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Holomorphic Torus Actions on Compact Locally Conformal Kähler Manifolds

Dedicated to the memory of Professor Katsuo Kawakubo

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Abstract. Given a torus action (T^2, M) on a smooth manifold, the orbit map $ev_x(t) = t \cdot x$ for each $x \in M$ induces a homomorphism $ev_*: \mathbb{Z}^2 \to H_1(M; \mathbb{Z})$. The action is said to be Rank-k if the image of ev_* has rank k ($\leqslant 2$) for each point of M. In particular, if ev_* is a monomorphism, then the action is called homologically injective. It is known that a holomorphic complex torus action on a compact Kähler manifold is homologically injective. We study holomorphic complex torus actions on compact non-Kähler Hermitian manifolds. A Hermitian manifold is said to be a locally conformal Kähler if a lift of the metric to the universal covering space is conformal to a Kähler metric. We shall prove that a holomorphic conformal complex torus action on a compact locally conformal Kähler manifold M is Rank-1 provided that M has no Kähler structure.

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1. Introduction

We study compact non-Kähler Hermitian manifolds which admit holomorphic complex torus actions. By a holomorphic complex torus action $\mu : T^1_{\mathbb{C}} \times M \to M$ we mean an effective action of a complex torus $T^1_{\mathbb{C}}$ on a complex manifold M such that the action μ is holomorphic. For each $x \in M$, the orbit map $ev_x : T^1_{\mathbb{C}} \longrightarrow M$ defined by $ev_x(t) = t \cdot x$ induces a homology homomorphism $ev_* : \mathbb{Z}^2 \to H_1(M; \mathbb{Z})$. If the image of ev_* has rank k ($\leqslant 2$) for each $x \in M$, we call it a Rank-k torus action. Especially when Rank $= 2$, i.e. ev_* is injective, the action is said to be homologically injective. The rank of the torus action is motivated by the work of Harvey and Lawson [6]. It is known that every holomorphically isometric action of a complex torus on a compact Kähler manifold is homologically injective (for example, [2]). The purpose of this note is to study which kind of holomorphic complex torus actions occur on compact non-Kähler Hermitian manifolds. A Hermitian manifold M is called a locally conformal Kähler manifold if M admits a Hermitian metric whose

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lift to the universal covering space $\tilde{M}$ is conformal to a Kähler metric on $\tilde{M}$. We refer to [5] for the detail of locally conformal Kähler manifolds. For example, some of compact elliptic surfaces and Inoue surfaces are such locally conformal Kähler manifolds. We shall prove the following main theorem:

THEOREM. *Let $(T^1_{\mathbb{C}}, M, g)$ be a holomorphic complex torus action on a compact locally conformal Kähler manifold M acting as a group of conformal transformations with respect to the Hermitian metric g. If M of real dimension at least 4 does not admit any Kähler structure, then the action is Rank-1.*

A holomorphic transformation f is conformal if $f^*g = \lambda \cdot g$ for some positive function $\lambda: M \to \mathbb{R}$. Note that the holomorphic isometries of g are contained in the group of holomorphic conformal transformations. Concerning Kähler structures on complex manifolds, we remark that keeping the complex structure fixed, a compact locally conformal Kähler manifold admits some Kähler metric if and only if it is globally conformal to Kähler (cf. [5, 9]).

Some elliptic surfaces admit such torus actions. For example, the usual Hopf manifold admits a Rank-1 holomorphic torus action generated by the Lee and anti-Lee vector field (compare [5]).

2. Conformal Kähler Actions

Let $\tilde{M}$ be the universal covering space of M and $\pi = \pi_1(M)$ its fundamental group. Let J be a lift of the complex structure of M to $\tilde{M}$ and $\tilde{g}$ the lift of g to $\tilde{M}$. If M is a locally conformal Kähler manifold, then there exists a Kähler metric h on $\tilde{M}$ such that $\tilde{g} = a \cdot h$ for some positive function a on $\tilde{M}$. Denote by Ω the fundamental 2-form of h (*i.e.*, $\Omega(X, Y) = h(X, JY)$). A holomorphic conformal transformation f of $(\tilde{M}, \tilde{g}, J)$ satisfies $f^*\tilde{g} = b \cdot \tilde{g}$. Letting $\mu = (f^*a)^{-1} \cdot b \cdot a$, we have $f^*\Omega = \mu \cdot \Omega$. Since Ω is Kähler and dim $M \geqslant 4$, μ must be constant. Thus, f becomes a holomorphically homothetic transformation onto $\tilde{M}$ itself. Let G be the group of all holomorphic homothetic transformations of $\tilde{M}$. Then there exists a map ρ which assigns to each element of G a positive number. It is easy to see that $\rho: G \longrightarrow \mathbb{R}^+$ is a (continuous) homomorphism.

LEMMA 1. *Let $(T^1_{\mathbb{C}}, M, g)$ be a holomorphic conformal complex torus action on a compact locally conformal Kähler manifold. Then either one of circles in $T^1_{\mathbb{C}}$ lifts to a nontrivial homothetic transformations on $(\tilde{M}, \Omega)$. In particular, if $\tilde{T}^1_{\mathbb{C}}$ is a lift of the action $T^1_{\mathbb{C}}$ to $\tilde{M}$, then $\rho(\tilde{T}^1_{\mathbb{C}}) = \mathbb{R}^+$.*

Proof. Suppose not. Then $\tilde{T}^1_{\mathbb{C}}$ acts as isometries with respect to h, i.e., $\rho(\tilde{T}^1_{\mathbb{C}}) = 1$, $\tilde{t}^*\Omega = \Omega$ for every $\tilde{t} \in \tilde{T}^1_{\mathbb{C}}$. Let $\xi, J\xi$ be the vector fields induced by $\tilde{T}^1_{\mathbb{C}}$. Consider the map $\tau: \tilde{M} \to \mathbb{R}$ defined by $\tau(\tilde{x}) = \Omega(J\xi_{\tilde{x}}, \xi_{\tilde{x}})$. As above, $\tau(\tilde{t} \cdot \tilde{x}) = \tau(\tilde{x})$ for $\tilde{t} \in \tilde{T}^1_{\mathbb{C}}$. In particular, $\xi f = J\xi f = 0$.

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Denote the orthogonal space by

$$\{\xi, J\xi\}^{\perp} = \{W \in T\tilde{M} \mid h(\xi, W) = h(J\xi, W) = 0\}.$$

Choose an arbitrary vector field $V \in \{\xi, J\xi\}^{\perp}$. By definition,

$$3d\Omega(V, \xi, J\xi) = V\Omega(\xi, J\xi) + \xi\Omega(J\xi, V) + J\xi\Omega(V, \xi) - \\ - \Omega([V, \xi], J\xi) - \Omega([\xi, J\xi], V) - \Omega([J\xi, V], \xi).$$

Let $\{\phi_\theta\}_{\theta\in\mathbb{R}}, \{\psi_\theta\}_{\theta\in\mathbb{R}}$ be 1-parameter subgroups generated by $\xi, J\xi$ respectively. Since ϕ_θ is holomorphic and

$$[V, \xi] = \lim_{\theta\to 0} \frac{V - \phi_{-\theta*}V}{\theta},$$

we have $\Omega([V, \xi], J\xi) = 0$. Similarly, $\Omega([J\xi, V], \xi) = 0$. As Ω is Kähler, $V\tau = 0$. Hence, τ is a constant map on $\tilde{M}$.

On the other hand, the fundamental group π also acts on $\tilde{M}$ as a group of homothetic transformations; $\gamma^*\Omega = \rho(\gamma)\cdot\Omega$ for $\gamma \in \pi$. There exists an element $\gamma \in \pi$ such that $\rho(\gamma) \neq 1$, otherwise M would admit a Kähler structure. As π centralizes $\tilde{T}^1_{\mathbb{C}}$, we have

$$\tau(\gamma\cdot\tilde{x}) = \Omega(\gamma_* J\xi_{\tilde{x}}, \gamma_*\xi_{\tilde{x}}) = \rho(\gamma)\tau(\tilde{x}) \neq \tau(\tilde{x}),$$

which yields a contradiction. □

COROLLARY 2. *There exists a circle S^1 on $T^1_{\mathbb{C}}$ whose lift is isomorphic to $R = \{\phi_\theta\}_{\theta\in\mathbb{R}}$ so that ${\phi_\theta}^*\Omega = \mathrm{e}^\theta\cdot\Omega$ up to a constant multiple.*

As R acts properly and freely on $\tilde{M}$, the action $(\pi, \tilde{M})$ induces a properly discontinuous action (Γ, Y) where $Y = \tilde{M}/R$. There is an equivariant principal bundle:

$$(Z, R) \to (\pi, \tilde{M}) \xrightarrow{P} (\Gamma, Y).$$

LEMMA 3. *There exists a section $s: Y \longrightarrow \tilde{M}$ such that $\tau(s(y)) \equiv$ const. for every $y \in Y$. In particular, for any $y \in Y$, $\Omega(J\xi_{s(y)}, \xi_{s(y)}) = 1$ (up to a constant multiple.)*

Proof. Let $\tau: \tilde{M} \longrightarrow \mathbb{R}$ be as above. Choose a point $p \in \tilde{M}$ and $\tau(p) = a \in \mathbb{R}^+$. Denote $N = \{z \in \tilde{M} \mid \tau(z) = a\}$. By Corollary 2,

$$\tau(\phi_\theta(x)) = {\phi_\theta}^*\Omega(J\xi_x, \xi_x) = \mathrm{e}^\theta\cdot\tau(x).$$

For each $z \in N$, we have

$$\tau_*(\xi_z) = \xi\tau = \lim_{\theta\to 0} \frac{\tau(\phi_\theta(z)) - \tau(z)}{\theta} = \lim_{\theta\to 0} \frac{e^\theta - 1}{\theta}\cdot\tau(z) = a \neq 0.$$

Hence z is a regular point of τ. Thus $N = \tau^{-1}(a)$ is a codimension 1-regular submanifold of $\tilde{M}$. Let $P': N \longrightarrow Y$ be the restriction of the projection P to N.

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For any $y \in Y$, there is $w \in \tilde{M}$ with $P(w) = y$. Choose e^t such that $a \cdot \tau(w)^{-1} = e^t$. Then $\tau(\phi_t(w)) = e^t\tau(w) = a$, and so $\phi_t(w) \in N$ and $P'(\phi_t(w)) = y$, P' is surjective. Similarly, P' is injective. Since P' is a restriction of P, $P' : N \to Y$ is homeomorphic. Define a section $s : Y \to \tilde{M}$ to be the inverse of P'. As $s(Y) = N$, we note that $\tau(s(y)) = a$ for $y \in Y$. □

Let $\tilde{T}^1_{\mathbb{C}} = R \times \operatorname{Ker} \rho$ and Δ a subgroup of covering translations of the lift: $\tilde{T}^1_{\mathbb{C}} \to T^1_{\mathbb{C}}$. Choose a point $\tilde{x} \in \tilde{M}$ such that $p(\tilde{x}) = x$ and let $ev_{\tilde{x}} : \tilde{T}^1_{\mathbb{C}} \to \tilde{M}$ be the evaluation map $ev_{\tilde{x}}(\tilde{t}) = \tilde{t} \cdot \tilde{x}$ as before. Consider the following commutative diagrams:

$$
\begin{array}{ccccc}
\Delta & \xrightarrow{\iota} & \pi & & \\
\downarrow & & \downarrow & & \\
\tilde{T}^1_{\mathbb{C}} & \xrightarrow{ev_{\tilde{x}}} & \tilde{M} & & \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \\
T^1_{\mathbb{C}} & \xrightarrow{ev_x} & M \Rightarrow T^1_{\mathbb{C}} & \xrightarrow{ev_x} & M
\end{array}
$$

$$
\begin{array}{ccccc}
\pi_1(\tilde{T}^1_{\mathbb{C}}) & \xrightarrow{ev_\#} & 1 & & \\
{\scriptstyle p_*}\downarrow & & {\scriptstyle p_*}\downarrow & & \\
\pi_1(T^1_{\mathbb{C}}) & \xrightarrow{ev_\#} & \pi_1(M) & \xrightarrow{\nu} & H_1(M) \\
{\scriptstyle \partial_\#}\downarrow & & {\scriptstyle \approx}\downarrow & & {\scriptstyle \|}\downarrow \\
\Delta & \xrightarrow{\iota} & \pi & \xrightarrow{\nu} & \pi/[\pi, \pi].
\end{array}
$$

Here $ev_* = \nu \circ ev_\# : \mathbb{Z}^2 = \pi_1(T^1_{\mathbb{C}}) \longrightarrow H_1(M)$ and ν is the canonical projection.

Case I. If $\operatorname{Ker} \rho = S^1$, then $\Delta = Z \subset R$. By the commutative diagram, $ev_*(\mathbb{Z}^2) = \nu(Z)$. As ρ factors through $\pi/[\pi, \pi]$, $\rho \circ ev_*(\mathbb{Z}^2) = \rho(Z) \approx \mathbb{Z}$ because $Z \subset R$ with $\rho(R) = \mathbb{R}^+$. Thus the action is Rank-1.

Case II (geometric part). We consider the case that $\tilde{T}^1_{\mathbb{C}} = R^2$. Then $\Delta = Z^2$. As R^2 acts properly and freely on $\tilde{M}$, the action $(\pi, \tilde{M})$ induces a properly discontinuous action (Q, W) where $W = \tilde{M}/R^2$ for which the following diagram is commutative:

$$
\begin{array}{ccccc}
(Z, R) & \xrightarrow{=} & (Z, R) & & \\
\downarrow & & \downarrow & & \\
(Z^2, R^2) & \longrightarrow & (\pi, \tilde{M}) & \xrightarrow{\mu} & (Q, W) \\
\downarrow & & \downarrow{\scriptstyle P} & & \| \downarrow \\
(Z, R) & \longrightarrow & (\Gamma, Y) & \longrightarrow & (Q, W).
\end{array}
$$

PROPOSITION 4. *Let* $(Z, R) \longrightarrow (\Gamma, Y) \longrightarrow (Q, W)$ *be the induced equivariant principal bundle. There exists a contact form* ω *on* Y *with the following properties:*

(1) ω *is a connection form on the principal bundle* $R \to Y \longrightarrow W$.
(2) ω *is* Γ*-invariant.*

As a consequence, if there exists a subgroup Q' *of finite index in* Q *which acts freely on* W*, then there is a principal circle bundle* $S^1 \to Y/\Gamma' \longrightarrow W/Q'$ *whose Euler class is nonzero in* $H^2(W/Q'; \mathbb{Z})$.

Proof. Let $s : Y \to \tilde{M}$ be a section as in Lemma 3. Define a 1-form ω on Y to be $\omega(V_y) = \Omega(\xi_{s(y)}, \tilde{V}_{s(y)})$ where

$$\tilde{V}_{s(y)} \in \{\xi\}^{\perp} = \{X \in T\tilde{M} \mid h(\xi, X) = 0\}$$

such that $P_*(\tilde{V}_{s(y)}) = V_y$. It is easy to check that ω is well defined. The 1-parameter group R induces a vector field η on Y. Since $\xi, J\xi$ generate R^2, $P_* J\xi = \eta$. As $J\xi \in \{\xi\}^{\perp}$, Lemma 3 implies that $\omega(\eta_y) = 1$. In particular, ω is a connection form on the principal bundle $R \to Y \longrightarrow W$. Using the interior product operator ι, we can write $P^*\omega(\tilde{V}_{s(y)}) = \iota_{\xi}\Omega(\tilde{V}_{s(y)})$. Then

$$dP^*\omega = d \cdot \iota_{\xi}\Omega = (L_{\xi} - \iota_{\xi} \cdot d)\Omega = L_{\xi}\Omega = \Omega.$$

For this,

$$L_{\xi}\Omega = \lim_{\theta \to 0} \frac{\phi_{\theta}^*\Omega - \Omega}{\theta} = \lim_{\theta \to 0} \frac{e^{\theta} - 1}{\theta} \cdot \Omega = \Omega$$

as before. Thus, $P^*(d\omega)^n = \Omega^n$ so that $\omega \wedge (d\omega)^n \neq 0$ for each point of Y, i.e., ω is a contact form. Moreover,

$$d\omega(\eta, V) = P^* d\omega(J\xi, \tilde{V}) = \Omega(J\xi, \tilde{V}) = h(\xi, \tilde{V}) = 0.$$

Hence η is a characteristic vector field for ω.

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We prove that ω is invariant under the action of Γ. For $\gamma \in \Gamma$, choose $u \in \pi$ with $P(u) = \gamma$. As $P(u \cdot s(y)) = \gamma \cdot y$, there exists $n \in Z$ such that $u \cdot s(y) = n \cdot s(\gamma \cdot y)$. Then note that $(n^{-1} \circ u)_* \tilde{V}_{s(y)} \in T_{s(\gamma \cdot y)}\tilde{M}$ for which

$$h((n^{-1} \circ u)_* \tilde{V}, \xi) = \rho(n^{-1} \circ u) \cdot h(\tilde{V}, (u^{-1} \circ n)_* \xi) = \rho(n^{-1} \circ u) \cdot h(\tilde{V}, \xi) = 0.$$

Thus, we have $(n^{-1} \circ u)_* \tilde{V}_{s(y)} \in \{\xi\}^{\perp}_{s(\gamma \cdot y)}$.

Since $P_*((n^{-1} \circ u)_* \tilde{V}_{s(y)}) = \gamma_* V_y$, a calculation shows that

$$\begin{aligned} \rho(u) \cdot \Omega(\xi_{s(y)}, \tilde{V}_{s(y)}) &= u^* \Omega(\xi_{s(y)}, \tilde{V}_{s(y)}) = \Omega(\xi_{n \cdot s(\gamma \cdot y)}, u_* \tilde{V}_{s(y)}) \\ &= \rho(n) \cdot \Omega(\xi_{s(\gamma \cdot y)}, (n^{-1} \circ u)_* \tilde{V}_{s(y)}). \end{aligned}$$

On the other hand, we note that $\rho(u) = \rho(n)$. For this,

$$\rho(u) = \tau(u \cdot s(y)) = \tau(n \cdot s(\gamma \cdot y)) = \rho(n) \cdot \tau(s(\gamma \cdot y)) = \rho(n).$$

Hence,

$$\omega(V_y) = \Omega(\xi_{s(y)}, \tilde{V}_{s(y)}) = \Omega(\xi_{s(\gamma \cdot y)}, (n^{-1} \circ u)_* \tilde{V}_{s(y)}) = \omega(\gamma_* V_y).$$

That is, $\gamma^* \omega = \omega$, ω is Γ-invariant. □

Remark 5. It is easy to see that P_* maps $\{\xi, J\xi\}^{\perp}$ isomorphically onto Null ω. As the complex structure leaves invariant $\{\xi, J\xi\}^{\perp}$, it induces a complex structure J on Null ω. So the pair (ω, J) is a pseudo-Hermitian structure on Y in which the characteristic vector field η induces a principal R-action on Y. Such a pseudo-Hermitian structure (ω, J) endowed with η is called standard (or equivalently, almost regular). Compare [1, 7, 8].

We recall the following algebraic result from [3]:

PROPOSITION 6. *Let $1 \to \mathbb{Z}^k \longrightarrow G \longrightarrow H \to 1$ be a central group extension. If the canonical projection $\nu : \mathbb{Z}^k \to G/[G, G]$ is injective, then G has a splitting subgroup G' of finite index, i.e., $G' \cong \mathbb{Z}^k \times H'$. Especially, the representative cocycle for the above extension has finite order in $H^2(H; \mathbb{Z}^k)$.*

Case III (algebraic part). Given a properly discontinuous action (Q, W), we can define the direct product action $(\mathbb{Z}^2 \times Q, \mathbb{R}^2 \times W)$:

$$(n, \alpha)(x, w) = (x + n, \alpha(w)).$$

PROPOSITION 7. *Suppose that $ev_* = \nu \circ \iota : \mathbb{Z}^2 \longrightarrow H_1(M; \mathbb{Z}) = \pi/[\pi, \pi]$ is injective. Passing to a subgroup Q' of finite index in Q, the product action $(\mathbb{R}^2, (\mathbb{Z}^2 \times Q', \mathbb{R}^2 \times W))$ is equivariantly diffeomorphic to the original action $(R^2, (\pi', \tilde{M}))$. Here π' is a subgroup of π which occurs as an extension $1 \to \mathbb{Z}^2 \longrightarrow \pi' \longrightarrow Q' \to 1$ restricted to Q'. In particular, Q' acts freely on W.*

Proof. Let $1 \to \mathbb{Z}^2 \xrightarrow{ev_\#} \pi \longrightarrow Q \to 1$ be the induced central group extension. We identify $ev_\#(n) = n$. A (central) group extension can be represented by a 2-cocycle $[f] \in H^2(Q; \mathbb{Z}^2)$. In fact, choosing a section $q : Q \longrightarrow \pi$, f is described as $f(\alpha, \beta) \cdot q(\alpha\beta) = q(\alpha)q(\beta)$ for $\alpha, \beta \in Q$. Then π is the product $\mathbb{Z}^2 \times Q$ with group law:

$$(n, \alpha)(m, \beta) = (n + m + f(\alpha, \beta), \alpha\beta).$$

Recall some facts about the injective Seifert fibering from [3].

Let $(Z^2, R^2) \to (\pi, \tilde{M}) \xrightarrow{\mu} (Q, W)$ be an equivariant principal bundle as above. Choosing a (continuous) section $\tilde{q} : W \to \tilde{M}$, the correspondence $(x, w) \longrightarrow x \cdot \tilde{q}(w)$ gives a diffeomorphism of the product $\mathbb{R}^2 \times W$ onto $\tilde{M}$ for which the action of $\mathbb{R}^2$ is equivalent to the original R^2-action. We obtain the corresponding action of π on $\mathbb{R}^2 \times W$. In fact, for an element $(0, \alpha) \in \mathbb{Z}^2 \times Q$, as the projection μ maps $\mu((0, \alpha) \cdot (0, w)) = \alpha \cdot w$, we can write the action $(0, \alpha) \cdot (0, w) = (\chi(\alpha)(\alpha \cdot w), \alpha \cdot w)$ for a function $\chi : Q \longrightarrow \mathrm{Map}(W, \mathbb{R}^2)$. Let $\gamma = (n, \alpha)$ be any element of π. As π centralizes $\tilde{T}^1_{\mathbb{C}} = R^2$,

$$\gamma \cdot (x, w) = (n + x + \chi(\alpha)(\alpha \cdot w), \alpha \cdot w). \tag{1}$$

The equality $((0, \alpha) \cdot (0, \beta))(0, w) = (f(\alpha, \beta), \alpha\beta)(0, w)$ implies that

$$f(\alpha, \beta) = \chi(\beta)(\beta \cdot w) + \chi(\alpha)(\alpha\beta \cdot w) - \chi(\alpha\beta)(\alpha\beta \cdot w).$$

Viewed χ as an element of the cochain complex $C^1(Q; \mathrm{Map}(W, \mathbb{R}^2))$, we have $f = \delta^1\chi$.

On the other hand, by Proposition 6, $[f]$ is a torsion element in $H^2(Q; \mathbb{Z}^2)$. Say, some $\ell \in \mathbb{Z}$, $\ell \cdot f = \delta^1\lambda'$ for a 1-cochain $\lambda' : Q \longrightarrow \mathbb{Z}^2$, i.e.,

$$f(\alpha, \beta) = \frac{1}{\ell}\lambda'(\beta) + \frac{1}{\ell}\lambda'(\alpha) - \frac{1}{\ell}\lambda'(\alpha\beta).$$

Put $\lambda = (1/\ell)\lambda'$ so that $f = \delta^1\lambda$ in $C^1(Q; \mathrm{Map}(W, \mathbb{R}^2))$. Here $\mathbb{R}^2$ is viewed as a constant map in $\mathrm{Map}(W, \mathbb{R}^2)$. Using λ, we have another action of π in this case:

$$\gamma \cdot (x, w) = (n + x + \lambda(\alpha), \alpha \cdot w).$$

Since

$$f = \delta^1\chi = \delta^1\lambda, [-\lambda + \chi] \in H^1(Q; \mathrm{Map}(W, \mathbb{R}^2)).$$

As W is simply connected and the quotient W/Q is compact, we know that $H^1(Q; \mathrm{Map}(W, \mathbb{R}^2)) = 0$. (See [4].) And so there exists an element $h \in H^0(Q; \mathrm{Map}(W, \mathbb{R}^2))$, i.e., a map $h : W \longrightarrow \mathbb{R}^2$ such that

$$-\lambda(\alpha) + \chi(\alpha)(w) = (\delta^0 h)(\alpha)(w) = h(\alpha^{-1} \cdot w) - h(w).$$

We have

$$-\lambda(\alpha) + \chi(\alpha)(\alpha \cdot w) = h(w) - h(\alpha \cdot w). \tag{2}$$

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Using h, we define a diffeomorphism $H\colon \mathbb{R}^2 \times W \longrightarrow \mathbb{R}^2 \times W$ to be $H((x, w)) = (x + h(w), w)$.

Let $1 \to [Q, Q] \to Q \longrightarrow Q/[Q, Q] \to 1$ be the exact sequence. Noting $\lambda(1) = 0$, we see that $\lambda([Q, Q]) \subset \mathbb{Z}^2$. A calculation shows that

$$\lambda(\alpha^\ell) = \ell \cdot \lambda(\alpha) - \{f(\alpha, \alpha) + f(\alpha^2, \alpha) + \ldots + f(\alpha^{\ell-1}, \alpha)\} \in \mathbb{Z}^2.$$

As Q is finitely generated, $Q/[Q, Q] = \mathbb{Z}^k \oplus \langle$a torsion group$\rangle$ for some k. Let Q' be a subgroup of Q which maps onto $\ell \cdot \mathbb{Z}^k$. Then Q' is of finite index in Q satisfying that $\lambda(Q') \subset \mathbb{Z}^2$. If $1 \to \mathbb{Z}^2 \to \pi' \longrightarrow Q' \to 1$ is the induced group extension, then there is an isomorphism Φ of π' onto the direct product $\mathbb{Z}^2 \times Q'$ by setting $\Phi(n, \alpha') = (n + \lambda(\alpha'), \alpha')$. Then $(\Phi, H) : (\pi', \mathbb{R}^2 \times W) \longrightarrow (\mathbb{Z}^2 \times Q', \mathbb{R}^2 \times W)$ is an equivariant diffeomorphism between the original action $(\pi', \mathbb{R}^2 \times W)$ and the product action $(\mathbb{Z}^2 \times Q', \mathbb{R}^2 \times W)$. In fact, using (1), (2),

$$\begin{aligned}\Phi(n, \alpha')H(x, w) &= ((n + \lambda(\alpha')) + x + h(w), \alpha' w)\\ &= (n + x + \chi(\alpha')(\alpha' w) + h(\alpha' w), \alpha' w)\\ &= H(n + x + \chi(\alpha')(\alpha' w), \alpha' w)\\ &= H((n, \alpha')(x, w)).\end{aligned}$$

Since π' acts freely on $\mathbb{R}^2 \times W$, Q' acts freely on W. □

3. Proof of the Theorem

We have only to check Case II of the above discussion. By Corollary 2, $ev_*\colon \mathbb{Z} \longrightarrow \pi/[\pi, \pi] = H_1(M; \mathbb{Z})$ is injective at least. So in order to complete the proof of the main Theorem, it suffices to show that $ev_* : \mathbb{Z}^2 \longrightarrow H_1(M; \mathbb{Z})$ cannot be injective. Suppose that $ev_* = \nu \circ \iota : \mathbb{Z}^2 \longrightarrow H_1(M; \mathbb{Z})$ is injective. Proposition 7 implies that there exists a subgroup Q' of finite index in Q for which $\pi' \cong \mathbb{Z}^2 \times Q'$, and Q' acts freely on W. Since the isomorphism Φ preserves the center, it induces an isomorphism:

$$\Gamma' \cong \mathbb{Z} \times Q' \tag{$*$}$$

However, applying Proposition 4, the principal bundle $S^1 \to Y/\Gamma' \longrightarrow W/Q'$ has a nonzero Euler class; equivalently, by taking the homotopy exact sequence of the fundamental groups, $1 \to \mathbb{Z} \xrightarrow{ev_\#} \Gamma' \longrightarrow Q' \to 1$ is a nontrivial group extension, which contradicts $(*)$.

Remark 8. When we work with the locally conformal symplectic structure invariant under the complex structure, the same result holds whenever the function $\tau(\tilde{x}) = \Omega(\xi_{\tilde{x}}, J\xi_{\tilde{x}})$ is nonzero everywhere.

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