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ARTICLE

Ramsey upper density of infinite graphs

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Abstract

For a fixed infinite graph H, we study the largest density of a monochromatic subgraph isomorphic to H that can be found in every two-colouring of the edges of $K_{\mathbb{N}}$. This is called the Ramsey upper density of H and was introduced by Erdős and Galvin in a restricted setting, and by DeBiasio and McKenney in general. Recently [4], the Ramsey upper density of the infinite path was determined. Here, we find the value of this density for all locally finite graphs H up to a factor of 2, answering a question of DeBiasio and McKenney. We also find the exact density for a wide class of bipartite graphs, including all locally finite forests. Our approach relates this problem to the solution of an optimisation problem for continuous functions. We show that, under certain conditions, the density depends only on the chromatic number of H, the number of components of H and the expansion ratio $|N(I)|/|I|$ of the independent sets of H.

Keywords: Ramsey theory; infinite graphs; upper density

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1. Introduction

Let $K_{\mathbb{N}}$ be the complete graph on the natural numbers. Let H be a countably infinite graph (meaning that the vertex set has the same cardinality as $\mathbb{N}$). Suppose that the edges of $K_{\mathbb{N}}$ are coloured red or blue. We can find a monochromatic subgraph $H' \subseteq K_{\mathbb{N}}$ isomorphic to H. For example, using Ramsey's Theorem, we can produce H' by finding a bijection between $V(H)$ and the vertices of a monochromatic infinite clique. Out of all possible subgraphs H', we want to find one which maximises its density. To measure the density, we use the following definition:

Definition 1.1. Let $S \subseteq \mathbb{N}$. We define the upper density of S (in this paper shortened to density) as

$$\bar{d}(S) = \limsup_{n\to\infty} \frac{|S \cap [n]|}{n}.$$

If $H' \subseteq K_{\mathbb{N}}$, we define $\bar{d}(H') = \bar{d}(V(H'))$.

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We are interested in an extremal question: if H is a fixed graph, what is the maximum density of H' that we can find in every red-blue colouring of $K_{\mathbb{N}}$? We call this value the Ramsey upper density of H.

Definition 1.2. Let H be a countably infinite graph. We define its Ramsey upper density $\rho(H)$ as the supremum of the values of λ for which, for every two-colouring of $E(K_{\mathbb{N}})$, there exists a monochromatic subgraph $H' \subseteq K_{\mathbb{N}}$, isomorphic to H, with $\bar{d}(H') \geq \lambda$.

The study of this parameter was initiated by Erdős and Galvin [8] for the particular case $H = P_\infty$, the one-way infinite path. They proved that $2/3 \leq \rho(P_\infty) \leq 8/9$. After some improvements on these bounds in [6, 10], the exact value

$$\rho(P_\infty) = \frac{12+\sqrt{8}}{17} \approx 0.87226$$

was determined by Corsten, DeBiasio, Lamaison and Lang [4]. The parameter $\rho(H)$ for general H was first introduced by DeBiasio and McKenney [6].

Our aim in this paper is to give bounds on $\rho(H)$ for a wider family of graphs H. These results can be found further down in the introduction, although some of the more general bounds, with a more involved statement, are left for later. As it will turn out, three parameters play an important role in the value of $\rho(H)$: its chromatic number, the number of components and the expansion properties of its independent sets.

1.1 Notation

An infinite graph is *locally finite* if every vertex has finite degree. The bounds that we will show in this paper apply only to locally finite graphs H.

Given $S \subseteq V(H)$, we will denote by $N(S) = (\cup_{v \in S} N(v)) \setminus S$ the set of vertices outside S with a neighbour in S. We let $\mu(H, n)$ be the minimum value of $|N(I)|$, where I is an independent set in H of size n. We say that a set I is *doubly independent* if both I and $N(I)$ are independent.

We say that a family $\{S_1, S_2, \ldots\}$ of subsets of $V(H)$ is *concentrated* in at most k components if there are components $C_1, C_2, \ldots, C_s$ of H, with $s \leq k$, such that all but finitely many sets S_i intersect some component C_j. We say that $V(H)$ is concentrated in at most k components if $\{\{v\} : v \in V(H)\}$ is concentrated in at most k components.

On some occasions we will use $C \in \{R, B\}$ to designate a colour (red or blue). When this happens, $\bar{C}$ will denote the other colour. In a graph G with coloured vertices, we will use C_G to refer to the set of vertices of colour C. If there is no ambiguity, we will omit the subindex G.

If F is a finite graph, we denote by $\omega \cdot F$ the graph obtained by taking the disjoint union of a countably infinite number of copies of F.

Finally, we define a function $f(x)$ which will be crucial in relating the values of $\rho(H)$ and $|N(I)|/|I|$, where I is an independent or a doubly independent set of H. Unfortunately, there is no satisfying intuition for why this particular choice of $f(x)$, and not another, is behind the relation between these two parameters. It is interesting however that the same function $f(x)$ arises from the study of upper bounds and lower bounds for $\rho(H)$.

Since the definition of $f(x)$ is quite complicated and its comprehension is not essential to the appreciation of our results, we encourage the reader to skip it for now. For the reading of the introduction, knowing that such a function exists is enough. Of course, for the reading of the proofs, the precise definition becomes necessary.

Definition 1.3. Let $\gamma \in (-1, 1)$. For a continuous function $g(x) : [0, +\infty) \to \mathbb{R}$, define

$$\Gamma^+_\gamma(g, t) = \min\{x : \gamma x + g(x) \geq t\}, \quad \Gamma^-_\gamma(g, t) = \min\{x : \gamma x - g(x) \geq t\},$$

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where we take the minimum of the empty set to be $+\infty$. We define $h(\gamma)$ to be the infimum, over all 1-Lipschitz[1] functions g with $g(0)=0$, of

$$h(\gamma)=\inf_g \limsup_{t\to\infty} \frac{\Gamma_\gamma^+(g,t)+\Gamma_\gamma^-(g,t)}{t}. \tag{1}$$

Define $f:(0,+\infty)\to\mathbb{R}$ as

$$f(\lambda)=1-\frac{1}{\frac{2\lambda}{(1+\lambda)^2}h\left(\frac{\lambda-1}{\lambda+1}\right)+\frac{2\lambda}{1+\lambda}}.$$

We define $f(0)=1$ and $f(+\infty)=1/2$ (by (2) below, we have $\lim_{t\to 0} f(t)=1$ and $\lim_{t\to+\infty} f(t)=1/2$.)

In an appendix to this paper, which can be found in the arXiv version (arXiv:2003.06329), we prove some properties of $f(x)$, including the following bounds:

$$\frac{x+1}{2x+1}\le f(x)\le \begin{cases} \dfrac{2x^2+3x+7+2\sqrt{x+1}}{4x^2+4x+9} & \text{for} \quad 0\le x<3, \\[2ex] \dfrac{x+1}{2x} & \text{for} \quad x\ge 3. \end{cases} \tag{2}$$

The upper bound is sharp for $x\in[0,1]$, and we conjecture[2] that it is sharp everywhere. Observe that $f(1)=(12+\sqrt{8})/17=\rho(P_\infty)$.

1.2 Results

We will now give a few bounds on $\rho(H)$, some of which apply for all locally finite graphs and some of which apply only for particular families. In many cases the specific results follow from other results which are more general, but which have more involved statements. These will be stated in later sections.

For locally finite graphs, knowing the chromatic number and the number of components is enough to determine $\rho(H)$ up to a factor of 2.

Theorem 1.4. *Let H be a locally finite graph.*

1. *If H has infinitely many components, then* $\rho(H)\ge 1/2$.
2. *If H has finitely many components:*
   (a) *If H has infinite chromatic number, then* $\rho(H)=0$.
   (b) *If H has finite chromatic number, then*

$$\min\left\{\frac{b}{2(\chi(H)-1)},\frac{1}{2}\right\}\le\rho(H)\le\min\left\{\frac{b}{\chi(H)-1},1\right\},$$

where b is the number of infinite components of H.

This theorem answers a question in [6], which asks whether for every Δ there exists a constant $c>0$ such that every graph with maximum degree at most Δ has Ramsey upper density at least c:

Corollary 1.5. *If H has maximum degree at most* Δ, *then* $\rho(H)\ge 1/(2\Delta)$.

[1] A 1-Lipschitz function is a function satisfying $|f(x)-f(y)|\le|x-y|$ for every x,y in the domain.
[2] An extended abstract for this paper, published in *Acta Math. Univ. Comenianae* for EUROCOMB 2019, stated this as proved. Since then, a mistake in the proof has been found.

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Let P_∞^k be the kth power of the infinite path, that is, the graph on $\mathbb{N}$ in which x and y are connected if $|x-y|\le k$. Elekes, Soukup, Soukup and Szentmiklóssy [7] showed that, in every two-colouring of $K_\mathbb{N}$, the vertex set can be partitioned into at most 2^{2k-1} monochromatic copies of P_∞^k plus a finite set, and the number of copies can be reduced to four for P_∞^2. DeBiasio and McKenney [6] pointed out that this implies $\rho(P_\infty^2)\ge 1/4$ and $\rho(P_\infty^k)\ge 2^{1-2k}$. Theorem 1.4 improves the bound for $k\ge 3$ to $\rho(P_\infty^k)\ge 1/(2k)$.

The case 2a in Theorem 1.4 connects to a result of Corsten, DeBiasio and McKenney [5]. While we show that every locally finite graph H with finitely many components and infinite chromatic number has $\rho(H)=0$, they show that these graphs are '2-Ramsey-dense', as they call it (see Corollary 1.7 in their paper). This property means that, in every two-colouring of $E(K_\mathbb{N})$, there exists a monochromatic copy of H with positive upper density. Of course this is not a contradiction, because there exists a *sequence* of colourings in which the density of the densest monochromatic copy of H tends to 0.

While no graph H is known for which the lower bound in 2b is tight and not equal to 1/2, the upper bound is tight in the following example. Let T be the tree formed by an infinite path $v_1v_2v_3\ldots$, in which we attach i leaves to v_i for every $i\in\mathbb{N}$. Then $\rho(b\cdot T+K_a)=b/(a-1)$ for every $1\le b<a$, where $b\cdot T+K_a$ denotes the disjoint union of b copies of T and an a-clique. The lower bound will follow from Theorem 3.1.

Another upper bound that applies to all locally finite graphs is related to the expansion of its independent sets:

Theorem 1.6. *Let H be a locally finite graph. Then*

$$\rho(H)\le f\left(\liminf_{n\to\infty}\frac{\mu(H,n)}{n}\right).$$

There are many graphs for which the bound in Theorem 1.6 is tight. The following theorem captures some of them.

Theorem 1.7. *Let H be a locally finite forest, or a locally finite bipartite graph in which every orbit of the automorphism group acting on V(H) has infinite size. Then*

$$\rho(H)=f\left(\liminf_{n\to\infty}\frac{\mu(H,n)}{n}\right).$$

This is a particular case of a more general condition on bipartite graphs that is sufficient for Theorem 1.6 to be tight. That condition is stated later as Theorem 4.1. The following corollaries illustrate some examples of graphs for which Theorem 1.7 applies:

Corollary 1.8. *Let T_k be the infinite k-ary tree, that is, the rooted tree in which every vertex has k children. Then $\rho(T_k)=f(k)$.*

Corollary 1.9. *Let* Grid_d *be the infinite d-dimensional grid, that is, the graph on $\mathbb{Z}^d$ where two vertices are connected if they are at Euclidean distance 1. Then $\rho(\mathrm{Grid}_d)=f(1)=(12+\sqrt{8})/17\approx 0.87226$.*

Corollary 1.10. *Let F be a finite bipartite graph. Then*

$$\rho(\omega\cdot F)=f\left(\min_{\substack{I \text{ indep. in } F\\ I\neq\emptyset}}\frac{|N(I)|}{|I|}\right).$$

In particular, we have $\rho(C_{2k})=f(1)$ for every $k\ge 2$, and for every $1\le a\le b$ we have

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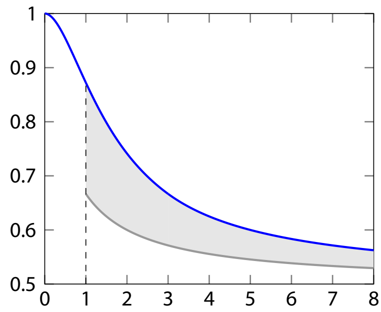

Figure 1. Plot of the function $f(x)$ on the interval $[0, 1]$, and the upper and lower bounds elsewhere. The conjectured value is given in blue.

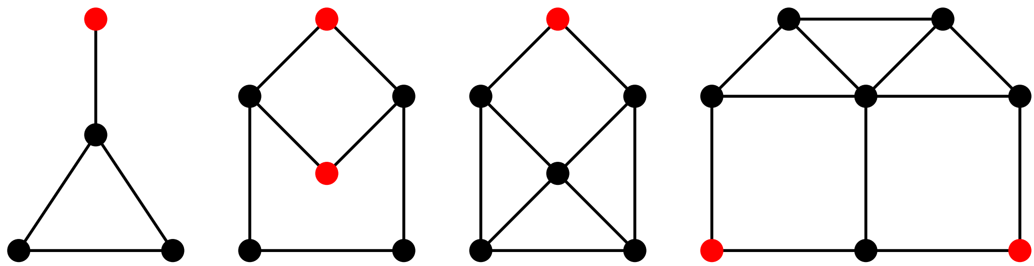

Figure 2. Four non-bipartite graphs F for which $\rho(\omega \cdot F)$ equals $f(1)$, $f(1)$, $f(2)$ and $f(3/2)$ respectively, with their doubly independent sets indicated.

$$\rho(\omega \cdot K_{a,b}) = f\left(\frac{a}{b}\right) = \frac{2\left(\frac{a}{b}\right)^2 + 3\left(\frac{a}{b}\right) + 7 + 2\sqrt{\frac{a}{b}+1}}{4\left(\frac{a}{b}\right)^2 + 4\left(\frac{a}{b}\right) + 9}.$$

In a finite bipartite graph F, there is always an independent set satisfying $|N(I)| \leq |I|$ (one of the two partition classes has this), so the value of $\rho(\omega \cdot F)$ always falls on the range in which $f(x)$ is known explicitly.

Finally, we will give two more lower bounds in the particular case of infinite factors $\omega \cdot F$. The first one is analogous to Corollary 1.10:

Theorem 1.11. *Let F be a finite connected graph, and let $I \subseteq V(F)$ be a non-empty doubly independent set. Then $\rho(\omega \cdot F) \geq f\left(\frac{|N(I)|}{|I|}\right)$.*

If the independent set $I \subseteq V(F)$ that minimises $|N(I)|/|I|$ is doubly independent, then Theorems 1.6 and 1.11 together give the exact value for $\rho(\omega \cdot F)$. This is always true in bipartite graphs, giving another reason why Corollary 1.10 holds. Figure 2 shows four non-bipartite graphs F for which this holds. If the graph F does not contain any non-empty doubly independent sets (such as K_3), the following lower bound can be used:

Theorem 1.12. *For every finite graph F, we have*

$$\rho(\omega \cdot F) \geq \frac{|V(F)|}{2|V(F)| - \alpha(F)}.$$

This theorem gives the best known lower bound for $\rho(\omega \cdot K_3)$. Combining Theorems 1.12 with 1.6 and (2) we obtain

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$$3/5 \le \rho(\omega \cdot K_3) \le f(2) \le \frac{21+\sqrt{12}}{33} \approx 0.74133.$$

This paper is organised as follows: we prove the general upper bounds in Section 2, and the general lower bounds in Section 3, besides a lemma that is instead proved in the appendix (the bulk of this proof is a rather long series of calculations without any interesting ideas behind). In Section 4, we discuss the application of the general bounds to particular families of graphs and obtain the remaining results above. In Section 5 we state some open questions. In the appendix, available in the arXiv version of the paper, we prove some properties of $f(x)$.

2. General upper bounds

We will prove two upper bounds in this section: the first one implies the upper bound from items 2a and 2b from Theorem 1.4, while the second one is Theorem 1.6. In both cases we will construct a colouring of $E(K_{\mathbb{N}})$ in which no dense monochromatic copy of H exists.

Theorem 2.1. *Let H be a locally finite graph with chromatic number at least a, such that $V(H)$ is concentrated in at most b components. There exists a two-colouring of $E(K_{\mathbb{N}})$ in which every monochromatic copy of H has density at most $b/(a-1)$.*

Proof. Consider the colouring of $E(K_{\mathbb{N}})$ in which the edge uv is red iff $a-1$ divides $v-u$. The graph formed by the blue edges has chromatic number $a-1$ and thus does not contain H as a subgraph. Every monochromatic copy H' of H in this colouring is red.

The red graph consists of $a-1$ cliques $C_1, \dots, C_{a-1}$, each with $\bar{d}(C_i) = 1/(a-1)$. The b components that concentrate $V(H')$ must be each contained in a clique C_i. Because modifying finitely many elements does not affect the density of a set, we have $\bar{d}(H') \le \bar{d}(C_1 \cup \dots \cup C_b) = b/(a-1)$. We conclude that $\rho(H) \le b/(a-1)$. □

Next we will prove Theorem 1.6. As before, the goal is to construct a two-colouring of $E(K_{\mathbb{N}})$ without dense monochromatic copies of H.

The intuition behind the construction to prove Theorem 1.6 is as follows: suppose that we are trying to find a red copy H' of H. If we have a blue clique K which has fewer than k vertices neighbouring K through some red edge, and $\mu(H,t) = k$, then we know that fewer than t vertices from K can be in H', because those vertices correspond to an independent set in H. Our goal is to find a construction that maximises the number of vertices from $[n]$ that can be excluded from a potential red or blue H' using this method.

This same approach was used, in the case of the infinite path, by Erdős and Galvin [8]. The improvement that Corsten, DeBiasio, Lamaison and Lang [4] made over their colouring came from a two-step construction: we start with an infinite set of vertices, we first decide the colour of the edges between them and then we choose the element of $\mathbb{N}$ that will correspond to each vertex. The same two-step technique is used here.

Proof of Theorem 1.6. Denote $\lambda = \liminf_{n\to\infty} \frac{\mu(H,n)}{n}$. We assume that $\lambda > 0$, as otherwise the statement becomes the trivial inequality $\rho(H) \le 1$.

Let $\epsilon > 0$. Let g be a 1-Lipschitz function such that the upper limit in (1) is less than $h(\gamma) + \epsilon$, for $\gamma = \frac{\lambda-1}{\lambda+1}$. Take an infinite set of vertices $v_1, v_2, \dots$ and arrange them from left to right in this order. Colour these vertices red and blue, in such a way that, for every $n \in \mathbb{N}$, among the n leftmost vertices there are exactly $\lfloor (n+g(n))/2 \rfloor$ red vertices (this is possible because g is 1-Lipschitz). Form a two-coloured complete graph by giving each edge the colour of its leftmost endpoint.

There must be infinitely many vertices of each colour. Indeed, if there are finitely many red vertices then the non-decreasing function $\frac{x+g(x)}{2}$ is bounded, while if there are finitely many blue vertices the non-decreasing function $\frac{x-g(x)}{2}$ is bounded. Note that if $x \pm g(x)$ is bounded, then

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$\gamma x \pm g(x)$ has an upper bound, as $\gamma \leq 1$. In the first case $\Gamma_\gamma^+(g, n) = +\infty$ for every n large enough, while in the second case $\Gamma_\gamma^-(g, n) = +\infty$.

Let the red vertices be $r_1, r_2, r_3, \ldots$ and the blue vertices be $b_1, b_2, b_3, \ldots$, according to the left-to-right order. Let α_i be the smallest value such that r_{α_i} has at most $\lambda(\alpha_i - i)$ blue vertices to its left, and β_i the smallest value such that b_{β_i} has at most $\lambda(\beta_i - i)$ red vertices to its left. The following discussion will not only prove the existence of α_i and β_i, but also give a bound on them.

For a fixed value of i, let $w = \frac{2}{1+\lambda}(\lambda i + 2\lambda + 2)$. Let $z^+ = \Gamma_\gamma^+(g, w)$ and $z^- = \Gamma_\gamma^-(g, w)$. By continuity of g and definition of Γ_γ^+ and Γ_γ^-, we have

$$g(z^+) = w - \gamma z^+, \quad g(z^-) = \gamma z^- - w.$$

One can check that the following identity holds by substution of $g(z^-)$:

$$\frac{z^- + g(z^-)}{2} + 2 = \lambda\left(\frac{z^- - g(z^-)}{2} - i - 2\right).$$

Among the $\lfloor z^- \rfloor$ leftmost vertices there are $\left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor$ red vertices and $\lfloor z^- \rfloor - \left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor$ blue vertices. Observe that

$$\left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor \leq \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \leq \frac{z^- + g(z^-)}{2} < \lambda\left(\frac{z^- - g(z^-)}{2} - i - 2\right)$$
$$\leq \lambda\left(\lfloor z^- \rfloor - \left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor - i\right).$$

If the last blue vertex among those $\lfloor z^- \rfloor$ is b_τ, then the number of red vertices to its left is less than $\lambda(\tau - i)$, meaning that $\beta_i \leq \tau$, and in particular β_i exists. Hence

$$\beta_i \leq \lfloor z^- \rfloor - \left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor \leq \frac{z^- - g(z^-)}{2} + 2 = \frac{1-\gamma}{2} z^- + \frac{w}{2} + 2.$$

Analogously, one has the identity

$$\frac{z^+ - g(z^+)}{2} + 2 = \lambda\left(\frac{z^+ + g(z^+)}{2} - i - 2\right)$$

and the inequality

$$\lfloor z^+ \rfloor - \left\lfloor \frac{\lfloor z^+ \rfloor + g(\lfloor z^+ \rfloor)}{2} \right\rfloor \leq \lambda\left(\left\lfloor \frac{\lfloor z^+ \rfloor + g(\lfloor z^+ \rfloor)}{2} \right\rfloor - i\right).$$

Hence we find

$$\alpha_i \leq \left\lfloor \frac{\lfloor z^+ \rfloor + g(\lfloor z^+ \rfloor)}{2} \right\rfloor \leq \frac{z^+ + g(z^+)}{2} = \frac{1-\gamma}{2} z^+ + \frac{w}{2}.$$

Adding the two values together, for i large enough we have

$$\alpha_i + \beta_i \leq \frac{1-\gamma}{2}(z^+ + z^-) + w + 2 \leq \left(\frac{1-\gamma}{2} h(\gamma) + \epsilon + 1\right) \frac{2\lambda}{1+\lambda} i + o(i). \quad (3)$$

Let $\phi : \mathbb{N} \to \{v_1, v_2, \ldots\}$ be an arbitrary bijection satisfying $\phi([\alpha_j + \beta_j]) = \{r_1, r_2, \ldots, r_{\alpha_j}, b_1, b_2, \ldots, b_{\beta_j}\}$ for every j. The function ϕ defines a colouring of $E(K_\mathbb{N})$, where the colour of the edge ij is the colour of the edge $\phi(i)\phi(j)$.

Let R and B be the sets of positive integers i whose image $\phi(i)$ is red or blue, respectively. Let $H' \subseteq K_\mathbb{N}$ be a monochromatic copy of H in this colouring. Suppose that H' is red. Let n be a

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positive integer, and let $B_n = V(H') \cap [n] \cap B$. Because the vertices of B_n form a monochromatic blue clique in our colouring of $E(K_{\mathbb{N}})$, the set B_n must be independent in H'.

Let j be the minimum value such that $\phi(B \cap [n]) \subseteq \{b_1, b_2, \ldots, b_{\beta_j}\}$. We claim first that there are at least $(1 - O(\epsilon))j$ vertices in $[n]$ which do not belong to H'. Indeed, let $B'_n = V(H') \cap \phi^{-1}(\{b_1, b_2, \ldots, b_{\beta_{j-1}}\}) \subseteq B_n$. From the construction of the colouring, the vertices that are connected to a vertex of $\{b_1, b_2, \ldots, b_{\beta_{j-1}}\}$ through a red edge are precisely the red vertices to the left of $b_{\beta_{j-1}}$, of which there are at most $\lambda(\beta_{j-1} - (j-1))$. This means that $\mu(H, |B'_n|) \leq \lambda(\beta_{j-1} - (j-1))$. For j large enough, this implies $|B'_n| \leq (1 + o(1))(\beta_{j-1} - (j-1))$, and since the vertices in $\phi^{-1}(\{b_1, b_2, \ldots, b_{\beta_{j-1}}\})$ which are not in B'_n are not in $V(H')$,

$$|[n] \setminus V(H')| \geq \beta_{j-1} - |B'_n| \geq (1 + o(1))(j-1) - o(1)\beta_{j-1}.$$

Since $\beta_j \leq \alpha_j + \beta_j = O(j)$ by (3), the right hand side in the inequality above is $(1 - o(1))j$.

Observe next that we cannot have $\beta_j = \beta_{j+1}$. This is because $b_{\beta_j - 1}$, which is to the left[3] of b_{β_j}, has more than $\lambda((\beta_j - 1) - j) = \lambda(\beta_j - (j+1))$ red vertices to its left. We thus have, by minimality of j, that $b_{\beta_{j+1}} \notin \phi([n])$, and by construction of ϕ we have $\phi([n]) \subset \{r_1, r_2, \ldots, r_{\alpha_{j+1}}, b_1, b_2, \ldots, b_{\beta_{j+1}}\}$. This leads to the desired bound:

$$\frac{|V(H') \cap [n]|}{n} \leq 1 - \frac{(1 - o(1))j}{n} \leq 1 - \frac{(1 - o(1))j}{\alpha_{j+1} + \beta_{j+1}} \leq 1 - \frac{1 - o(1)}{\left(\frac{1-\gamma}{2} h(\gamma) + \epsilon + 1\right) \frac{2\lambda}{1+\lambda}}$$

which for ϵ small enough and n large enough can take values arbitrarily close to $f(\lambda)$.

The case in which H' is monochromatic blue is analogous. Indeed, besides the direction of the rounding, it is equivalent to taking the function $-g(x)$ instead of $g(x)$. □

3. General lower bounds

In this section we will prove three lower bounds. One is item 1 from Theorem 1.4, another is the lower bound of item 2b in the same theorem, and the final one is the following, which will be used in the proof of Theorems 4.1 and 1.11:

Theorem 3.1. *Let H be a locally finite graph, a, b, r, s be positive integers with $a > b$, and $\Psi : V(H) \to [a]$ be a proper colouring. Suppose that there exists an infinitely family of pairwise disjoint doubly independent sets $I_1, I_2, \ldots$ in H, not concentrated in fewer than b components and each I_i contained in a single component of H, such that $|I_i| = r$, $|N(I_i)| \leq s$ and $\Psi(v) = a$ for all $v \in N(I_i)$. Then*

$$\rho(H) \geq \frac{b}{a-1} f\left(\frac{s}{r}\right).$$

As an example of a graph whose Ramsey density can be computed from Theorem 3.1, but not from the other lower bounds mentioned in this paper, let $H = b \cdot T + K_a$ be the graph described shortly before the statement of Theorem 1.6. We can define a proper colouring $\Psi : V(H) \to [a]$ in which the vertices of K_a all receive different colours, and the trees T are properly two-coloured with colours $\{1, a\}$. Then for every $r \in \mathbb{N}$ there exist infinitely many pairwise disjoint independent sets I, in every T-component, where $N(I)$ is a single vertex with colour a (just take r leaves of a vertex with label greater than r and colour a). Theorem 3.1 then tells us that $\rho(b \cdot T + K_a) \geq \frac{b}{a-1} f\left(r^{-1}\right)$ for all $r \in \mathbb{N}$, and so $\rho(b \cdot T + K_a) \geq \frac{b}{a-1}$. We will have equality here, as this matches the upper bound from Theorem 1.42b.

Another example is the graph $2 \cdot P_\infty + K_3$ (disjoint union of two infinite paths and a triangle). The graph can be properly coloured with colours $\{1, 2, 3\}$ in a way that both paths use only

[3] We cannot have $\beta_j = 1$ for $j \geq 2$, because then b_1 would have at most $\lambda(1 - j) < 0$ red vertices to its left.

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colours $\{1, 3\}$. Then for every r, each P_∞-component contains infinitely many pairwise disjoint independent sets I with $|I| = r$, $|N(I)| = r + 1$ and $N(I)$ being monochromatic in colour 3 (just take r consecutive vertices receiving colour 1). By Theorem 3.1, we have $\rho(2 \cdot P_\infty + K_3) \geq f\left(\frac{r+1}{r}\right)$ and, by continuity of $f(x)$, we have $\rho(2 \cdot P_\infty + K_3) \geq f(1)$, which matches the upper bound from Theorem 1.6.

However, for every graph for which we know that Theorem 3.1 produces the correct lower bound, we either have $a - 1 = b$ or $s/r \to 0$, as in the two examples above.

We start with the proof of Theorem 1.41. This result follows easily from the infinite version of Ramsey's theorem:

Proof of Theorem 1.41. Let $\chi : E(K_\mathbb{N}) \to \{R, B\}$ be an edge-colouring. Let $\mathcal{F}$ be an inclusion-maximal family of pairwise disjoint monochromatic infinite cliques in χ. Then $\mathbb{N} \setminus V(\mathcal{F})$ is finite, because otherwise by Ramsey's theorem there would be an infinite monochromatic clique in χ restricted to $\mathbb{N} \setminus V(\mathcal{F})$, contradicting the maximality of $\mathcal{F}$. Let $\mathcal{F}_R$ and $\mathcal{F}_B$ be the families of red and blue cliques in $\mathcal{F}$. Since $\bar{d}(V(\mathcal{F}_R) \cup V(\mathcal{F}_B)) = 1$, we have $\max\{\bar{d}(V(\mathcal{F}_R)), \bar{d}(V(\mathcal{F}_B))\} \geq 1/2$. Wlog assume $\bar{d}(\mathcal{F}_R)) \geq 1/2$. We can suppose that $\mathcal{F}_R$ contains infinitely many cliques, because otherwise we can take one clique $K \in \mathcal{F}_R$ and divide it into infinitely many infinite cliques. Let $K_1, K_2, \ldots,$ be the cliques in $\mathcal{F}_R$. We can partition the vertex set of H into infinitely many parts $S_1, S_2, \ldots,$ each of which is made up of infinitely many components of H. Now take any $\Phi : V(H) \to V(K_\mathbb{N})$ which is a bijection from each S_i to each K_i. The image of H is a monochromatic graph H' and $\bar{d}(V(H')) = \bar{d}(\mathcal{F}_R) \geq 1/2$. □

The proofs of Theorems 1.42b and 3.1 will both be (partially) algorithmic: given a colouring $\chi : E(K_\mathbb{N}) \to \{R, B\}$, we will define an algorithm that constructs a dense monochromatic copy of H. The algorithms will be similar, so we will first prove Theorem 3.1 and then explain how to adapt the proof to Theorem 1.42b.

Let H, a, b, r, s, Ψ be as in Theorem 3.1, and let $\chi : E(K_\mathbb{N}) \to \{R, B\}$. Our goal is to find a copy of H in $K_\mathbb{N}$ with density at least $b/(a-1)f(s/r)$. In order to find such a copy of H, it will be helpful to also colour the vertices of $K_\mathbb{N}$, in a way that encodes information about how the vertices are connected through red or blue edges. The following colouring is a variant of one used in [7].

We denote by $N_C(v)$ the set of vertices connected to v through an edge of colour C. When C is a colour that is either red or blue, we denote the other colour by $\bar{C}$.

Definition 3.2. Let $\chi : E(K_\mathbb{N}) \to \{R, B\}$ be a colouring, and let a be a positive integer. An a-good colouring of $V(K_\mathbb{N})$ is a partition $\mathbb{N} = \cup_{i=1}^{a}(R_i \cup B_i) \cup X$ into $2a + 1$ classes (some of which might be empty), where X is finite, with the following properties:

- For every colour $C \in \{R, B\}$, every $1 \leq i \leq a - 1$ and every nonempty finite subset $S \subseteq C_i$, the set $(\cap_{v \in S} N_C(v)) \cap C_i$ is infinite.
- For every colour $C \in \{R, B\}$, every $1 \leq i \leq a - 1$ and every nonempty finite subset $S \subseteq C_a \cup (\cup_{j=i+1}^{a-1} \bar{C}_j)$, the set $(\cap_{v \in S} N_C(v)) \cap \bar{C}_i$ is infinite.

The colouring from [7] is constructed using ultrafilters. We define ours algorithmically, even though ultrafilters would have worked just as well, in order to make the properties of this colouring more intuitive and, in the process, avoiding an appeal to the axiom of choice.

We call each class R_i a *shade of red* and each class B_i a *shade of blue*. X can be seen as a residual set, which can be removed without affecting the density of the graph. The choice of a is related to the chromatic number of the monochromatic subgraphs that we can find in this graph. Indeed, say that we want to find a red clique of size a containing $v \in R_i$. If $i \leq a - 1$, then we can set $v = v_1$ and then iteratively select $v_2, v_3, \ldots, v_a \in R_i$, each adjacent to the previous ones through a red edge. If $i = a$, we can set $v = v_a$ and then iteratively select $v_{a-1}, v_{a-2}, \ldots, v_1$, with $v_j \in B_j$, each adjacent to the previous ones through a red edge.

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We denote by $K_{r,s}^C$ a complete bipartite graph in which all edges have colour C, all vertices in the part of size s have colour C and all vertices in the part of size r have colour $\bar{C}$. These subgraphs will be used to embed the sets $I_i \cup N(I_i)$ in our coloured graph.

The proof of Theorem 3.1 will have three main steps, which are captured by these lemmas:

Lemma 3.3. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be a colouring, and let a be a positive integer. There exists an a-good colouring in which at least two of $(R_a \cup B_{a-1})$, $(B_a \cup R_{a-1})$ and X are empty.*

Lemma 3.4. *Let $\chi : E(K_{\mathbb{N}}) \cup V(K_{\mathbb{N}}) \to \{R, B\}$ be a colouring, and let r, s be positive integers. There exists a colour C and a subgraph $W \subseteq K_{\mathbb{N}}$, with $\bar{d}(W) \geq f(s/r)$, in which every component is either an isolated vertex with colour C, or a $K_{r,s}^C$. Furthermore, if $V(K_{\mathbb{N}})$ is further subdivided into finitely many shades, then W can be taken in such a way that each $K_{r,s}^C$ only uses one shade of each colour.*

Lemma 3.5. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be an edge-colouring, let $a \geq a' \geq b$ be positive integers. Let $\mathbb{N} \to \{R_1, \ldots, R_a, B_1, \ldots, B_a, X\}$ be an a-good colouring in which at most a' shades of each colour are non-empty. Let $W \subseteq K_{\mathbb{N}}$ be a subgraph in which every component is either an isolated vertex with colour C, or a $K_{r,s}^C$ which uses only one shade of each colour. Let H be a graph satisfying the conditions of Theorem 3.1 for some positive integers r and s (except possibly $a > b$). Then there exists a monochromatic $H' \subseteq K_{\mathbb{N}}$ of colour C, $H' \simeq H$, with $\bar{d}(H') \geq b/a'\bar{d}(W)$.*

It is straightforward to combine these three lemmas to deduce Theorem 3.1:

Proof of Theorem 3.1. Let $\chi : E(K_{\mathbb{N}})$ be given. Apply Lemma 3.3 to this edge-colouring to obtain an a-good colouring where at most $a - 1$ shades of each colour are non-empty. Assign the colour red to the vertices in X. Apply Lemma 3.4 to obtain C and W. Remove from W every component which uses a vertex of X (this does not affect $\bar{d}(W)$ because it only removes finitely many vertices). By Lemma 3.5, we can find a monochromatic $H' \subseteq K_{\mathbb{N}}$ with $\bar{d}(H') \geq b/(a-1)\bar{d}(W) \geq b/(a-1)f(s/r)$. □

Proof of Lemma 3.3. For each vertex v, we will denote by $c(v)$ and $s(v)$ the colour and the shade that we assign to it, respectively. The colour assigned to a vertex might change while the algorithm is running, but the shade of each vertex is final once assigned and it will match the colour that the vertex has at that time.

At some points, the shade assigning algorithm will call the basic colouring algorithm to colour an infinite set $V = \{v_1, v_2, \ldots\}$ of vertices. We will first describe this algorithm.

Basic colouring algorithm: First, the colour $c(v_1)$ is assigned, satisfying that $N_{c(v_1)}(v_1) \cap V$ is infinite. Once the colours of $v_1, \ldots, v_{n-1}$ have been assigned, assuming by induction that $(\cap_{i=1}^{n-1} N_{c(v_i)}(v_i)) \cap V$ is infinite, the colour $c(v_n)$ is chosen so that $(\cap_{i=1}^{n} N_{c(v_i)}(v_i)) \cap V$ is infinite.

The colouring produced satisfies that $(\cap_{i=1}^{n} N_{c(v_i)}(v_i)) \cap V$ is infinite for every n. We say that a colour C is *dominant* in this colouring if, for every n, $(\cap_{i=1}^{n} N_{c(v_i)}(v_i)) \cap V$ contains infinitely many vertices v with $c(v) = C$. Observe that at least one of the colours is dominant.

Now we define the shade assigning algorithm:

1. For every $v \in \mathbb{N}$, start with $c(v)$ and $s(v)$ unassigned.
2. If finitely many vertices v remain with $s(v)$ unassigned, assign $s(v) = X$ and END.
3. Let V be the set of vertices without a shade. Colour V with the basic colouring algorithm. Choose a colour C that is dominant. Let i be the minimum value such that C_i is empty. For every $v \in V$ with $c(v) = C$, set $s(v) = C_i$.
4. If $i = a - 1$, set $s(v) = \bar{C}_a$ for every $v \in V$ with $c(v) = \bar{C}$ and END. If $i \neq a - 1$, return to Step 2.

The algorithm runs the loop 2–4 at most $2a-3$ times before ending. Whenever a set C_i with $i \le a-1$ is defined, the colour C is dominant in the corresponding colouring, meaning that in particular $(\cap_{v\in S} N_C(v)) \cap C_i$ is infinite for every finite non-empty $S \subseteq C_i$, as it is a superset of the colour C vertices of $(\cap_{i=1}^n N_{c(v_i)}(v_i)) \cap V$ for n large enough. For the same reason, for any finite subset S of vertices whose shade is not assigned when C_i is defined, we have that $(\cap_{v\in S} N_{\bar{C}}(v)) \cap C_i$ is infinite. If C_a is defined at some point in the algorithm (namely at the end), then $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_{a-1}, C_a$ are defined in this order. This proves that the colouring that we obtained is a-good.

To conclude the proof of Lemma 3.3, simply observe that X is nonempty only if the algorithm terminates at Step 2, the set $(R_a \cup B_{a-1})$ is nonempty only if the algorithm terminates at Step 4 with $C = B$ and $(B_a \cup R_{a-1})$ is nonempty only if the algorithm terminates at Step 4 with $C = R$. □

The proof of Lemma 3.4 divides $K_{\mathbb{N}}$ into infinitely many finite graphs and then combines the regularity lemma and a max flow/min cut argument, to reduce the problem to an optimisation problem equivalent to (1). We will now state the lemmas that we will need for this:

Lemma 3.6 (Regularity Lemma [9]). *For every $\epsilon > 0$ and $m_0, \ell \ge 1$ there exists $M = M(\epsilon, m_0, \ell)$ such that the following holds. Let G be a graph on $n \ge M$ vertices whose edges are coloured in red and blue and let $d > 0$. Let $\{W_i\}_{i\in[\ell]}$ be a partition of $V(G)$. Then there exists a partition $\{V_0, \ldots, V_m\}$ of $V(G)$ and a subgraph H of G with vertex set $V(G) \setminus V_0$ such that the following holds:*

1. $m_0 \le m \le M$;
2. $\{V_i\}_{i\in[m]}$ *refines* $\{W_i \cap V(H)\}_{i\in[\ell]}$;
3. $|V_0| \le \epsilon n$ *and* $|V_1| = \cdots = |V_m| \le \lceil \epsilon n \rceil$;
4. $\deg_H(v) \ge \deg_G(v) - (d+\epsilon)n$ *for each* $v \in V(G) \setminus V_0$;
5. $H[V_i]$ *has no edges for* $i \in [m]$;
6. *all pairs* (V_i, V_j) *are* ϵ*-regular and with density either 0 or at least* d *in each colour in* H.

The max flow-min cut result that we will use can be seen as a weighted version of König's Theorem:

Lemma 3.7. *Let G be a finite bipartite graph on $V = (X, Y)$, and let r, s be positive integers. There exists a unique value of D for which both of these exist:*

- *A function $h : E(G) \to \mathbb{N} \cup \{0\}$ such that $\sum_{e\ni v} h(e) \le r$ if $v \in X$, $\sum_{e\ni v} h(e) \le s$ if $v \in Y$ and $\sum_{e\in E(G)} h(e) = D$.*
- *A vertex cover Z of G such that $r|Z \cap X| + s|Z \cap Y| = D$.*

Proof. Take an orientation of every edge in G from X to Y and give it an infinite capacity. Connect every vertex in X to a source σ through an edge with capacity r, and every vertex in y to a sink τ through an edge with capacity s. Let D be the maximum flow in this network. D is the maximum value for which a function h as in the statement exists (by the integrality theorem, there exists a maximum flow in which the flow of every edge is an integer). D is also the minimum value for which a cut (C_1, C_2) with $\sigma \in C_1$ and $\tau \in C_2$ exists. Observe that (C_1, C_2) is a cut with finite capacity iff $(C_2 \cap X) \cup (C_1 \cap Y)$ is a vertex cover of G, in which case the capacity of the cut is $r|C_2 \cap X| + s|C_1 \cap Y|$. Our lemma follows from the Ford–Fulkerson theorem. □

The next lemma that we will introduce requires the definition of two parameters, which up to a change of coordinates are equivalent to Γ^+_γ and Γ^-_γ. The change of coordinates is a rotation of the axes by 45 degrees, in the following sense: if we denote the 1-Lipschitz function from Definition 1.3 as g and the non-decreasing function below as g', then the point (x, y) is in the graph of g' if and only if $(x+y, x-y)$ is in the graph of g.

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Definition 3.8. Let $g:[0,+\infty)\to[0,+\infty)$ be a continuous, non-decreasing function. Let λ, t be positive real numbers. We define the following two parameters:

$$\ell_\lambda^+(g,t)=\min\left\{x:g(\lambda x)-x\geq t\right\},\quad \ell_\lambda^-(g,t)=\min\left\{x:x-\frac{g(x)}{\lambda}\geq t\right\},$$

where we take the minimum of the empty set to be $+\infty$.

Lemma 3.9. *For $\lambda,\epsilon>0$ there exists $\gamma>0$ with the following property: for every non-decreasing continuous function $g:[0,+\infty)\to[0,+\infty)$ with $g(0)=0$ and every $m>0$ there exists $t\in[\gamma m,m]$ such that*

$$\frac{\ell_\lambda^+(g,t)+\ell_\lambda^-(g,t)}{t}\geq\frac{f(\lambda)}{1-f(\lambda)}-\epsilon.$$

The proof of Lemma 3.9 can be found in the appendix. Combining Lemmas 3.7 and 3.9, we can obtain the following:

Lemma 3.10. *For every $\epsilon,r,s>0$ there exists $\gamma,\eta>0$ and N for which the following hold: for every graph G on $[n]$, with $n>N$ and $\delta(G)\geq(1-\eta)n$, and for every total colouring $\chi:V(G)\cup E(G)\to\{R,B\}$, there exists $t\in[\gamma n,n]$, a colour C, and $h:E(G)\to\mathbb{N}\cup\{0\}$, such that the following hold:*

- *For every edge $e=uv$, if $h(e)>0$ then $\chi(e)=C$ and $\chi(u)\neq\chi(v)$.*
- *$\sum_{e\ni v}h(e)\leq r$ for every v with $\chi(v)=C$ and $\sum_{e\ni v}h(e)\leq s$ for every v with $\chi(v)=\bar{C}$.*
- *$\frac{|C\cap[t]|}{t}+\frac{\sum_{v\in(\bar{C}\cap[t])}\sum_{e\ni v}h(e)}{st}\geq f(s/r)-\epsilon.$*

Proof. Let $\lambda=s/r$. Our constants will follow the hierarchy

$$\eta,N^{-1}\ll\gamma\ll\kappa\ll\xi\ll\epsilon,\lambda.$$

That is, after ϵ and λ are given we pick ξ small enough, after fixing ξ we pick κ small enough, and so on.

For every red vertex v, we define its blue degree $d_B(v)$ as the number of blue vertices w such that vw is blue. Let $v_1,v_2,\ldots,v_{|R|}$ be the set of red vertices, sorted from smallest to largest blue degree, and let $d_i=d_B(v_i)$. Define additionally $d_0=0$ and $d_k=d_{|R|}$ for $k>|R|$. Let $g:[0,+\infty)\to[0,+\infty)$ be the function that satisfies $g(k)=d_k$ for every integer k and which is linear between every pair of consecutive integers.

By Lemma 3.9 there exists $\tau\in[\gamma n,\kappa n]$ for which $\frac{\ell_\lambda^+(g,\tau)+\ell_\lambda^-(g,\tau)}{\tau}\geq\frac{f(\lambda)}{1-f(\lambda)}-\xi$. Adding 1 on each side of the expression, $\frac{\ell_\lambda^+(g,\tau)+\ell_\lambda^-(g,\tau)+\tau}{\tau}\geq\frac{1}{1-f(\lambda)}-\xi$. Let $t=\left(\frac{1}{1-f(\lambda)}-\xi\right)\tau$. Then, since $t\leq\ell_\lambda^+(g,\tau)+\ell_\lambda^-(g,\tau)+\tau$, we have either $|R\cap[t]|<\ell_\lambda^-(g,\tau)$ or $|B\cap[t]|\leq\ell_\lambda^+(g,\tau)+\tau$. We consider both cases, in the former we will have $C=B$ and in the latter (mostly) $C=R$:

Case 1: $|R\cap[t]|<\ell^-(g,\tau)$. Let $R'=R\cap[t]$. Let G' be the graph of blue edges in G between R' and B. Let h, Z and D be as in Lemma 3.7 applied to G', with $X=B$ and $Y=R'$. Suppose that $D\leq s(|R'|-\tau)$. Every vertex $v\in R'\setminus Z$ must have all its blue neighbours in $B\cap Z$, and so $d_B(v)\leq|B\cap Z|$. Therefore

$$d_{|R'|-|Z\cap R'|}\leq|Z\cap B|=\frac{D-s|Z\cap R'|}{r}\leq\frac{s}{r}(|R'|-|Z\cap R'|-\tau).$$

Setting $x=|R'|-|Z\cap R'|$, this expression rearranges to $x-\frac{g(x)}{\lambda}\geq\tau$, so by definition of ℓ_λ^- this means that $x\geq\ell_\lambda^-(g,\tau)$. But this is a contradiction, because $x\leq|R'|<\ell_\lambda^-(g,\tau)$. This means that we have $D>s(|R'|-\tau)$, and

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$$\frac{|B\cap[t]|}{t}+\frac{D}{st}\geq\frac{t-|R'|}{t}+\frac{s(|R'|-\tau)}{st}=1-\frac{\tau}{t}=1-\frac{1}{\frac{1}{1-f(\lambda)}-\xi}\geq f(\lambda)-\epsilon.$$

Case 2: $|B\cap[t]|\leq\ell^+(g,\tau)+\tau$. Let $B'=B\cap[t]$. Let G' be the graph of red edges between R and B'. Let h, Z and D be as in Lemma 3.7 applied to G', with $X=R$ and $Y=B'$. Suppose that $D<s(|B'|-\tau-\eta n-\frac{1}{\lambda})$. Every edge between $R\setminus Z$ and $B'\setminus Z$ is blue. Every vertex v has at most ηn vertices to which it is not connected, and so $d_B(v)\geq|B'\setminus Z|-\eta n$ for all $v\in R\setminus Z$.

If $R\setminus Z$ is empty, then $r|R|\leq D\leq s|B'|\leq st\leq s\frac{\tau}{1-f(\lambda)}\leq s\frac{\kappa}{1-f(\lambda)}n$. This leads to $|R|\leq\frac{\kappa s}{r(1-f(\lambda))}n\leq(1-f(\lambda))n$ and $|B|\geq f(\lambda)n$. In this case we can take $t'=n$, $h=0$ and $C=B$ for Lemma 3.10. Thus we can assume that $R\setminus Z$ is not empty, and so $d_{|R\cap Z|+1}\geq|B'\setminus Z|-\eta n$.

$$\begin{aligned}d_{|R\cap Z|+1}&\geq|B'|-|B'\cap Z|-\eta n\geq|B'|-\frac{D-r|R\cap Z|}{s}-\eta n\\&=\frac{s|B'|-D}{s}+\frac{1}{\lambda}|R\cap Z|-\eta n\geq\tau+\eta n+\frac{1}{\lambda}+\frac{1}{\lambda}|R\cap Z|-\eta n\\&\geq\tau+\frac{1}{\lambda}(|R\cap Z|+1).\end{aligned}$$

Setting $x=\frac{1}{\lambda}(|R\cap Z|+1)$, this expression rearranges to $g(\lambda x)-x\geq\tau$, so by definition of ℓ_λ^+ this means that $x\geq\ell_\lambda^+(g,\tau)$. On the other hand, $x=\frac{|R\cap Z|+1}{\lambda}\leq\frac{D}{s}+\frac{1}{\lambda}<|B'|-\tau-\eta n-\frac{1}{\lambda}+\frac{1}{\lambda}<|B'|-\tau\leq\ell_\lambda^+(g,\tau)$, which is a contradiction. This means that we have $D\geq s(|B'|-\tau-\eta n-\frac{1}{\lambda})$, and

$$\frac{|R\cap[t]|}{t}+\frac{D}{st}\geq\frac{t-|B'|}{t}+\frac{s(|B'|-\tau-\eta n-\frac{1}{\lambda})}{st}\geq1-\frac{\tau}{t}-\frac{\eta}{\gamma}-\frac{1}{\lambda\gamma N}\geq f(\lambda)-\epsilon.$$ □

To prove Lemma 3.4, we apply the regularity lemma to the graph and use Lemma 3.10. We also use the fact that, by the Kővári-Sós-Turán theorem, every large enough dense bipartite graph contains a large complete bipartite subgraph:

Proof of Lemma 3.4. Let $\lambda=s/r$. We first claim that, for every $\epsilon>0$, there exists $\gamma(\epsilon)>0$ and $N(\epsilon)$ such that, for every $n>N$, there exist $t\in[\gamma n,n]$, a colour C and a subgraph $\mathcal{F}\subseteq K_{\mathbb{N}}$ contained in $[n]$ in which every component is either an isolated vertex of colour C or a $K_{r,s}^C$ using only a shade of each colour, with

$$\frac{|V(\mathcal{F})\cap[t]|}{t}\geq f(\lambda)-\epsilon.$$

Let a be the total number of shades (from both colours). Our constants will follow the hierarchy

$$N^{-1}\ll M^{-1}\ll\rho\ll\delta\ll\zeta\ll\gamma,\eta\ll\epsilon,r^{-1},s^{-1},a^{-1}.$$

Let G be the restriction of our colouring to $[n]$. Take a partition of $[n]$ into $\ell=a\lceil\rho^{-1}\rceil$ parts $\{Z_1,\ldots,Z_\ell\}$, such that each Z_i is contained in one shade, and $\max Z_i-\min Z_i<\rho n$. Applying Lemma 3.6 to G with $d=2\delta$, we find a subgraph $H\subseteq G$ and a partition $[n]=\{V_0,V_1,\ldots,V_m\}$, with $\ell\leq m\leq M$, as in the statement of Lemma 3.6, replacing ϵ with δ.

We suppose that the labelling of the parts is such that $\min V_1<\min V_2<\cdots<\min V_m$. We define an auxilliary graph H' as follows: the vertex set is $[m]$. The colour of every vertex i is the same as the colour of each of its vertices in G. Between any two vertices ij, we draw an edge if the bipartite graph V_iV_j is nonempty in H, and we colour it in the most dense colour in V_iV_j.

Let $y=|V_1|=\cdots=|V_m|$. Then $\frac{(1-\delta)n}{m}\leq y\leq\frac{n}{m}$. The minimum degree in H' is at least $(1-\eta)m$. Indeed, given i and $v\in V_i$, we have $d_{H'}(i)\geq\frac{d_H(v)-\delta n}{y}\geq\frac{d_G(v)-4\delta n}{y}\geq(1-\frac{4\delta}{1-\delta})m>(1-\eta)m$.

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Apply Lemma 3.10 to H', with parameters $\epsilon/2, r, s$ to obtain a colour C, a function $h : E(H') \to \mathbb{N}$ and a value $\tau \in [\gamma m, m]$ as in the statement of Lemma 3.10, replacing t with τ. Subdivide each V_i with colour C into r parts $V_{i,1}, \ldots, V_{i,r}$, each of size at least $\lfloor y/r \rfloor$ and each V_i with colour $\bar{C}$ into s parts $V_{i,1}, \ldots, V_{i,s}$, each of size at least $\lfloor y/s \rfloor$. Construct a matching $\mathcal{M}$ of pairs $(V_{i,k}, V_{j,k'})$, where for any fixed values of i and j, the number of pairs $(V_{i,k}, V_{j,k'})$ in $\mathcal{M}$ is $h(ij)$.

Within each pair $(V_{i,k}, V_{j,k'})$, where V_i has colour C and V_j has colour $\bar{C}$, find a maximum family $\mathcal{F}_{i,k,j,k'}$ of disjoint copies of $K^C_{r,s}$. Since $N \gg M, \delta^{-1}, r, s$, and therefore $\delta y \gg r, s$, then $\min\{|V_{i,k} \setminus V(\mathcal{F}_{i,k,j,k'})|, |V_{j,k'} \setminus V(\mathcal{F}_{i,k,j,k'})|\} < \delta y$. That is because otherwise the bipartite graph between $V_{i,k} \setminus V(\mathcal{F}_{i,k,j,k'})$ and $V_{j,k'} \setminus V(\mathcal{F}_{i,k,j,k'})$ would have density at least δ in the edges of colour C, and for δy large enough this implies the existence of a copy of $K^C_{r,s}$, which would contradict the maximality of $\mathcal{F}_{i,k,j,k'}$.

Let $\mathcal{F}$ be the union of all families $\mathcal{F}_{i,k,j,k'}$. Let $t = \min V_\tau$. We will now bound $\frac{|(V(\mathcal{F}) \cup C) \cap [t]|}{t}$. If $v \geq t + \rho n$, and $v \in V_i$ with $i \neq 0$, then $\min V_i > \max V_i - \rho n \geq v - \rho n \geq t = \min V_\tau$, and thus $i > \tau$. This means that $|(\cup_{i=1}^{\tau} V_i) \setminus [t]| \leq \rho n$, and $t \geq \tau y - \rho n \geq \frac{(1-\delta)\tau n}{m} - \rho n$. On the other hand, if $v \leq t$ then either $v \in V_0$ or $v \in V_i$ with $\min V_i \leq v \leq t = \min V_\tau$, and thus $i \leq \tau$. This implies that $t \leq \sum_{i=0}^{\tau} |V_i| \leq \delta n + \tau y \leq \delta n + \frac{\tau n}{m}$.

Every V_i with colour C and $i \in [\tau]$ will trivially be contained in $(V(\mathcal{F}) \cup C) \cap (\cup_{i=1}^{\tau} V_i)$. For any V_i with colour $\bar{C}$ and $i \in [\tau]$, there are $\sum_{e \ni i} h(e)$ parts $V_{i,k}$ which are paired up with a different part $V_{j,k'}$. We either have $|V_{i,k} \setminus V(\mathcal{F})| \leq \delta y$ or $|V_{j,k'} \setminus V(\mathcal{F})| \leq \delta y$. In the first case, $|V_{i,k} \cap V(\mathcal{F})| \geq \lfloor y/s \rfloor - \delta y \geq (1/s - 1/y - \delta)y$. In the second case, $|V_{j,k'} \cap V(\mathcal{F})| \geq \lfloor y/r \rfloor - \delta y \geq (1/r - 1/y - \delta)y$. But $\mathcal{F}$ is a family of copies of $K_{r,s}$, so $|V_{i,k} \cap V(\mathcal{F})| = \frac{r}{s}|V_{j,k'} \cap V(\mathcal{F})| \geq (1/s - \lambda^{-1}(1/y + \delta))y$. In either case we have $|V_{i,k} \cap V(\mathcal{F})| \geq (1 - \zeta)y/s$.

Putting our bounds together:

$$
\begin{aligned}
\frac{|(V(\mathcal{F}) \cup C_G) \cap [t]|}{t} &\geq \frac{|(V(\mathcal{F}) \cup C_G) \cap (\cup_{i=1}^{\tau} V_i)| - \rho n}{t} \\
&\geq \frac{y|C_{H'} \cap [\tau]|}{t} + (1-\zeta)\,\frac{y}{s}\,\frac{\sum_{v \in (\bar{C}_{H'} \cap [\tau])} \sum_{e \ni v} h(e)}{t} - \frac{\rho n}{t} \\
&\geq (1-\zeta)\,\frac{\tau y}{t}\left(\frac{|C_{H'} \cap [\tau]|}{\tau} + \frac{\sum_{v \in (\bar{C}_{H'} \cap [\tau])} \sum_{e \ni v} h(e)}{s\tau}\right) - \frac{\rho n}{t} \\
&\geq (1-\zeta)\,\frac{\tau y}{t}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\rho}{\frac{\tau(1-\delta)}{m} - \rho} \\
&\geq (1-\zeta)\,\frac{\tau y}{\delta n + \tau y}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\rho}{\gamma(1-\delta) - \rho} \\
&\geq (1-\zeta)\,\frac{1}{1 + \delta\frac{n}{my}\frac{m}{\tau}}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{4} \\
&\geq (1-\zeta)\,\frac{1}{1 + \frac{\delta}{(1-\delta)\gamma}}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{4} \\
&\geq f(\lambda) - \epsilon.
\end{aligned}
$$

To conclude the proof of our initial claim, notice that $t \geq \left(\frac{(1-\delta)\tau}{m} - \rho\right) n \geq ((1-\delta)\gamma - \rho)n \geq \gamma' n$ for a constant $\gamma' > 0$.

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We are now ready to construct W. Take a sequence $f(s/r) > \epsilon_1 > \epsilon_2 > \cdots > 0$ with $\epsilon_i \to 0$. Start by applying the claim with $\epsilon = \epsilon_1$ and $n_1 = N(\epsilon)$ to obtain a subgraph $\mathcal{F}_1$ with colour C_1 with density at least $f(s/r) - \epsilon_1$ in $[t_1]$. Now proceed by induction and set $n_i = \max\{N(\epsilon_i/2), 2n_{i-1}(r + s)/(\epsilon_i\gamma(\epsilon_i/2))\}$. Applying the claim with $\epsilon = \epsilon_i/2$ we find a subgraph $\mathcal{F}_i{}'$ with colour C_i contained in $[n_i]$ and with density at least $f(s/r) - \epsilon_i/2$ in $[t_i]$, for some $t_i \in [\gamma(\epsilon)n_i, n_i]$. Remove from $\mathcal{F}_i{}'$ all components that intersect $[n_{i-1}]$ (this represents at most $n_{i-1}(r+s)$ vertices) to obtain $\mathcal{F}_i$. Then $\mathcal{F}_i$ is disjoint from all previous $\mathcal{F}_j$, and by the choice of n_i, it still has density at least $f(s/r) - \epsilon_i$ in $[t_i]$.

Select a colour C such that $C_i = C$ for infinitely many i. Let $W = \cup_{C_i=C}\mathcal{F}_i$. Then by construction $\bar{d}(W) \geq f(s/r)$, since the t_i tend to infinity, and the components of W are isolated vertices of colour C or $K_{r,s}^C$. This concludes the proof of Lemma 3.4. □

Finally, we prove Lemma 3.5 by defining an algorithm that constructs a monochromatic H'. This algorithm uses enough components from W (mapping to them either single vertices of H or sets $I_i \cup N(I_i)$) to keep a fraction of its density and takes advantage of the properties of the a-good colouring to map the remaining vertices of H.

Proof of Lemma 3.5. Without loss of generality, assume that C is red, let S_j denote the vertices in W of shade R_j, plus the blue vertices contained in a copy of $K_{r,s}^R$ in W in which the red side has shade R_j. Removing from W the finite sets S_j does not affect its density, so suppose that each S_j is either empty or infinite. We will show that there exists a set $J \subseteq [a]$, of size b, such that $\bar{d}(\cup_{j\in J} S_j) \geq b/a'\bar{d}(W)$.

By definition of density, there exists a sequence $n_1 < n_2 < \ldots$ of positive integers such that $|V(W) \cap [n_i]|/n_i \to \bar{d}(W)$. For each i there exists a subset $J_i \subseteq [a]$ of b indices such that

$$\frac{|(\cup_{j\in J_i} S_j) \cap [n_i]|}{n_i} \geq \frac{b}{a'}\frac{|V(W) \cap [n_i]|}{n_i}.$$

For infinitely many i, the set J_i is the same, which we denote J. By taking an appropriate subsequence of $n_1, n_2, \ldots$, we can suppose without loss of generality that $J_i = J$ for all i and that $n_{i+1}/n_i \to \infty$. Let $\mathcal{F}_i$ be the union of components from W contained in some S_j with $j \in J$, which contain a vertex from $[n_i]$ but no vertex from $[n_{i-1}]$. Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be the family of doubly independent sets. We can suppose that the elements in $\mathcal{I}$ are such that the sets $I_i \cup N(I_i)$ are pairwise disjoint. Indeed, because H is locally finite, each $I_i \cup N(I_i)$ intersects finitely many sets $I_j \cup N(I_j)$, so we can find an infinite subfamily $\mathcal{I}'$ by including in it only the sets I_i such that $I_i \cup N(I_i)$ does not intersect a set $I_j \cup N(I_j)$ for some $j < i$ with $I_j \in \mathcal{I}'$. This does not change the components in which $\mathcal{I}$ is concentrated.

Let $J' \subseteq J$ be the set of indices in j for which S_j is non-empty. We assign to each component $\mathcal{C} \subseteq H$ a number $\kappa(\mathcal{C}) \in J'$, in such a way that for every $j \in J'$ there are infinitely many sets I_i in components with $\kappa(\mathcal{C}) = j$. Indeed, if finitely many components intersect $\mathcal{I}$, there are at least b components that contain infinitely many elements of $\mathcal{I}$, so give different values of $\kappa(\mathcal{C})$ to $|J'| \leq b$ of them, whereas if there are infinitely many components that intersect $\mathcal{I}$, we can assign each value of J' to infinitely many of them. The purpose of $\kappa(\mathcal{C})$ will be to identify the shade of red to be used in the vertices while embedding $\mathcal{C}$ in the red edges of χ.

We will define an injective graph homomorphism $\Phi: H \to K_{\mathbb{N}}$ which maps edges to red edges, whose image contains $\mathcal{F}_i$ for infinitely many i. This is enough to prove Theorem 3.1, because for infinitely many large enough values of i we have

$$\frac{|\Phi(V(H)) \cap [n_i]|}{n_i} \geq \frac{|V(\mathcal{F}_i) \cap [n_i]|}{n_i} \geq \frac{b}{a'}\frac{|V(W) \cap [n_i]|}{n_i} - \frac{(r+s)n_{i-1}}{n_i}$$
$$\geq \frac{b}{a'}\bar{d}(W) - o(1).$$

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We will define Φ in steps. Recall that Ψ here denotes the proper vertex colouring of H from Theorem 3.1. On every step, we will define the image of finitely many vertices of H. After every step, the following conditions must hold. Let u, v be two adjacent vertices in some component $\mathcal{C}$ of H, such that $\Phi(v)$ is defined and $\Phi(u)$ is not. Then:

- If $\kappa(\mathcal{C}) \neq a$, then $\Phi(v) \in R_{\kappa(\mathcal{C})}$.
- If $\kappa(\mathcal{C}) = a$ and $\Psi(v) = a$, then $\Phi(v) \in R_a$.
- If $\kappa(\mathcal{C}) = a$ and $\Psi(v) \neq a$, then $\Phi(v) \in B_{\Psi(v)}$ and $\Psi(u) < \Psi(v)$.

The algorithm will consist of two operations that alternate: defining the image of a vertex $v \in V(H)$ and adding some $\mathcal{F}_i$ to the image of Φ. If we identify $V(H)$ with $\mathbb{N}$ and always apply the first operation to the least vertex v with undefined $\Phi(v)$, at the end of the algorithm $\Phi(v)$ will be defined for every vertex in $V(H)$.

Define the image of a vertex $v \in V(H)$**:** Suppose first that $v \in \mathcal{C}$ and $\kappa(\mathcal{C}) = k \neq a$. Let $w_1, \ldots, w_q$ be the neighbours of v which have $\Phi(w_i)$ defined. By our invariant, the vertices $\Phi(w_i)$ all have shade R_k, and therefore there are infinitely many vertices x in shade R_k which are connected to every $\Phi(w_i)$ through a red edge. Select one such x which is not yet in the image of Φ and set $\Phi(v) = x$.

Now suppose that $\kappa(\mathcal{C}) = a$. Let $\Psi(v) = k$. For every edge uw of H, define an orientation $\overrightarrow{uw}$ such that $\Psi(u) < \Psi(w)$. Let T be the set of vertices that can be reached from v in this orientation. Because T is connected, does not contain a path of length greater than a, and the degree of every vertex is finite, T is finite. Also, T does not have an oriented cycle. Observe that, by our invariant, if $\overrightarrow{uw}$ is an edge and $\Phi(u)$ is defined, then $\Phi(w)$ is defined.

Now define $\Phi(w)$ for every $w \in T$ for which the image is still undefined, in decreasing order of $\Psi(w)$. If $\Psi(w) = a$, choose an arbitrary vertex $x \in R_a$ which is not yet the image of any vertex and set $\Phi(w) = x$. If $\Psi(w) = k < a$, then for every $w' \in N^+(w)$ the image $\Phi(w')$ is defined and in $B_{k+1} \cup \cdots \cup B_{a-1} \cup R_a$. By the properties of a-good colourings, there are infinitely many vertices[4] $x \in B_k$ which are connected to every $\Phi(w')$ through a red edge. Choose one which is not yet in the image of Φ and set $\Phi(w) = x$.

Add some set $\mathcal{F}_i$ **to the image:** Select some $\mathcal{F}_i$ which is so far disjoint with the image of Φ. For each $K^R_{r,s}$ component $Z \subseteq \mathcal{F}_i \cap S_j$, choose a doubly independent set $I \in \mathcal{I}$ in a component $\mathcal{C}$ with $\kappa(\mathcal{C}) = j$, such that no vertex from $I \cup N(I) \cup N(N(I))$ has a defined image. If $V(Z) = X \cup Y$ with $|X| = r$ blue and $|Y| = s$ red, then set Φ to be bijective from I to X, and injective from $N(I)$ to Y. The vertices v of $\mathcal{F}_i \cap S_j$ that remain outside of the image at this point all have shade R_j. For each of them, choose a vertex w with $\Psi(w) = a$ in a component $\mathcal{C}$ with $\kappa(\mathcal{C}) = j$ (there are infinitely many of these vertices), such that no vertex in $w \cup N(w)$ has a defined image and set $\Phi(w) = v$. After doing this for every isolated vertex in $\mathcal{F}_i \cap S_j$ for every $j \in J'$, the set $\mathcal{F}_i$ is contained in the image.

After both steps are applied alternatingly infinitely many times, the image of Φ is a monochromatic red graph $H' \subseteq K_{\mathbb{N}}$ which contains infinitely many sets $\mathcal{F}_i$, and therefore $\bar{d}(H') \geq b/a' \bar{d}(W)$. □

To prove the lower bound of Theorem 1.42b, we just need the following variant of Lemma 3.5. The proof is then analogous to the proof of Theorem 3.1, except we bypass completely the use of Lemma 3.4, and we only need that in the a-good colouring we have $\max\{\bar{d}(R), \bar{d}(B)\} \geq 1/2$.

Lemma 3.11. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be an edge-colouring, let $a \geq a' \geq b$ be positive integers. Let $\mathbb{N} \to \{R_1, \ldots, R_a, B_1, \ldots, B_a, X\}$ be an a-good colouring in which at most a' shades of each colour are non-empty. Let $C \in \{R, B\}$. Let H be a graph with chromatic number a and at least*

[4]If $N^+(w)$ is empty, how do we know that B_k is infinite? Because $\kappa(\mathcal{C}) = a$, we know that R_a is not empty. By the properties of a-good colourings, every vertex $y \in R_a$ has infinitely many red neighbours in B_k, and in particular B_k is infinite.

b infinite components. Then there exists a monochromatic $H' \subseteq K_{\mathbb{N}}$ of colour C, $H' \simeq H$, with $\bar{d}(H') \geq b/a'\bar{d}(C)$.

Proof. Let $\Psi : V(H) \to [a]$ be a proper colouring, in which in every component of H the most common colour is a. Without loss of generality suppose that C is red. As in the proof of Lemma 3.5, there exists J' with $|J'| \leq b$, such that $\bar{d}(\cup_{j\in J'} R_j) \geq b/a'$, and all R_j with $j \in J'$ are infinite. Let $\mathcal{F} = \cup_{j\in J'} R_j$. Define a function κ from the components of H to J' for which the pre-image of every value contains infinitely many vertices. The algorithm now alternates between **define the image of a vertex** $v \in V(H)$, as above, and **add a vertex of** $\mathcal{F}$ **to the image**. At the end of the procedure, we obtain a red $H' \subseteq K_{\mathbb{N}}$ isomorphic to H which contains $\mathcal{F}$ and thus has density at least $\bar{d}(\mathcal{F}) \geq b/a'\bar{d}(R)$.

Add a vertex of $\mathcal{F}$ **to the image:** Let $v \in \mathcal{F}$ be a vertex in R_j. Choose a vertex w in a component $\mathcal{C}$ with $\kappa(\mathcal{C}) = j$, such that no vertex in $w \cup N(w)$ has a defined image and with $\Psi(w) = a$ and set $\Phi(w) = v$. □

4. Bounds for particular families of graphs

The goal of this section is to prove the remaining results from Section 1. We will start by stating and proving the more general result for bipartite graphs, which will be used to imply Theorem 1.7.

Theorem 4.1. *Let H be a locally finite bipartite graph, and let $\lambda = \liminf_{n\to\infty} \frac{\mu(H,n)}{n}$. Suppose that for every $\lambda' > \lambda$ there exist infinitely many pairwise disjoint independent sets $I_1, I_2, \ldots$, all of the same (finite) size, with $\frac{|N(I_i)|}{|I_i|} \leq \lambda'$. Then $\rho(H) = f(\lambda)$.*

Proof. The upper bound follows from Theorem 1.6. We will show that, for every $\epsilon > 0$, we have $\rho(H) \geq f(\lambda) - \epsilon$. Our goal is to show that H satisfies the condition of Theorem 3.1 for $a = 2$, $b = 1$, a certain colouring Ψ and some doubly independent sets I'_i. Let $\Psi : V(H) \to \{1, 2\}$ be a proper colouring. Choose $\lambda' > \lambda$ such that $f(\lambda') > f(\lambda) - \epsilon$ (it exists by continuity of f). There exist infinitely many pairwise disjoint independent sets I_i, all with the same size, such that $\frac{|N(I_i)|}{|I_i|} \leq \lambda'$ (by the condition from the statement). Partition each set I_i into non-empty sets $I_{i,1}, \ldots, I_{i,k_i}$, where each vertex v is classified according to its colour by Ψ and the component it belongs to. If two vertices v, w have a common neighbour, then they are in the same component and $\Psi(u) = \Psi(v)$. For this reason, $|N(I_i)| = \sum_{j=1}^{k_i} |N(I_{i,j})|$. There exists some τ_i such that

$$\frac{|N(I_{i,\tau_i})|}{|I_{i,\tau_i}|} \leq \frac{\sum_{j=1}^{k_i} |N(I_{i,j})|}{\sum_{j=1}^{k_i} |I_{i,j}|} = \frac{|N(I_i)|}{|I_i|} \leq \lambda'.$$

Set $I'_i = I_{i,\tau_i}$. Set $r_i = |I'_i|$ and $s_i = |N(I'_i)|$. There is a pair (r, s) satisfying $(r, s) = (r_i, s_i)$ for infinitely many values of i. Considering only the values of i for which this equality holds, we have our set of independent sets. Note that, because H is bipartite, $N(I'_i)$ is monochromatic and thus independent, meaning that I'_i is doubly independent. If $\Psi(N(I'_i)) = 2$ does not hold for infinitely many i, replace Ψ with $\bar{\Psi} = 3 - \Psi$. We can now apply Theorem 3.1 to obtain $\rho(H) \geq f(s/r) \geq f(\lambda')$. □

We will prove Theorem 1.7 using Theorem 4.1. To do this we need to show that, in both graphs with infinite orbits and forests, the condition in the statement of Theorem 4.1 holds.

Proof of Theorem 1.7. Let $\lambda = \liminf\limits_{n\to\infty} \frac{\mu(H,n)}{n}$. Fix $\lambda' > \lambda$. We will show that, in both cases, there exist infinitely many pairwise disjoint independent sets $I_1, I_2, \ldots$, all with the same size, with $\frac{|N(I_i)|}{|I_i|} \leq \lambda'$.

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For graphs with infinite orbits: Choose n such that $\frac{\mu(H,n)}{n} < \lambda'$. Let I be an independent set of size n with $|N(I)| = \mu(H, n)$. We will show that there are infinitely many automorphisms $\sigma_i \in \mathrm{Aut}(H)$ such that the sets $\sigma_i(I)$ are pairwise disjoint. Then we can take $I_i = \sigma_i(I)$ to conlude the proof. We proceed by induction on n. For $n = 1$, if $I = \{v\}$, this is equivalent to the orbit of v being infinite.

Suppose that the result is true for $n - 1$. Suppose that we have already found $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that the sets $\sigma_i(I)$ are pairwise disjoint. Let $X = \cup_{i=1}^{k} \sigma_i(I)$. We will construct $\sigma_{k+1} \in \mathrm{Aut}(H)$ such that $\sigma_{k+1}(I)$ is disjoint from X. Choose $v \in I$. By the induction hypothesis, there is an infinite family $\{\tau_i\}_{i=1}^{\infty} \subseteq \mathrm{Aut}(H)$ such that the sets $\tau_i(I - v)$ are pairwise disjoint. If $\tau_i(v) \notin X$ for some i, then we can take $\sigma_{k+1} = \tau_i$, and we are done. Therefore, assume that $\tau_i(v) \in X$ for every i. By pigeonhole principle, there exists w such that $\tau_i(v) = w$ for infinitely many i. Choose $\phi \in \mathrm{Aut(H)}$ such that $\phi(w) \notin X$ (it exists because the orbit of w is infinite). The set $\phi^{-1}(X)$ intersects finitely many sets $\tau_i(I - v)$, therefore there exists some i with $\tau_i(I - v)$ disjoint from $\phi^{-1}(X)$ and $\tau_i(v) = w$. Putting this together, $\phi(\tau_i(I))$ is disjoint from X, as we wanted.

For forests: The following lemma will be used to find independent sets of bounded size with bounded expansion within larger independent sets:

Lemma 4.2. *For every $\lambda' > \lambda$ there exists $M = M(\lambda, \lambda')$ such that, for every independent set I in a forest with $|N(I)| \le \lambda|I|$, there exists $I' \subseteq I$ with $|N(I')| \le \lambda'|I'|$ and $|I'| \le M$.*

Knowing this lemma, choose $\lambda'' < \lambda''' \in (\lambda, \lambda')$ and set $M = M(\lambda''', \lambda')$. Suppose that we have already constructed pairwise disjoint independent sets $I_1, I_2, \ldots, I_k$ with $|I_i| \le M$ and $|N(I_i)| \le \lambda'|I_i|$. We will find a new set I_{k+1}, disjoint from the others. Let $S = \cup_{i=1}^{k} I_i$. There exists n large enough such that $\frac{n}{n-|S|} \le \frac{\lambda'''}{\lambda''}$. By definition of lim inf and $\mu(H, n)$, there exists an independent set I with $|I| \ge n$ and $|N(I)| \le \lambda''|I|$. Then

$$|N(I \setminus S)| \le |N(I)| \le \lambda''|I| \le \lambda''(|I \setminus S| + |S|) \le \lambda'''|I \setminus S|.$$

By our claim, there exists $I_{k+1} \subseteq I \setminus S$ such that $|I_{k+1}| \le M$ and $|N(I_{k+1})| \le \lambda'|I_{k+1}|$. Once we have constructed an infinite family of independent sets $I_1, I_2, \ldots$, simply take a pair (r, s) which is equal to $(|I_i|, |N(I_i)|)$ for infinitely many i (which is possible because this pair can only take finitely many values), and we are done.

Proof of Lemma 4.2. Let $\delta = \delta(\lambda, \lambda') > 0$ be small enough, which we will fix later. Let F be the forest with vertex set $I \cup N(I)$ and containing only the edges between I and $N(I)$ in our original graph. It is enough to prove our result in F. Denote $J = N(I)$. For every component of F take a vertex of I as the root.

There exists a set $S \subseteq V(F)$ with $|S| \le \delta|V(F)|$, satisfying that every component of $F \setminus S$ has size at most δ^{-1}. Indeed, start with $S = \emptyset$ and consider the set U of vertices whose component in $F \setminus S$ contains at least δ^{-1} vertices. The rooted forest structure in F induces a rooted forest structure in $F \setminus S$. Let U' be the set of vertices in $F \setminus S$ which have at least $\delta^{-1} - 1$ descendants. If $U \neq \emptyset$ then $U' \neq \emptyset$, because the root of the largest component will be in U'. Select a minimal vertex v in U' and add it to S. This removes v and all its descendants from U and thus reduces the size of U by at least δ^{-1}. After at most $\delta|V(F)|$ steps, U will be empty.

Let X be the union of S and the parents of the vertices of $S \cap J$. This set has $|X| \le 2|S| \le 2\delta|V(F)|$, and every component T of $F \setminus X$ (which has the structure of a rooted tree) is adjacent to at most one vertex in $X \cap J$, in which case that vertex is the parent (in F) of the root of T. This is because a vertex v in T cannot have a child in $X \cap J$, as that child would be in $S \cap J$ and v would be its parent, and hence in X. As a consequence, every component of $F \setminus (X \cap I)$ contains at most one vertex from $X \cap J$.

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Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be the components of $F \setminus (X \cap I)$. Then

$$\frac{\sum_{j=1}^{k} |C_i \cap J|}{\sum_{j=1}^{k} |C_i \cap I|} = \frac{|J|}{|I| - |X \cap I|} \le \frac{|N(I)|}{|I| - 2\delta(|I| + |N(I)|)} \le \frac{\lambda}{1 - 2\delta(1+\lambda)} =: \lambda''.$$

There exists some component C_i such that $|C_i \cap J| \le \lambda''|C_i \cap I|$. If $C_i \cap I$ has size not greater than $M := 2\delta^{-1}$, then set $I' = C_i \cap I$ and we are done, because $N(I') \subseteq C_i \cap J$. Otherwise C_i has size greater than $2\delta^{-1}$, hence it must contain a (unique) vertex $v \in X \cap J$. Let $C'_1, C'_2, \ldots, C'_r$ be the components obtained from C_i by removing v, labelled in decreasing order of $|C'_j \cap J|/|C'_j \cap I|$. Consider the minimum integer t such that $\sum_{j=1}^{t} |C'_j \cap I| \ge \delta^{-1}$. Because every component in $F \setminus X$ has size at most δ^{-1}, we have $\sum_{j=1}^{t} |C'_j \cap I| \le \sum_{j=1}^{t-1} |C'_j \cap I| + \delta^{-1} \le 2\delta^{-1} = M$. Set $I' = \cup_{j=1}^{t} (C'_i \cap I)$. Then

$$\frac{|N(I')|}{|I'|} = \frac{1 + \sum_{j=1}^{t} |C'_j \cap J|}{\sum_{j=1}^{t} |C'_j \cap I|} \le \delta + \frac{\sum_{j=1}^{r} |C'_j \cap J|}{\sum_{j=1}^{r} |C'_j \cap I|} \le \delta + \lambda''.$$

This proves Lemma 4.2, for $\delta > 0$ small enough such that $\delta + \lambda'' < \lambda'$. □

Next we will prove Corollaries 1.8–1.10 as direct applications of Theorem 1.7:

Proof of Corollary 1.8. We will show that $\mu(T_k, n) = kn$. For every independent set I of size n, the set of children of the vertices of I has size kn and is contained in $N(I)$, thus $|N(I)| \ge kn$. Equality can hold, for example for $I = \{v_1, \ldots, v_n\}$ where v_1 is the root of T_k and v_{i+1} is a grandchild of v_i. We therefore have $\mu(T_k, n) = kn$. Since T_k is a forest, Theorem 1.7 applies and $\rho(T_k) = f(k)$. □

Proof of Corollary 1.9. Let I be an independent set. The set $I + (1, 0, \ldots, 0)$ is contained in $N(I)$, so $|N(I)| \ge |I|$ and $\mu(\mathrm{Grid}_d, n) \ge n$ for all n. On the other hand, let I_k be the set of vertices in $[2k]^d$ with odd sum of coordinates. I_k is an independent set of size $(2k)^d/2$, and $I_k \cup N(I_k)$ is contained in $[2k+2]^d - (1, 1, \ldots, 1)$. Since I_k and $N(I_k)$ are disjoint,

$$\frac{|N(I)|}{|I|} = \frac{|I \cup N(I)|}{|I|} - 1 \le \frac{(2k+2)^d}{(2k)^d/2} - 1,$$

which tends to 1 as $k \to \infty$. We have $\liminf_{n\to\infty} \frac{\mu(\mathrm{Grid_d}, n)}{n} = 1$. The graph Grid_d is vertex-transitive, so by Theorem 1.7 we have $\rho(\mathrm{Grid}_d) = f(1)$. □

Proof of Corollary 1.10. The graph $\omega \cdot F$ satisfies that every orbit of the automorphism group on $V(\omega \cdot F)$ is infinite (because it intersects every copy of F), so we are in the setting of Theorem 1.7. We need to show that $\liminf_{n\to\infty} \frac{\mu(\omega\cdot F, n)}{n} = \min_{I \subseteq V(F) \text{ indep.}} \frac{|N(I)|}{|I|}$.

Let I be an independent set in F that minimises $\frac{|N(I)|}{|I|}$, and let $J \subseteq V(\omega \cdot F)$ be an independent set of size n. Partition J into independent sets $J_1, J_2, \ldots, J_m$, according to the component in which the vertices are contained. Then

$$\frac{|N(J)|}{|J|} = \frac{\sum_{i=1}^{m} |N(J_i)|}{\sum_{i=1}^{m} |J_i|} \ge \min \frac{|N(J_i)|}{|J_i|} \ge \frac{|N(I)|}{|I|}.$$

This implies that $\frac{\mu(\omega\cdot F, n)}{n} \ge \frac{|N(I)|}{|I|}$. Equality holds infinitely many times, since for all n divisible by I we can take the union of the sets I in $n/|I|$ different copies of F. Therefore $\rho(\omega \cdot F) = f\left(\liminf_{n\to\infty} \frac{\mu(\omega\cdot F, n)}{n}\right) = f\left(\frac{|N(I)|}{|I|}\right)$.

In an even cycle C_{2k}, each independent set satisfies $|N(I)| \ge |I|$, because C_{2k} contains a perfect matching. Since each chromatic class in the bipartition satisfies $|N(I)| = |I|$, we have $\rho(\omega \cdot C_{2k}) = f(1)$. For $1 \le a \le b$, in $K_{a,b}$, every independent set has size at most b, and its neighbourhood has size at least a. Both inequalities are sharp if I is the side of the bipartition with size b. Thus $\frac{|N(I)|}{|I|} \geq \frac{a}{b}$, and $\rho(\omega \cdot K_{a,b}) = f(a/b)$. □

Next we will deduce Theorem 1.11 from Theorem 3.1:

Proof of Theorem 1.11. Let $a = |V(F)|$, and let $b = a - 1$. Let $\Psi : V(F) \to [a]$ be a colouring that assigns the value a to every vertex in $N(I)$, and where the remaining vertices in F all get different values in $[a-1]$. Because I is doubly independent, this is a proper colouring. Ψ extends to a colouring of $\omega \cdot F$, by colouring all copies of F equally.

Let $I_1, I_2, \ldots$ be the sets I of all copies of F. Each I_i is contained in a component of F, $\Psi(N(I_i)) = a$ and the family of sets I_i is not concentrated in fewer than b components. Thus, by Theorem 3.1, setting $r = |I|$ and $s = |N(I)|$, we have $\rho(\omega \cdot F) \geq f\left(\frac{|N(I)|}{|I|}\right)$. □

Finally, we will prove Theorem 1.12 using a result of Burr, Erdős and Spencer [3] for the Ramsey number of $n \cdot F$:

Theorem 4.3 ([3]). *Let F_1, F_2 be two finite graphs without isolated vertices. The two-colour Ramsey number $R(n \cdot F_1, n \cdot F_2)$ satisfies*

$$R(n \cdot F_1, n \cdot F_2) = (|V(F_1)| + |V(F_2)| - \min\{\alpha(F_1), \alpha(F_2)\})n + O(1),$$

where $\alpha(G)$ is the size of the largest independent set in G. In particular, $R(n \cdot F, n \cdot F) = (2|V(F)| - \alpha(F))n + O(1)$.

Proof of Theorem 1.12. Let $\chi : E(K_{\mathbb{N}}) \to \{R.B\}$ be a colouring. Let $n_1, n_2, \ldots$ be an increasing sequence of positive integers with $n_{i+1}/n_i \to \infty$. Let k_i be the maximum value such that $R(k_i \cdot F, k_i \cdot F) \leq n_{i+1} - n_i$. By Theorem 4.3, we have

$$k_i = \left(\frac{1}{2|V(F)| - \alpha(F)} + o(1)\right)(n_{i+1} - n_i) = \left(\frac{1}{2|V(F)| - \alpha(F)} + o(1)\right) n_{i+1}.$$

There exist a family $\mathcal{F}_i$ of k_i monochromatic disjoint copies of F with vertices in $[n_i + 1, n_{n+1}]$, all with the same colour C_i. Choose a colour C which is equal to C_i for infinitely many i. Then $H' = \cup_{C_i = C} \mathcal{F}_i$ is a copy of $\omega \cdot F$ with

$$\limsup_{n\to\infty} \frac{|V(H) \cap [n]|}{n} \geq \limsup_{i: C_i = C} \frac{k_i |V(F)|}{n_{i+1}} = \frac{|V(F)|}{2|V(F)| - \alpha(F)}.$$

□

5. Open problems and remarks

In a previous version of this paper, we asked to improve the bounds on $\rho(\omega \cdot K_3)$: using the results in this paper, from Theorem 1.12 we find a lower bound of 3/5, and from Theorem 1.6 we find an upper bound of $f(2) \leq (21 + \sqrt{12})/33 \approx 0.74133$. This was answered in a later paper of Balogh and the author [1], which shows that $\rho(\omega \cdot K_3) = f(2)$, and proves an explicit lower bound $f(2) \geq 1 - 1/\sqrt{7} \approx 0.62203$.

As noted in the introduction, f depends on the solution of a certain optimisation problem of Lipschitz functions. It would be helpful to remove such dependency and obtain a closed formula for f (perhaps the upper bound of (2)). In particular, if the upper bound of (2) is sharp then we can observe that the behaviour of $f(x)$ changes at $x = 3$, which corresponds to $\limsup \mu(H, n)/n = 3$ (the independent sets have similar expansion ratio as in the infinite ternary graph). The cause of this is that the optimal colouring of $K_{\mathbb{N}}$ changes.

Problem 5.1. Find a closed formula for $f(x)$. In particular, prove or disprove that it matches the upper bound from (2).

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References

[1] Balogh, J. and Lamaison, A. (2020) Ramsey upper density of infinite graph factors. https://arxiv.org/abs/2010.13633

[2] Burke, J. R. and Webb, W. A. (1981) Asymptotic behavior of linear recurrences. *Fibonacci Quart.* **19**(4) 318–321.

[3] Burr, S. A., Erdős, P. and Spencer, J. H. (1975) Ramsey theorems for multiple copies of graphs. *Trans. Amer. Math. Soc.* **209** 87–99.

[4] Corsten, J., DeBiasio, L., Lamaison, A. and Lang, R. (2019) Upper density of monochromatic infinite paths. *Adv. Comb* **4** 16.

[5] Corsten, J., DeBiasio, L. and McKenney, P. (2020) Density of monochromatic infinite subgraphs ii. https://arxiv.org/abs/2007.14277

[6] DeBiasio, L. and McKenney, P. (2019) Density of monochromatic infinite subgraphs. *Combinatorica* **39**(4) 847–878.

[7] Elekes, M., Soukup, D. T., Soukup, L. and Szentmiklóssy, Z. (2017) Decompositions of edge-colored infinite complete graphs into monochromatic paths. *Discrete Math.* **340**(8) 2053–2069.

[8] Erdős, P. and Galvin, F. (1993) Monochromatic infinite paths. *Discrete Math.* **113**(1-3) 59–70.

[9] Komlós, J. and Simonovits, M. (1996) Szemerédi's Regularity Lemma and its applications in graph theory, *Combinatorics, Paul Erdős is Eighty*, Vol. 2, Miklós, D., Sós, V. T. and Szőnyi, T., Bolyai Society Mathematical Studies, pp. 295–352.

[10] Lo, A., Sanhueza-Matamala, N. and Wang, G. (2018) Density of monochromatic infinite paths. *Electron. J. Combin.* **25**(4) #P4.29.

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