, can be expressed in terms of an integral

$$G(f) = \int fg \, d\mu ,$$

with $g(x) \in L^q$, $p^{-1} + q^{-1} = 1$, and $\|G\| = \underline{N}^\infty(g)$.

When μ is not totally σ-finite the extension may fail in several ways. There may exist a function $g(x)$ defined on X

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and such that $g(x) = g_e(x)$ a.e. in e for each $e \in S_f$ but $g(x)$ may fail to be measurable or $g(x)$ may be measurable but the integral expression be invalid in certain sets of infinite measure (Example 3). In these cases we say that the finite Radon-Nikodym theorem has a local extension to X. There may exist no function defined on X with $g(x) = g_e(x)$ a.e. in e for each $e \in S_f$ (Example 4).

Let $G \in (L^p)^*$. If $e \in S_f$, $\chi_e \in L^p$. If $|G(e)|/\mu(e) > 1/n$,

$$\frac{1}{n}\mu(e) < |G(\chi_e)| \leq \|G\| \underline{N}^p(\chi_e) = \|G\| [\mu(e)]^{1/p}$$

$$[\mu(e)]^{1/q} \leq n \|G\| .$$

If $1 < p < \infty$ this result can be used to show that there exist disjoint sets s_i in S_f^+, $X_1 = \bigcup_1^\infty s_i$ with $G(\chi_e) = 0$ if $e \subset X = X_1$, $e \in S_f$. Then G can be identified with $g(x) = \Sigma_1^\infty g_{s_i}(x)$ and it can be shown that $(\underline{L}^p)^* = \underline{L}^q$ if $1 < p < \infty$. This method fails when $p = 1$, $q = \infty$ and it is well known that $(\underline{L}^1)^*$ and $\underline{L}^\infty$ need not be isometric.

In § 3 we consider three elementary examples illustrating some of the differences between totally σ-finite and non totally σ-finite measure spaces. In § § 2 and 3 we introduce a local theory which permits a description of $(\underline{L}^1)^*$ in terms of integrals when the measure space permits the local extension of the Radon-Nikodym theorem. An analogous description is possible in every case for the Bourbaki theory of Radon measures on locally compact topological spaces [1].

Example 4 ([3], p. 131) shows that a local extension need not be possible and that the local theory is not adequate to describe $(\underline{L}^1)^*$ for general non-topological measure spaces. J. T. Schwartz [5] has characterized $(\underline{L}^1)^*$ in general in terms of measures. In § 4 we study decompositions of X into sets of S_f^+ and the problem of characterizing $(L^1)^*$ in terms of integrals in the general case. The principal results are presented in Lemma 4.1 and Theorem 4.1.

2. Definitions and notation. We adopt in general the definitions and notation in [3]. Let (X, S, μ) denote an arbitrary measure space, where X is a space of points x, S a relatively complemented, countably additive class of measurable sets (a σ-ring) and μ a positive countably additive measure on S. When $X \in S$ the space is complemented and S is called a σ-algebra. The measure is called σ-finite (totally σ-finite) if S is a σ-ring (σ-algebra) and every set in S can be expressed as a union of a countable collection of measurable sets of finite measure.

If S is a σ-algebra we let S_f and S_f^+ denote the collections of measurable sets of finite, and finite positive measure respectively, S' the σ-ring generated by S_f.

Let R^X and $\bar{R}^X$ denote the spaces of real and extended real valued functions on X. A function f is called null if $N(f) = \{x: f(x) \neq 0\}$ is a null set (i.e. has measure zero). The relation $f = g$ if $f - g$ is null is an equivalence relation. It is usual to extend the equivalence classes to include functions that are not defined in some null set. A set A (function f) will be called locally null if for each $e \in S_f$, $A \cap e$ $(f\chi_e)$ is null. The relation $f = g$ if $f - g$ is locally null is an equivalence relation. We can thus also consider spaces where the points are equivalence classes of functions modulo locally null functions. We let $\underline{\bar{R}}^X$, $\underline{\bar{R}}_\ell^X$ denote the spaces of equivalence classes of extended real valued functions on X modulo null and locally null functions respectively. When μ is σ-finite or totally σ-finite, null and locally null coincide.

If S is a σ-algebra, a function f in $\bar{R}^X$ is measurable if the inverse of each extended Borel set on the real line is measurable. If S is a σ-ring, f is measurable if the intersection of the inverse of every extended Borel set with $N(f)$ is

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measurable. A function f in $\bar{R}^X$ will be called locally measurable if, for each $e \in S_f$, $f\chi_e$ is a measurable function. Locally measurable functions have the same combinatory properties as measurable functions. We denote by M, M_f the spaces of measurable and locally measurable functions, by $\underline{M}$ and $\underline{M}_\ell$ the subspaces of $\bar{R}^X$, $\bar{R}^X_\ell$ of equivalence classes of measurable and locally measurable functions modulo null and locally null functions respectively. We note that if f is locally measurable then $f\chi_A$ is measurable for every $A \in S'$. Thus if μ is σ-finite or totally σ-finite local measurability implies measurability.

For each $f \in M$, $\underline{N}^p(f) = [\int |f|^p d\mu]^{1/p}$, $1 \leq p < \infty$; $N^\infty(f) = \text{ess. sup.} |f(x)|$ is defined with $0 \leq \underline{N}^p(f) \leq \infty$. Since every null function h is measurable and $\underline{N}^p(h) = 0$, $\underline{N}^p$ is also defined naturally on $\underline{M}$. We denote by L^p the space of functions f in M with $\underline{N}^p(f) < \infty$, by $\underline{L}^p$ the Banach space of points f of $\underline{M}$ with $\underline{N}^p(\hat{f}) < \infty$. Thus, as sets of points,

$$\underline{L}^p \subset \underline{M} \subset \underline{\bar{R}}^X .$$

When μ is not σ-finite or totally σ-finite we define

$$\underline{N}^p_\ell(f) = \sup_{e \in S_f} \underline{N}^p(f), \quad 1 \leq p \leq \infty .$$

We can write

$$\bar{\int} |f| d\mu \quad \text{for} \quad \underline{N}^1_\ell(f) .$$

Then $\bar{\int}$ is defined for every locally measurable function whereas $\int$ is defined only for those locally measurable functions that are measurable. We denote by L^p_ℓ, $\underline{L}^p_\ell$ the analogues of the spaces L^p and $\underline{L}^p$ using local measurability instead of measurability. Then, as sets of points,

$$\underline{L}^p \subset \underline{M}_\ell \subset \underline{\bar{R}}_\ell \ .$$

The spaces $\underline{L}^p$ are vector spaces normed by $\underline{N}_\ell^p$.

Definition. A set A will be called purely infinite if it contains $A' \in S$ with $\mu(A') = \infty$ and if $\mu(A') = 0$ or ∞ for every $A' \in S$, $A' \subset A$.

3. Elementary examples. The following trivial example illustrates some of the pathology introduced by purely infinite sets.

Example 1. X consists of two points $x_1, x_2, S = \underline{P}(X)$ (the collection of all subsets of X), μ is the measure for which $\mu(x_1) = 1$, $\mu(x_2) = \infty$. (We do not distinguish between x_1 and x_2 as points of X and as one point subsets of X.)

In the example the set x_2 is purely infinite, S a σ-agebra and μ is not totally σ-finite.

Definition. Suppose that ν is a signed measure absolutely continuous with respect to a measure μ, $\nu << \mu$. Then a measurable function $g(x)$ will be called a Radon-Nikodym derivative (RN-derivative) of ν with respect to μ if

$$\nu(A) = \int_A g(x) d\mu$$

for every $A \in S$. A locally measurable function $g(x)$ will be called a local RN-derivative of ν with respect to μ if

$$\nu(e) = \int_e g(x) d\mu$$

for every $e \in S_f$.

Let $\nu << \mu$ in Example 1. Then $g(x_1) = \nu(x_1)$, $g(x_2) = 0$ is a unique RN-derivative if $\nu(x_2) = 0$, there exists no RN-derivative if $0 < |\nu(x_2)| < \infty$ and $g(x_1) = \nu(x_1)$, $g(x_2) = \pm a$, $0 < a \leq \infty$, is an RN derivative if $\nu(x_2) = \pm \infty$.

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The functions $g(x_1) = \nu(x_1)$, $-\infty \leq g(x_2) \leq \infty$ are local RN-derivatives for every $\nu << \mu$.

Let $|f(x_1)| < \infty$, $f(x_2) = 0$. Then $f \in L^p$, $1 \leq p \leq \infty$ and every $f \in L^p$ is of this form if $1 \leq p < \infty$. The equivalence class of f in $\underline{L}^p$, $1 \leq p \leq \infty$ coincides with f since $g \equiv 0$ is the only null function. Now $f \in L^p_\ell$, $1 \leq p \leq \infty$ and its equivalence class $\hat{f}_\ell \in \underline{L}^p$ consists of the functions $f(x) + g(x)$ where $g(x)$ is locally null, i.e. $g(x_1) = 0$, $-\infty \leq g(x_2) \leq \infty$. $\underline{N}^p(\hat{f}) = \underline{N}^p_\ell(\hat{f}_\ell)$ and the correspondence between $\hat{f}$ and $\hat{f}_\ell$, $1 \leq p < \infty$, shows that $\underline{L}^p$ and $\underline{L}^p_\ell$ are isometric. For the case $p = \infty$, $f(x) + g(x)$ determines different points in L^∞ for different finite values of $g(x_2)$ and $\hat{f}_\ell$ corresponds to the one point with $g(x_2) = 0$. Thus $\underline{L}^\infty_\ell$ is isometric to a subspace of $\underline{L}^\infty$.

For $1 \leq p < \infty$ the situation illustrated by the example is typical. If $\hat{f} \in \underline{L}^p$, $f \in \hat{f}$ is in L^p_ℓ and determines an equivalence class $\hat{f}_\ell$ in $\underline{L}^p$ with $\underline{N}^p(\hat{f}_\ell) = \underline{N}^p(\hat{f})$. If $\hat{f}_\ell \in \underline{L}^p_\ell$ and $f \in \hat{f}$, then since $N^p_\ell(f) = \sup_{e \in S_f} N^p(f\chi_e)$, there is an increasing sequence $e_n \in S_f$ with $N^p(f\chi_{e_n}) \geq N^p(f) - 1/n$, $n = 1, 2, \ldots$. If $A = \cup_1^\infty e_n$, $f\chi_A \in L^p$, and $N^p(\hat{f\chi_A}) = N^p_\ell(\hat{f}_\ell)$. <u>We have thus shown that for every measure space</u> (X, S, μ), $\underline{L}^p$ <u>and</u> $\underline{L}^p_\ell$, $1 < p < \infty$, <u>are isometric.</u> Thus $\underline{L}^p_\ell$, $1 < p < \infty$ is a Banach space.

Returning to the example we note that $S_f = S'$ consists of the empty set and the set x_1. Here $L^p(X, S, \mu)$ and $L^p(X, S', \mu)$ coincide if $1 \leq p < \infty$ but $\underline{L}^\infty(X, S', \mu)$ and $\underline{L}^\infty_\ell(X, S, \mu)$ are isometric, noting that $f(x)$ is non measurable (S') unless it vanishes at x_2.

We next consider the topological dual of $\underline{L}^p$, $1 \leq p < \infty$. If $g \in \overline{R}^X$, g determines an element G of $(\underline{L}^p)^*$ if $|g(x_1)| < \infty$ by

$$G(f) = \int fg \, d\mu = f(x_1)g(x_1),$$

noting that $f(x_2) = 0$ and using the convention for extended real numbers that $0 \cdot \infty = 0$ (or we could replace $\int$ by $\overline{\int}$). Each g in the same equivalence class modulo locally null functions determines the same G with $\|G\| = \underline{N}^{\infty}_{\ell}(\hat{g}_{\ell})$. Conversely if $G \in (\underline{L}^p)^*$ let $g(x_1) = G(\chi_{x_1})$, $-\infty \leq g(x_2) \leq \infty$. If $f \in L^p$,

$$G(f) = G(f\chi_{x_1}) = f(x_1)G(\chi_{x_1}) = f(x_1)g(x_1).$$

All such g belong to the same equivalence class $\hat{g}_{\ell}$ in $\underline{L}^q_{\ell}$ and $\underline{N}^q_{\ell}(\hat{g}_{\ell}) = \|G\|$. Thus for this example

$$(\underline{L}^p)^* = \underline{L}^q_{\ell} = \underline{L}^q, \quad 1 < p < \infty$$

$$(\underline{L}^1)^* = \underline{L}^{\infty}_{\ell} \subset \dot{\underline{L}}^{\infty}.$$

Example 2. $X = (0, 1)$, $S = \underline{P}(X)$, $\mu(A)$ denotes the number of points in A if A is finite, $= \infty$ otherwise.

S is a σ-algebra, μ not σ-finite. The empty set is the only null set and $A \in S_f$ implies that A is finite. Measurability and local measurability coincide, every $\nu << \mu$ has a unique RN-derivative and $(\underline{L}^1)^* = \underline{L}^{\infty} = \underline{L}^{\infty}_{\ell}$. We note that $L^p(X, S, \mu) = L^p(X, S', \mu)$ if $1 \leq p < \infty$ but that $L^{\infty}(X, S, \mu) = L^{\infty}_{\ell}(X, S, \mu) \supset L^{\infty}(X, S', \mu)$ since f is not measurable (S') if

$\{x: f(x) \neq 0\}$ is not countably infinite.

Example 3. ([4], p. 36). With X and μ the same as in Example 2 let S denote the Lebesgue measurable subsets of X.

Let ν denote Lebesgue measure on S. Then $\nu << \mu$. Suppose that ν has an RN-derivative $g(x)$ with respect to μ. Since $\mu(x) = 1$, $\nu(x) = 0$ for every one point set $x \in X$, $g(x) \equiv 0$. This is not compatible with

$$1 = \nu(X) = \int_X g(x)\, d\mu .$$

We note that every locally null function is a local RN-derivative for ν with respect to μ.

If $f \in L^1$ the points where $f(x) \neq 0$ are at most countable say x_i, $i = 1, 2, \ldots$ and $\underline{N}^1(f) = \Sigma_1^\infty |f(x_i)|$. It is easy to show that $(\underline{L}^1)^*$ is isometric to B, the space of bounded functions on X, by

$$G(f) = \int gf\, d\mu , \text{ with } \|G\| = \sup_{x \in X} |g(x)| .$$

Here gf is measurable for every g since the product vanishes outside a countable collection of points. The only null or locally null set is the empty set. Thus $L^1 = \underline{L}^1$, $L^\infty_\ell = \underline{L}^\infty_\ell$. We note that every element of $\bar{R}^X$ is locally measurable and $B = \underline{L}^\infty_\ell$. Since there are bounded non-measurable functions, viz. the characteristic function of a non-Lebesgue measurable set,

$$\underline{L}^\infty \subset \underline{L}^\infty_\ell = (\underline{L}^1)^* .$$

We note also that $\underline{L}^\infty(X, S, \mu) \supset \underline{L}^\infty(X, S', \mu)$.

In the above examples $\underline{L}^\infty_\ell$ is a Banach space as the topological dual of the Banach space $\underline{L}^1$. In the general case completeness can be shown by a slight modification of the argument of [2], Theorem 3.1. There is no loss of generality in assuming that each $g_i(P)$ in [2], Theorem 3.1 is defined and finite everywhere in X since the equivalence class of each element in L_ℓ^∞ contains such functions. If $e_\infty = \{P: g_0(P) = \infty\}$ $\underline{N}_\ell^\infty(g_0) < \infty$ implies that, for each $e \in S_f$, $N_\ell^\infty(g_0 \chi_e) < \infty$ whence $e \cap e_\infty$ is null. Thus e_∞ is locally null. The remainder of the argument in Theorem 3.1 goes through verbatim.

The above examples illustrate two properties of totally σ-finite measures that may become invalid for arbitrary measures -

1. We call $A \in S$ finitely regular if

$$\mu(A) = \sup_{\substack{e \subset A \\ e \in S_f}} \mu(e) .$$

Since μ is monotone every $e \in S_f$ is finitely regular. Thus if A is not finitely regular

$$\mu(A) = \infty , \quad \sup_{\substack{e \subset A \\ e \in S_f}} \mu(e) = a < \infty .$$

There then exists an increasing sequence $\{e_n\}$ of sets in S_f with $\lim_n \mu(e_n) = a$. If $A' = \bigcup_1^\infty e_n$, A' is measurable and $A - A'$ is measurable and purely infinite. Thus every set that is not finitely regular contains a purely infinite measurable subset.

2. If μ is totally σ-finite $X = \bigcup_1^\infty e_n$ with $e_n \in S_f^+$. If $A \cap e \in S$ for every $e \in S_f$, $A = A \cap X = \bigcup_1^\infty A \cap e_n \in S$. Example 3 shows that in the general case there may exist non-measurable sets A with $A \cap e \in S$ for every $e \in S_f$. Replacing measurability by local measurability restores this property in the general case.

Purely infinite sets can always be removed by changing the measure from μ to μ_o where

$$\mu_o(A) = \sup_{\substack{e \subset A \\ e \in S_f}} \mu(e) .$$

Then $\mu_o = \mu$ on the finitely regular sets but purely infinite measurable sets become null sets. We note that, even if μ is complete, μ_o will not be complete unless every subset of every purely infinite measurable set is measurable. In Example 1, $\underline{L}_\ell^p(X, S, \mu) = \underline{L}^p(X, S, \mu_o)$, $1 \leq p \leq \infty$. In Example 3, $\mu = \mu_o$ and the local theory is needed to characterize $(\underline{L}^1)^*$ by integrals.

Purely infinite sets may also be deleted by retaining the measure μ but replacing the σ-algebra S by a suitable σ-ring. S' was used in Example 1 and is in general adequate in studying $\underline{L}^p$, $1 \leq p < \infty$, but was inadequate for the description of $(\underline{L}^1)^*$ in Examples 2 and 3. The collection S'' of sets in S that contain no purely infinite measurable subsets is a σ-ring. In Examples 2 and 3 it coincides with S .

The local theory has effectively neglected purely infinite sets, restored property 2, and has provided a natural way of describing $(\underline{L}^1)^*$ in terms of integrals in all three examples. In the general case each element of $\underline{L}_\ell^\infty$ determines a continuous linear functional on $\underline{L}^1$ (compare Theorem 4.1 below). However $\underline{L}_\ell^\infty$ can be isometric to a proper subset of $(\underline{L}^1)^*$. When the measure space permits a local extension of the Radon-Nikodym theorem it can be shown that $(\underline{L}^1)^*$ and $\underline{L}_\ell^\infty$ are isometric.

4. $(\underline{L}^1)^*$ for general measure spaces. The following example ([3], p. 131) shows that a local extension of the Radon-Nikodym theorem is not always possible and that $(\underline{L}^1)^*$ and

$\underline{L}_\ell^\infty$ need not be isometric.

Example 4. Let X, Y be sets of points with cardinal numbers α, β respectively where $\alpha \geq$ the first non-countable cardinal and $\beta > \alpha$. We consider the Cartesian product $X \times Y$, call the set of points $A_{y_0} = \{(x, y_0), x \in X\}$, a horizontal line, the set $A_{x_0} = \{(x_0, y), y \in Y\}$ a vertical line.

A set A will be called full on a horizontal or vertical line if it consists of all but at most countably many points of the line. We let S denote the collection of sets which intersect each horizontal or vertical line in a full set or a countable set. Then S is a σ-algebra. For each $A \in S$ let $\mu(A)$ equal the number of horizontal lines on which A is full plus the number of vertical lines on which A is full. Then $(X \times Y, S, \mu)$ is a complemented measure space that is not totally σ-finite. If $\nu(A)$ denotes the number of horizontal lines on which A is full ν is a positive measure, $\nu << \mu$.

Since for each $x \in X$, $\mu(A_x) = 1 < \infty$ and $\nu << \mu$ on the subsets of A_x, the classical finite Radon-Nikodym theorem implies the existence of $g_x(P)$, $P \in A_x$, with

$$\nu(e) = \int_e g_x(P)\, d\mu$$

for every measurable subset e of A_x. Since the measurable subsets of A_x are either full or countable, $g_x(P) = 0$ for all but at most countably many points of A_x. Similarly for each $y \in Y$ there exists $g_y(P)$, $P \in A_y$ with

$$\nu(e) = \int g_y(P) d\mu$$

for every $e \subset A_y$, $e \in S$, and $g_y(P) = 1$ almost everywhere in A_y. The collection of functions g_x, $x \in X$; g_y, $y \in Y$ determine an element of $(L^1)^*$ by the argument of Theorem 4.1 below. Suppose now that there exists a locally measurable

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function $g(P)$ with

$$g(P) = g_x(P) \text{ a.e. in each } A_x ,$$

$$= g_y(P) \text{ a.e. in each } A_y .$$

If $A = \{P \in X \times Y : g(P) \neq 0\}$, as in [3] the x-conditions imply that A has cardinal number $> \beta$, the y-conditions that A has cardinal number $\leq \alpha$ giving a contradiction.

We note that in Example 4, $X = \bigcup_{x \in X} A_x$, $A_x \in S_f^+$, $x \in X$, $A_x \cap A_{x'} = \theta$, $x \neq x'$. Thus there exists a measurable function $f(P)$ defined on all of X, coinciding with $g_x(P)$ a.e. in each A_x, namely the function $g(P) = 0$. However each $A_y \in S_f^+$ but $\mu(A_x \cap A_y) = 0$, $x \in X$ and

$$1 = \nu(A_y) \neq \int_{A_y} g(P) d\mu = 0 .$$

We note also that $X = (\bigcup_{x \in X} A_x) \cup (\bigcup_{y \in Y} A_y) = \bigcup_{\lambda \in \Lambda} A_\lambda$, where Λ is the set of all x and y indices. Here $\mu(A_\lambda \cap A_{\lambda'}) = 0$, $\lambda \neq \lambda'$ and, for any $A \in S_f^+$,

$$\mu(A) = \sum_{\lambda \in \Lambda} (A \cap A_\lambda) ,$$

where $\sum_{\lambda \in \Lambda} \mu(A \cap A_\lambda)$ means the supremum of all finite sums of this form. In this case the functions $g_{A_\lambda}(P)$, $\lambda \in \Lambda$, determine a $g_A(P)$ for every $A \in S'$ but it is impossible to define $g(P)$ on X coinciding a.e. with $g_{A_\lambda}(P)$ in A_λ for each $\lambda \in \Lambda$.

The following lemma shows that similar decompositions are possible for every complemented measure space.

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Lemma 4.1. To each complemented measure space (X, S, μ) correspond two decompositions (D) and (ND) where, for both

$$X = X_1 \cup X_2 , \quad \text{with } X_1 \text{ purely infinite or locally null,}$$

$$X_1 \cap X_2 = \theta ;$$

$$X_2 = \bigcup_{\lambda \in \Lambda} e_\lambda , \; e_\lambda \in S_f^+ ,$$

and where, for

(D) $e_\lambda \cap e_{\lambda'} = \theta , \quad \lambda \neq \lambda'$;

(ND) $\mu(e_\lambda \cap e_{\lambda'}) = 0 , \quad \lambda \neq \lambda'$ and $\mu(e) = \sum_{\lambda \in \Lambda} \mu(e \cap e_\lambda)$

for every $e \in S_f$.

If μ is a complete measure X is locally null in both cases. The decompositions (D) and (ND) are not in general unique.

Proof. We verify (ND), (D) being simpler and similar. We consider all collections of sets from S_f^+ with pairwise intersections null and partially order these collections by inclusion.

If $C_\alpha, \alpha \in A$, is a chain in this partially ordered set, let C' denote the collection of all elements e_λ in any C_α. If $e_\lambda \neq e_{\lambda'}$, $e_\lambda \in C_\alpha$, $e_{\lambda'} \in C_{\alpha'}$ for some α, α' and we can assume that $\alpha < \alpha'$. Then $e_\lambda, e_{\lambda'} \in C_{\alpha'}$ and $\mu(e_\lambda \cap e_{\lambda'}) = 0$. Thus C' is an upper bound for the chain. By Zorn's Lemma there is a maximal collection $C = \{e_\lambda, \lambda \in \Lambda\}$. If $e \in S_f^+$ maximality implies that $\mu(e \cap e_\lambda) > 0$ for some $\lambda \in \Lambda$.

Let $X_2 = \bigcup_{\lambda \in \Lambda} e_\lambda$, $X_1 = X - X_2$. Then X_1 can contain no set of S_f^+ and so is purely infinite or locally null.

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Let $e \in S_f^+$. Assume that $\mu(e \cap e_\lambda) > 0$ for uncountably many $\lambda \in \Lambda$. There would then exist $\delta > 0$ and a sequence λ_i, $i = 1, 2 \ldots$ with $\mu(e \cap e_{\lambda_i}) > 0$ whence, since $\mu(e_{\lambda_i} \cap e_{\lambda_j}) = 0$, $i \neq j$,

$$\infty = \Sigma_{i=1}^{\infty} \mu(e \cap e_{\lambda_i}) = \mu(e \cap (\cup_1^\infty e_{\lambda_i}) < \mu(e) < \infty ,$$

a contradiction. Thus for at most countably many λ say λ_j, $j = 1, 2, \ldots, \mu(e \cap e_{\lambda_j}) > 0$ and

$$\sum_{\lambda \in \Lambda} \mu(e \cap e_\lambda) = \Sigma_{j=1}^{\infty} \mu(e \cap e_{\lambda_j}) \leq \mu(e) .$$

Let $e' = e - \cup_1^\infty (e \cap e_{\lambda_j})$. Then $e' \in S_f$. If $e' \in S_f^+$ there exists $\lambda' \neq \lambda_j$, $j = 1, 2, \ldots,$ with $\mu(e \cap e_{\lambda'}) \geq \mu(e' \cap e_{\lambda'}) > 0$ giving a contradiction proving that $\mu(e') = 0$. If μ is complete, since $X_1 \cap e \subset e'$, $X_1 \cap e$ is in S with $\mu(X_1 \cap e) = 0$, i.e. X_1 is locally null.

The index set in (D) can be countable. Then $\bigcup_{\lambda \in \Lambda} e_\lambda$ is measurable, X_1 a measurable null or purely infinite set. Since $\chi_{e_\lambda} \in L^p$, $1 \leq p < \infty$, $\lambda \in \Lambda$, $\underline{L}^p$ cannot be separable when Λ is not countable.

Remark. If S is a σ-algebra and μ is not totally σ-finite, $\underline{L}^p$ $(1 \leq p < \infty)$, cannot be separable unless S contains a purely infinite set.

Definition. A measure μ on a complemented measure space (X, S, μ) will be called locally σ-finite if there exists a decomposition (D) with $\mu(e) = \sum_{\lambda \in \Lambda} \mu(e \cap e_\lambda)$ for every $e \in S_f$.

When μ is locally σ-finite the finite Radon-Nikodym Theorem extends locally to X by $g(x) = \sum_{\lambda \in \Lambda} g_{e_\lambda}(x)$, i.e. every $\nu << \mu$ has a local RN-derivative with respect to μ. We observe that if there is a decomposition (ND) with Λ of cardinal $\aleph_1$ (the smallest uncountable cardinal), μ is locally σ-finite. There is then a well ordering of the sets e_λ such that, for each $\lambda_0 \in \Lambda$, at most countably many λ precede λ_0. Thus $\bigcup_{\lambda<\lambda_0} e_\lambda \in S$, $e_{\lambda_0} \cap (\bigcup_{\lambda<\lambda_0} e_\lambda)$ is null and $e^1_{\lambda_0} = e_{\lambda_0} - \bigcup_{\lambda<\lambda_0} e_\lambda \in S^+_f$. Since $e^1_{\lambda_1} \cap e^1_{\lambda_2} = 0$ if $\lambda_1 \neq \lambda_2$, both (D) and (ND) apply for $X_2 = \bigcup_{\lambda \in \Lambda} e^1_\lambda$. Thus if $\alpha, \beta \leq \aleph_1$ in Example 4, μ is locally σ-finite.

We show how a decomposition (ND) can be used to characterize every $(\underline{L}^1)^*$ in terms of integrals. We fix a decomposition (ND) in Lemma 4.1, let g_Λ denote a collection of functions $g_\lambda(x), \lambda \in \Lambda$ where $g_\lambda(x)$ vanishes outside e_λ and is measurable. We define $\underline{N}^\infty_\Lambda(g_\Lambda) = \sup_{\lambda \in \Lambda} \underline{N}^\infty(g_\lambda)$ and denote by $\underline{L}^\infty_\Lambda$ the space of collections g_Λ with $\underline{N}^\infty_\Lambda(g_\lambda) < \infty$. It is easy to verify that $\underline{L}^\infty_\Lambda$ is a vector space semi-normed by $\underline{N}^\infty_\Lambda(\cdot)$. $\underline{N}^\infty_\Lambda(g_\Lambda) = 0$ if and only if $g_\lambda(x) = 0$ a.e. in e_λ for each $\lambda \in \Lambda$. If $\underline{L}^\infty_\Lambda$ is the corresponding normed space it is easy to show that it is a Banach space. We note that the space $\underline{L}^\infty_\Lambda$ does not depend on the choice of the (ND) decomposition. Where primes refer to a second decomposition, to each $e_{\lambda'}, \lambda' \in \Lambda'$, corresponds a countable sequence $e_{\lambda_i}, \lambda_i \in \Lambda$, with $e_{\lambda'} - e_{\lambda'} \cap (\bigcup_{i=1}^\infty e_{\lambda_i})$ null. Given g_Λ let $g_{\Lambda'} = \{ g_{\lambda'}(x) = \sup_i g_{\lambda_i}(x) \chi_{e_{\lambda'}} ; \lambda' \in \Lambda' \}$. Then $\underline{N}^\infty_{\Lambda'}(g_{\Lambda'}) \leq \underline{N}^\infty_\Lambda(g_\Lambda)$ and $\underline{N}^\infty_{\Lambda'}(\hat{g}_{\Lambda'}) \leq \underline{N}^\infty_\Lambda(\hat{g}_\Lambda)$. Now $g_{\Lambda'}$ determines g^*_Λ in a similar way and $\underline{N}^\infty_\Lambda(g_\Lambda - g^*_\Lambda) = 0$.

Thus $\underline{N}_{\Lambda'}^{\infty}(\hat{g}_{\Lambda'}) = \underline{N}_{\Lambda}^{\infty}(\hat{g}_{\Lambda})$. Reversing the roles of Λ, Λ' shows that $\underline{L}_{\Lambda}^{\infty}$ and $\underline{L}_{\Lambda'}^{\infty}$ are isometric.

If $f \in \underline{L}^1$, $e_f = \{x : f(x) \neq 0\}$ can be expressed as the union of a countable collection of sets of finite measure and therefore as the union of a null set and a countable collection $e_f \cap e_{\lambda_i}$, $\lambda_i \in \Lambda$. We define

$$g_f(x) = \sup_i g_{\lambda_i}(x), \quad x \in e_f; \quad = 0 \text{ elsewhere.}$$

Then $g_f(x) = g_{\lambda_i}(x)$ a.e. in $e_f \cap e_{\lambda_i}$, $i = 1, 2, \ldots$. We note that

$$g_{af}(x) = g_f(x), \quad a \in R;$$

$$g_{f_1+f_2}(x) = \sup\{g_{f_1}(x), g_{f_2}(x)\},$$

which coincides with $g_{f_i}(x)$ a.e. in e_{f_i}, $i = 1, 2$.

Theorem 4.1. For any complemented measure space (X, S, μ) $(\underline{L}^1)^*$ and $\underline{L}_{\Lambda}$ are isometric by the correspondence

$$(*) \qquad G(f) = \int f g_f \, d\mu,$$

with $\|G\| = \underline{N}_{/}^{\infty}(g_{/})$. If X is locally σ-finite $\underline{L}_{\Lambda}^{\infty} = \underline{L}_{\ell}^{\infty}$. If X is totally σ-finite $\underline{L}_{\Lambda}^{\infty} = \underline{L}_{\ell}^{\infty} = \underline{L}^{\infty}$.

Proof. Let $g_{\Lambda} \in \underline{L}_{\Lambda}^{\infty}$. If f, f_1, $f_2 \in \underline{L}^1$, $\alpha \in R$ and $G(f)$ is defined by $(*)$,

$$G(\alpha f) = \int \alpha f g_{\alpha f} \, d\mu = \alpha \int f g_f \, d\mu = \alpha G(f),$$

$$G(f_1 + f_2) = \int (f_1 + f_2) g_{f_1+f_2} \, d\mu = \int f_1 g_{f_1} \, d\mu + \int f_2 g_{f_2} \, d\mu$$

$= G(f_1) + G(f_2)$.

Thus G is linear on $\underline{L}^1$. Since

$$|G(f)| = |\int fg_f d\mu| \leq \underline{N}^1(f)\underline{N}^\infty(g_f) \leq \underline{N}^1(f)\underline{N}^\infty_\Lambda(g_\Lambda) ,$$

$G \in (\underline{L}^1)^*$ and $\|G\| \leq \underline{N}^\infty_\Lambda(g_\Lambda)$. Let $a = \underline{N}^\infty_\Lambda(g_\Lambda)$. Given $\varepsilon > 0$ there exists $\lambda \in \Lambda$ with $\underline{N}^\infty(g_\lambda) > a - \varepsilon/2$ and a measurable subset e' of e_λ with $|g_\lambda(x)| > a - \varepsilon$ in e' . Now $f_0(x) = [\mu(e')]^{-1}\chi_{e'} \in L^1$ and $\underline{N}^1(f_0) = 1$. Thus

$$\|G\| \geq |\int f_0 g_{f_0} d\mu| > a - \varepsilon .$$

Since ε is arbitrary $\|G\| = \underline{N}^\infty_\Lambda(g_\Lambda)$.

Conversely if $G \in (\underline{L}^1)^*$ the classical theory applied to the restriction of G to $L^1(e_\lambda)$ gives the existence of $g_\lambda(x) \in L^\infty(e_\lambda)$ with

$$G(f\chi_{e_\lambda}) = \int fg_\lambda d\mu , \quad \underline{N}^\infty(g_\lambda) \leq \|G\| .$$

We let $g_\Lambda = \{g_\lambda(x) , \lambda \in \Lambda \}$. Then $\underline{N}^\infty_\Lambda(g_\Lambda) \leq \|G\|$.

If $f \in L^1$, $e_f = n \cup_{i=1}^\infty (e_f \cap e_{\lambda_i})$ where n is null and $\lambda_i \in \Lambda$, $i = 1, 2, \ldots$. Thus

$$G(f\chi_{\cup_1^n e_{\lambda_i}}) = \int_{\cup_1^n e_{\lambda_i}} fg_f d\mu , \quad n = 1, 2, \ldots .$$

By continuity

$$G(f) = \int fg_f d\mu .$$

This implies that $\|G\| \leq \underline{N}_{\Lambda}^{\infty}(g_{\Lambda})$. Since the opposite inequality was established, $\|G\| = \underline{N}_{\Lambda}^{\infty}(g_{\Lambda})$.

If μ is locally σ-finite the condition that $\mu(e_{\lambda} \cap e_{\lambda'}) = 0$ if $\lambda \neq \lambda'$ in the lemma can be replaced by $e_{\lambda} \cap e_{\lambda'} = 0$, $\lambda \neq \lambda'$. The collection $g_{\Lambda} = \{ g_{\lambda}(x), \lambda \in \Lambda \}$ can then be replaced by the function

$$g(x) = \sum_{\lambda \in \Lambda} g_{\lambda}(x) = \sup_{\lambda \in \Lambda} g_{\lambda}(x).$$

Then $g(x)$ is locally measurable and in L_{ℓ}^{∞} and $\underline{L}_{\ell}^{\infty} = \underline{L}_{\Lambda}^{\infty}$. If μ is totally σ-finite $g(x)$ is measurable.

Example 2 shows that $(\underline{L}^{1})^{*} = \underline{L}^{\infty}$ is possible when μ is not totally σ-finite. We do not know if there are examples where μ is not locally σ-finite but $(\underline{L}^{1})^{*} = \underline{L}_{\ell}^{\infty}$.

Remark. $\underline{L}^{\infty}$, $\underline{L}_{\ell}^{\infty}$ are Banach subspaces of $\underline{L}_{\Lambda}^{\infty}$.

$\underline{L}^{\infty}$ is the conjugate of $\underline{L}^{1}$ for the length function determined by $\underline{N}^{1}$. By ([2], Theorem 3.1) it is a Banach space. We have shown above that $\underline{L}_{\ell}^{\infty}$ is always complete.

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