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SOME POINTWISE CONVERGENCE RESULTS IN $L^p(\mu)$, $1<p<\infty$

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Introduction. The theory of almost everywhere convergence has its roots in the poineering work of A. Kolmogorov, and today it constitutes one of the most captivating and challenging chapters in modern probability theory and analysis. Whereas some modes of convergence for sequences of measurable functions, e.g. convergence in norm, can be readily obtained by an intelligent exploitation of the various properties of the function spaces involved, a.e. convergence invariably requires a rather high, and sometimes surprising, degree of mathematical virtuosity. That this is indeed the case can easily be seen in some of the early work in the subject: Kolmogorov's study of sums of independent random variables, and in particular his strong law of large numbers; G. Birkhoff's celebrated ergodic theorem; and finally the famous martingale convergence theorem of J. L. Doob. The theory took a different turn in the 1950's which the work of E. Hopf [12], and later N. Dunford and J. Schwartz [10], who succeeded in proving pointwise ergodic theorems for contraction operators on the function space, $L^1(\mu)$ thus paving the way for the modern theory as we know it today. These early results were characterized by the fact that the operators in question were contractions on $L^\infty(\mu)$ as well, and subsequent work of R. V. Chacon [5] showed that additional properties were in fact necessary for a pointwise ergodic theorem in $L^1(\mu)$. In the case of $L^p(\mu)$, $1<p<\infty$, however, the situation is quite different in that the mean ergodic theorem holds in these spaces as well as some other desirable properties. It is not too surprising then that much of the recent work in a.e. convergence, and ergodic theory in particular, his taken place in these spaces, and the exposition of some of this work is the goal of the present article. We will thus confine ourselves to some pointwise convergence results in $L^p(\mu)$, $1<p<\infty$, culminating in the remarkable work in M. A. Akcoglu. Other important topics, in particular the Chacon–Ornstein theorem and the work of L. Carleson, receive a beautiful treatment in the recent book of A. Garsia [11].

General principles. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space which we assume to be finite, although many of the results are valid in the case $\mu(\Omega)=\infty$. Denote by

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$L^p(\mu)$, $1 \le p \le \infty$, the usual Banach space of real-valued measurable functions on Ω with norm $\|f\|_p = (\int |f|^p \, d\mu)^{1/p}$ for $1 \le p < \infty$, $\|f\|_\infty = \text{ess sup}\,|f(\omega)|$. We consider a sequence of bounded operators $\{T_k\}$ on $L^p(\mu)$. The basic problem in the theory of a.e. convergence is to find conditions on the operators T_k such that for each $f \in L^p(\mu)$ the sequence $\{T_k f\}$ converges a.e. Central to this study is the ergodic theory associated with a contraction operator on $L^p(\mu)$. Recall that a linear operator $T: L^p(\mu) \to L^p(\mu)$ is said to be a contraction if $\|T\|_p \le 1$, i.e. $\int |Tf|^p \le \int |f|^p$ for all $f \in L^p(\mu)$. For such an operator define for $k \ge 1$,

$$T_k = S_k(T) = \frac{1}{k} \sum_{j=0}^{k-1} T^j,$$

the averages of T. We say that the pointwise ergodic theorem holds for T if the sequence $\{S_k(T)f\}_{k \ge 1}$ converges a.e. for each $f \in L^p(\mu)$. The first general result of this type was the pointwise ergodic theorem of G. Birkhoff for contractions induced by measure preserving transformations, i.e. operators $T = T_\varphi$ of the form $T_\varphi f = f(\varphi)$ where $\varphi: \Omega \to \Omega$ is measurable and satisfies $\mu(\varphi^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$. Note that the operator T_φ has other desirable properties: T_φ is a contraction on all $L^p(\mu)$, $1 \le p \le \infty$, and T is positive, i.e. $T_\varphi f \ge 0$ a.e. whenever $f \ge 0$ a.e. These two properties played an important role in the subsequent development of the theory.

The first tractable proof of Birkhoff's theorem seems to be due to F. Riesz [19] and relies heavily on the particular form of the operator. Nowadays ergodic theory is approached by a different method using an important principle of Banach which we state as follows:

BANACH'S PRINCIPLE [Cf. 11]. Let $\{T_k\}$ be a sequence of bounded operators on $L^p(\mu)$, $1 \le p < \infty$. If $\sup_k |T_k f| < \infty$ a.e. for each $f \in L^p(\mu)$, then the set of $f \in L^p(\mu)$ for which $\{T_k f\}$ converges a.e. is closed in $L^p(\mu)$.

In the cases of greatest interest one can prove pointwise convergence on a dense set of functions in $L^p(\mu)$ and therefore Banach's principle becomes particularly useful. In fact, we have the following

THEOREM 1. *Let $T: L^p(\mu) \to L^p(\mu)$ be a contraction, $1 < p < \infty$. Then*

(i) *There is a contraction P on $L^p(\mu)$ satisfying $P = P^2 = TP = PT$ and such that $S_k(T)f \to Pf$ in $L^p(\mu)$ for each $f \in L^p(\mu)$.*

(ii) $S_k f \to Pf$ a.e. *on a dense set of $f \in L^p(\mu)$.*

Statement (i) is the so-called mean ergodic theorem a proof of which can be found in [21]. A proof of statement (ii) is contained in [13]. It is important to emphasize here that $1 < p < \infty$. Indeed, reflexivity plays an important role in the proofs, and the corresponding statements are false for $p = 1$ and $p = \infty$.

Combining Theorem 1 with Banach's principle we see that it suffices to prove that $\sup_k |T_k f| < \infty$ a.e. for each $f \in L^p(\mu)$ in order to prove the pointwise ergodic theorem for a contraction T on $L^p(\mu)$, $1 < p < \infty$. In actual practice, however, one usually proves a great deal more, at least in the reflexive case.

DEFINITION 1. A sequence of bounded operators $\{T_k\}$ on $L^p(\mu)$, $1 < p < \infty$, is said to satisfy a dominated estimate if there is a constant $C_p < \infty$ such that $\int (\sup_k |T_k f|)^p \le C_p \int |f|^p$ for all $f \in L^p(\mu)$.

In the case $T_k = S_k(T)$ where T is a contraction on $L^1(\mu)$ and $L^\infty(\mu)$, hence on all $L^p(\mu)$, $1 \le p \le \infty$ (theorem of M. Riesz. Cf. [11]) dominated estimates have been established with $C_p = (p/p-1)^p$. Since a dominated estimate involves the norms in the spaces $L^p(\mu)$, one would hope to somehow exploit the geometry of these spaces to obtain such a result. Or we might seek other conditions of a geometrical nature which are strong enough to give us pointwise convergence.

To this end let $\{T_k\}$ be a sequence of bounded operators on $L^p(\mu)$ and let $f \in L^p(\mu)$. Define for $k \ge 1$ and $n \ge 0$ the sets $B_k(n) = B_k(n, f) = \{T_1 f \le n, T_2 f \le n, \ldots, T_k f > n\}$. Note that the sets $\{B_k(n)\}_k$ are disjoint for each $n \ge 0$ and $\sum_{k=1}^N B_k(n) = \{\max_{1 \le k \le N} T_k f > n\}$. Since $T_k f > n$ on $B_k(n)$ we have $n\mu(\max_{1 \le k \le N} T_k f > n) \le \sum_{k=1}^N \int_{B_k(n)} T_k f$. This simple fact motivates the following

DEFINITION 2. Let $\{T_k\}_{k \ge 1}$ be a sequence of bounded operators on $L^p(\mu)$, $1 \le p < \infty$. We say that he sequence satisfies the dual maximum principle ($(DM)p$) if there is a constant $C_p < \infty$ such that $\|\sum_{k=1}^N T_k^* I_{B_k}\|_q \le C_p$ for all sequences $\{B_k\}_k$ of disjoint sets and for all integers $N \ge 1$.

Here the operator T_k^* on $L^q(\mu)$ is the adjoint of $T_k (1/p + 1/q = 1)$ and I_A is the indicator function of $A \in \mathcal{F}$.

It now follows that if the sequence $\{T_k\}$ satisfies $(DM)_p$, then $\sup_k |T_k f| < \infty$ a.e. for each $f \in L^p(\mu)$. Indeed, for each n we have

$$\mu\left(\max_{1 \le k \le N} T_k f > n\right) \le \frac{1}{n} \sum_{k=1}^N \int_{B_k(n)} T_k f = \frac{1}{n} \int f \sum_{k=1}^N T_k^* I_{B_k(n)}$$

$$\le \frac{1}{n} \|f\|_p \left\| \sum_{k=1}^N T_k^* I_{B_k} \right\|_q \le \frac{1}{n} \|f\|_p C_p$$

from Holder's inequality. Letting $N \to \infty$, we obtain

$$\mu\left(\sup_k T_k f > n\right) \le \frac{1}{n} \|f\|_p C_p; \quad \text{hence} \quad \mu\left(\sup_k T_k f = \infty\right)$$

$$= \lim_{n \to \infty} \mu\left(\sup_k T_k f > n\right) \le \lim_{n \to \infty} \frac{1}{n} \|f\|_p C_p = 0.$$

Similarly, replacing f by $-f$ yields

$$\mu\left(\sup_k T_k(-f)=\infty\right)=\mu(\inf T_kf=-\infty)=0$$

and therefore $\mu(\sup_k |T_kf|=\infty)=0$.

In actual fact $(DM)_p$ is equivalent to a stronger result as the following proposition shows [9]:

PROPOSITION 1. *The sequence* $\{T_k\}$ *of bounded operators on* $L^p(\mu)$, $1\le p<\infty$, *satisfies* $(DM)_p$ *if and only if* $\sup_{\|f\|_p\le 1}\int \sup_k |T_kf|<\infty$.

Thus far the main application of the dual maximum principle has been in $L^2(\mu)$ and we turn now to this case.

Pointwise convergence in $L^2(\mu)$. Because of the regular geometric properties of the Hilbert space $L^2(\mu)$ it is perhaps not too surprising that the first pointwise convergence results were proven in these spaces. Here the notions of self-adjoint and unitary operators play a fundamental role and it is for these operators that a pointwise ergodic theorem was first established by E.M. Stein. On the other hand, there is another important sequence of operators which is frequently studied, that of increasing sequences of orthogonal projections, i.e. operators $\{P_k\}$ which satisfy $P_k=P_k^*$ and $P_kP_n=P_{k\wedge n}$ where $k\wedge n=\min(k,n)$. In $L^2(\mu)$ Birkhoff's ergodic theorem and the martingale convergence theorem take very appealing forms: the operator T_φ defines a unitary operator so that Birkhoff's theorem becomes a pointwise ergodic theorem for a certain class of unitary operators in $L^2(\mu)$; and the martingale convergence theorem becomes a pointwise convergence theorem for a certain class of increasing sequences of orthogonal projections. Moreover, these operators share another important property, that of positivity. Therefore a first step in the generalization of the Birkhoff and martingale theorems in $L^2(\mu)$ would be to consider positive operators, and such a generalization was obtained by E. M. Stein [20].

THEOREM 2. (i) *Let* T *be a positive contraction on* $L^2(\mu)$ *which is self-adjoint or unitary. Then* $\{S_k(T)f\}_k$ *converges* a.e. *for all* $f\in L^2(\mu)$.

(ii) *Let* $\{P_k\}$ *be an increasing sequence of positive orthogonal projections on* $L^2(\mu)$. *Then* $\{P_kf\}$ *converges* a.e. *for all* $f\in L^2(\mu)$.

Stein established a dominated estimate for the sequence $\{S_k(T)\}_k$ in the self-adjoint case and from this he obtained a dominated estimate for the iterates $\{T^k\}_{k\ge 0}$ via the spectral theorem. As for (ii) we can give a quick proof as follows: Denote by P the (positive) strong limit of P_k and let $T_k=P-P_k$. Then $\{T_k\}$ forms a decreasing sequence of orthogonal projections and to prove a.e. convergence it clearly suffices to prove that $\{T_k\}$ satisfies $(DM)_2$. But

$$\begin{aligned}\int\left(\sum_{k=1}^{N} T_kI_{B_k}\right)^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\int T_iI_{B_i}T_jI_{B_j} = \sum_{i=1}^{N}\sum_{j=1}^{N}\int_{B_i} T_{i\vee j}I_{B_j}\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\int_{B_i}(P-P_{i\vee j})I_{B_j} \le \sum_{i=1}^{N}\sum_{j=1}^{N}\int_{B_i} PI_{B_j}\\ &= \int\left(P\sum_{k=1}^{N} I_{B_k}\right)^2 \le \int (P1)^2 \equiv C\end{aligned}$$

independent of N and the sequence $\{B_k\}$ of disjoint sets.

Of course, one would like to prove Theorem 2 without the positivity hypothesis but, unfortunately, all of the above statements would be false in this case as shown by the early work of D. Menshov [18] and more recently by D. L. Burkholder [4] and M. A. Akcoglu and L. Sucheston [1]. In other words, if one drops positivity another condition must be added. Note that the operators that figure in the Birkhoff and martingale theorems are defined on $L^1(\mu)$ and are contractions on this space. We now give a definition introduced in [9] which we state in general form although it will be applied in $L^2(\mu)$ only.

Definition 3. (i) Let $\{T_k\}$ be a sequence of bounded operators on $L^p(\mu)$, $1\le p<\infty$. The sequence is said to be uniformly absolutely continuous (u.a.c.) if there is an integrable function $g\ge 0$ such that

$$\int |T_kf| \le \int g\,|f| \quad \text{for all } f\in L^\infty(\mu).$$

(ii) A bounded operator T on $L^p(\mu)$ is u.a.c. if the sequence $\{T^k\}_{k\ge 0}$ is u.a.c.

Property u.a.c. is weaker than the $L^1(\mu)$-contraction property and is essentially a condition on the adjoint sequence $\{T_k^*\}$. In fact, it is not difficult to see that $\{T_k\}$ is u.a.c. if and only if there is a $g\in L^1(\mu)$ such that

$$\sup_k |T_k^*h| \le g \quad \text{a.e. for all} \quad h\in L^\infty(\mu) \quad \text{satisfying} \quad \|h\|_\infty \le 1.$$

Hence if, for example, $\|T_k^*\|_\infty \le M<\infty$ for all $k\ge 1$, then $\{T_k\}$ is u.a.c. Also, a sequence of positive operators $\{T_k\}$ is u.a.c. if and only if $\sup_k T_k^*1$ is integrable. We have now the following

Theorem 3 [9]. (i) *Let T be a contraction on $L^2(\mu)$. Suppose that either* (a) *T is self-adjoint and* u.a.c., *or* (b) *T is unitary and $T^*=T^{-1}$ is* u.a.c. *Then $\{S_k(T)f\}$ converges* a.e. *for all $f\in L^2(\mu)$*

(ii) *Let $\{P_k\}$ be an increasing sequence of orthogonal projections which is* u.a.c. *Then $\{P_kf\}$ converges* a.e. *for all $f\in L^2(\mu)$.*

The fact that one need only suppose that T^{-1} is u.a. c. in the unitary case was pointed out to the author by Louis-Paul Rivest. The proof of the above theorem consists of verifying $(DM)_2$ and uses a combinatorial argument due to

Stein. The result contains the important theorem of S. Kaczmarz [14] for orthogonal projections of the form $P_kf=\sum_{j=0}^{k}\alpha_j(f)\varphi_j$ whose Lebesgue functions are bounded. Here $\{\varphi_i\}$ is an orthonormal sequence of functions and $\alpha_j(f)=\int f\varphi_j$ is the usual Fourier coefficient. It is interesting to note that for sequences of projections of this form $(DM)_2$ is equivalent to the following property: $\exists C<\infty$ such that for all increasing sequences $A_j\nearrow$ of sets in $\mathcal{F}$, $\sum_j \alpha_j(I_{A_j})^2\le C$, i.e. $\exists f\in L^2(\mu)$ such that $\|f\|_2^2\le C$ and $\int f\varphi_j=\int_{A_j}\varphi_j$ for all $j\ge 0$.

We close this section with a pointwise convergence result of another type. Let $\{P_k\}$ be an increasing sequence of orthogonal projections on $L^2(\mu)$ and let

$$A_k=\frac{1}{k}\sum_{j=0}^{k-1}P_j.$$

It may be that the operators A_k are better behaved than $\{P_k\}$. For example, in the case of the trigonometric series on $L^2([-\pi,\pi])$ the operators A_k are positive and $L^\infty(\mu)$-contractions, and the famous Fejer–Lebesgue theorem states that $A_kf\to f$ a.e. for all $f\in L^1([-\pi,\pi])$. See also Kaczmarz [14]. For functions in $L^2(\mu)$ this result is contained in the following general theorem due to S. Lessard [16].

Theorem 4. *Let $\{P_k\}$ be an increasing sequence of orthogonal projections on $L^2(\mu)$. If the sequence $\{A_k\}$ is* u.a.c., *or if the operators A_k are positive, then $\{A_kf\}$ converges* a.e. *for each $f\in L^2(\mu)$.*

The proof again is a verification of $(DM)_2$ in the u.a.c. case. In the positive case it is an adaptation of E. M. Stein's technique [20].

The theorem of M. A. Akcoglu. In this section we assume that the measure space $(\Omega,\mathcal{F},\mu)$ is σ-finite. Since we now know that the pointwise ergodic theorem does not hold for arbitrary contractions on $L^2(\mu)$, the question arises as to what type of additional assumptions one would need for such a theorem in $L^p(\mu)$, $1<p<\infty$. The most natural property is that of positivity and most of the recent results centered around proving the pointwise ergodic theorem in this case (the theorems are, incidentally, false for $p=1$ and $p=\infty$). If T is a contraction on all $L^p(\mu)$ spaces the result is true even if T is not positive (theorem of Dunford and Schwartz. Cf. [10]. See also Hopf [12]). An important step was taken by A. Ionescu Tulcea [13] who proved a dominated estimate for positive contractions on $L^p(\mu)$, $1<p<\infty$, which are invertible isometries, and for nonpositive invertible isometries if $p\neq 2$. The proof uses a characterization of such operators by J. Lamperti [15] and employs an approximation argument which brings into play dominated estimates for operators of the form T_φ. A. de la Torre [7] has a simple proof of the dominated estimates for positive contractions which are invertible using the Hardy–Littlewood maximal function. Using the result of Ionescu Tulcea, R. V. Chacon, and J. Olsen [6] proved a dominated estimate for positive contractions T which have a strictly positive invariant function, i.e. a function $f>0$ a.e. such that $Tf=f$ a.e. But the general theorem remained elusive and resisted all attempts at proof until the remarkable result of M. A. Akcoglu [2].

THEOREM 5. *Let T be a positive $L^p(\mu)$-contraction, $1<p<\infty$. Then $\{S_k(T)f\}_k$ converges* a.e. *for each* $f\in L^p(\mu)$.

Akcoglu's proof of this theorem is ingenious: he first establishes a dominated estimate for positive contractions acting on a finite dimensional L^p space by showing that such operators can be dilated to a positive invertible isometry and then appealing to the Ionescu Tulcea result. Use is then made of conditional expectation operators to obtain the general result.

The outstanding open problem in the theory today is to find minimal conditions on a nonpositive contraction on $L^p(\mu)$, $1<p<\infty$, which are enough to give a pointwise ergodic theorem. The simplest case would be to prove such a theorem for an $L^2(\mu)$-contraction which is also a contraction on $L^\infty(\mu)$, or for which T^* is simply u.a.c., as is the case for unitary operators. It is conjectured that if T is an arbitrary $L^p(\mu)$ contraction such that T^* is u.a.c. then the pointwise ergodic theorem holds for T. A more brazen conjecture would be that for $p\neq 2$, the pointwise ergodic theorem holds for all $L^p(\mu)$-contractions, $1<p<\infty$.

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