21

Hausdorff and Packing Measures of Compactly Nonrecurrent Regular Elliptic Functions

From now on, throughout this chapter, and, in fact, throughout the entire book, H_e^t stands for the t-dimensional Hausdorff measure on $\mathbb{C}$ with respect to the Euclidean metric, whereas H_s^t refers to its spherical counterpart. The same convention is applied to the packing measures Π_e^t and Π_s^t. Note that the measures H_e^t and H_s^t as well as Π_e^t and Π_s^t are equivalent to the Radon–Nikodym derivative bounded away from zero and ∞ on compact subsets of $\mathbb{C}$. In particular, the Hausdorff dimension of any subset A of $\mathbb{C}$ has the same value no matter whether calculated with respect to the Euclidean or spherical metric; it will be denoted in what follows simply by $\mathrm{HD}(A)$. If H^t or Π^t is endowed with neither the subscript "e" nor the subscript "s," then it refers simultaneously to both the Euclidean as well as the spherical measures. As in the previous chapters, we keep

$$h = \mathrm{HD}(J(f)).$$

The goal of this chapter can be viewed as two-fold. The first is to provide a geometrical characterization of the h-conformal measure m_h, which, with the absence of parabolic points, turns out to be a normalized packing measure, and the second is to give a complete description of geometric, Hausdorff, and packing measures of the Julia sets $J(f)$. All of this is contained in the following theorem.

Theorem 21.0.1 *Let $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent regular elliptic function. If $h = \mathrm{HD}(J(f)) = 2$, then $J(f) = \mathbb{C}$. If $h < 2$, then*

(a) $\mathrm{H}_s^h(J(f)) = 0$.
(b) $\Pi_s^h(J(f)) > 0$.
(c) $\Pi_s^h(J(f)) < +\infty$ *if and only if* $\Omega(f) = \emptyset$.

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

In either Case (c) or if $\text{HD}(J(f)) = 2$*, the unique spherical h-conformal measure* m_h *coincides with the normalized packing measure* $\Pi_s^h/\Pi_s^h(J(f))$ *restricted to the Julia set* $J(f)$*.*

This theorem has an interesting story: for expanding rational functions f, we always have, essentially because of [**Bow2**], that

$$0 < \mathrm{H}^h(J(f)), \Pi^h(J(f)) < +\infty,$$

and these two measures coincided up to a multiplicative constant. Their probability version is then the unique h-conformal measure. If f is still a rational function but parabolic or, more generally, nonrecurrent, then (see [**DU5**] and [**U3**], respectively):

(a) $\mathrm{H}_h(J(f)) < +\infty$ and $\Pi_h(J(f)) > 0$.
(b) $\mathrm{H}_h(J(f)) = 0$ if and only if $h < 1$ and $\Omega(f) \neq \emptyset$.
(c) $\Pi_h(J(f)) = +\infty$ if and only if $h > 1$ and $\Omega(T) \neq \emptyset$.

So, the descriptions of Hausdorff and packing measures in the cases of both nonrecurrent rational functions and compactly nonrecurrent regular elliptic functions coincide except that, in the latter case, $\mathrm{H}_h(J(f)) \leq 1$ never holds. For other transcendental meromorphic and entire functions, even hyperbolic (expanding), the situation is generally less clear and varies from case to case; see, for example, [**UZ1**] and [**MyU2**].

As an immediate consequence of Theorem 21.0.1, we get the following.

Corollary 21.0.2 *Let* $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *be a compactly nonrecurrent regular elliptic function. If* $\Omega(f) = \emptyset$*, then the Euclidean h-dimensional packing measure* Π_e^h *is finite on each bounded subset of* $J(f)$*.*

21.1 Hausdorff Measures

We start with the proof of the first part of Theorem 21.0.1. Our first preparatory result is the following.

Lemma 21.1.1 *If* $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *is a compactly nonrecurrent elliptic function, then*

$$\bigcup_{j=1}^{\infty} f^{-j}(\infty) \backslash \overline{O_+(\mathrm{Crit}(f))} \neq \emptyset.$$

Proof Seeking contradiction, suppose that

$$f^{-1}(\infty) \subseteq \overline{O_+(\mathrm{Crit}(f))}.$$

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

So, for each $b \in f^{-1}(\infty)$, there exists $c_b \in \mathrm{Crit}(f) \cap J(f)$ such that $b \in \overline{O_+(c_b)}$. It then follows from Definition 18.2.1 that its items (1) and (3) are ruled out for c_b, whence item (2) must hold. We then conclude that

$$b \in O_+(f(c_b)). \tag{21.1}$$

Since then $O_+(f(c_b))$ is a finite set and since $f(\mathrm{Crit}(f))$ is also a finite set, we conclude that

$$\bigcup_{b \in f^{-1}(\infty)} O_+(f(c_b))$$

is a finite set. But, (21.1) implies that

$$f^{-1}(\infty) \subseteq \bigcup_{b \in f^{-1}(\infty)} O_+(f(c_b)).$$

Since $f^{-1}(\infty)$ is infinite, we arrived at a contradiction, and we are, thus, done. ■

Proof of part (a) of Theorem 21.0.1 By Lemma 21.1.1, there exists $b \in f^{-1}(\infty)\backslash\overline{O_+(\mathrm{Crit}(f))}$, say

$$b \in f^{-1}(\infty)\backslash\overline{O_+(\mathrm{Crit}(f))}.$$

Hence, there exists $\kappa > 0$ such that

$$B_e(b, 3\kappa) \cap O_+(\mathrm{Crit}(f)) = \emptyset. \tag{21.2}$$

Consider an arbitrary point $z \in \mathrm{Tr}(f)$. Then there exists an infinite increasing sequence $\{n_j\}_{j=0}^{\infty}$ such that

$$\lim_{j\to\infty} f^{n_j}(z) = b \quad \text{and} \quad |f^{n_j}(z) - b| < \kappa/2 \tag{21.3}$$

for every $j \geq 1$. It follows from this, (21.2), and Theorem 17.1.8 that, for every $j \geq 1$, there exists a holomorphic inverse branch

$$f_z^{-n_j}: B_e(f^{n_j}(z), 2\kappa) \longrightarrow \mathbb{C}$$

of f^{n_j} sending $f^{n_j}(z)$ to z. Let m_h be the unique h-conformal atomless measure proven to exist in Theorem 20.3.11. Using now Theorem 8.3.8 and Lemmas 8.3.13, 10.4.7, and 20.3.8, we conclude that

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

$$
\begin{aligned}
& m_{h,e}\big(B_e\big(z, 2K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z)-b|\big)\big) \\
&\quad \geq m_{h,e}\Big(f_z^{-n_j}\Big(B_e\big(f^{n_j}(z), 2|f^{n_j}(z)-b|\big)\Big)\Big) \\
&\quad \geq K^{-h} m_{h,e}\big(B_e\big(f^{n_j}(z), 2|f^{n_j}(z)-b|\big)\big)|(f^{n_j})'(z)|^{-h} \\
&\quad \geq K^{-h} m_{h,e}\big(B_e(b, |f^{n_j}(z)-b|)\big)|(f^{n_j})'(z)|^{-h} \\
&\quad \succeq |f^{n_j}(z)-b|^{(q_b+1)h-2q_b}|(f^{n_j})'(z)|^{-h} \\
&\quad = \big(2K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z)-b|\big)^h (2K)^{-h}|f^{n_j}(z)-b|^{q_b(h-2)}.
\end{aligned}
$$

Since $h < 2$, using (21.3), this implies that

$$
\overline{\lim_{r\to 0}} \frac{m_{h,e}(B_e(z,r))}{r^h} \geq \overline{\lim_{j\to\infty}} \frac{m_{h,e}\big(B_e\big(z, 2K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z)-b|\big)\big)}{\big(2K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z)-b|\big)^h} = +\infty.
$$

Hence,

$$
\mathrm{H}_e^h(\mathrm{Tr}(f)) = 0
$$

in view of Theorem 1.6.3(1). Since, by Theorem 20.3.11, $m_{h,e}(J(f)\setminus \mathrm{Tr}(f)) = 0$, it follows from Lemma 17.6.4 that $\mathrm{H}_e^h(J(f)\setminus\mathrm{Tr}(f)) = 0$. In conclusion,

$$
\mathrm{H}_e^h(J(f)) = 0,
$$

which completes the proof. ■

21.2 Packing Measure I

In this section, we shall prove Proposition 21.2.1, stated below. Its item (3) is just item (b) of Theorem 21.0.1, while item (1) contributes toward the last assertion of this theorem. We shall also prove Lemma 21.2.2, which establishes one side of item (c).

Proposition 21.2.1 *If $f:\mathbb{C}\longrightarrow\widehat{\mathbb{C}}$ is a compactly nonrecurrent regular elliptic function, then*

(1) *The h-conformal measure m_h is absolutely continuous with respect to the packing measure Π^h. Moreover,*
(2) *The Radon–Nikodym derivative $dm_s/d\Pi_s^h$ is uniformly bounded away from infinity. In particular,*
(3)

$$
\Pi^h(J(f)) > 0.
$$

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

Proof Since

$$J(f) \cap \omega\big(\mathrm{Crit}(f) \backslash \mathrm{Crit}(J(f))\big) = \Omega(f),$$

we conclude from Lemma 18.2.6 that there exists $y \in J(f)$ at a positive distance, and denote it by 8η, from $O_+(\mathrm{Crit}(f))$. Fix $z \in \mathrm{Tr}(f)$. Then there exists an infinite sequence $(n_j)_{j=1}^{\infty}$ of increasing positive integers such that $f^{n_j}(z) \in B_e(y, \eta)$ for every $j \geq 1$. Hence,

$$B_e(f^{n_j}(z), 4\eta) \cap O_+(\mathrm{Crit}(f)) = \emptyset.$$

Consequently,

$$\mathrm{Comp}\big(z, f^{n_j}, 4\eta\big) \cap \mathrm{Crit}(f^{n_j}) = \emptyset.$$

Hence, it follows from Lemmas 8.3.13 and 10.4.7 that

$$\liminf_{r \to 0} \frac{m_{h,e}(B_e(z,r))}{r^h} \leq B$$

for some constant $B \in (0, \infty)$ and all $z \in \mathrm{Tr}(f)$. Applying Lemma 20.3.4, we, therefore, get that

$$\liminf_{r \to 0} \frac{m_{h,s}(B_s(z,r))}{r^h} \leq 2^h B.$$

Therefore, by Theorem 1.6.4(1), the measure $m_{h,s}|_{\mathrm{Tr}(f)}$ is absolutely continuous with respect to $\Pi_s^h|_{\mathrm{Tr}(f)}$. Since, by Theorem 20.3.11, $m_{h,s}(J(f) \backslash \mathrm{Tr}(f)) = 0$, we are done. ■

Lemma 21.2.2 *If* $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *is a compactly nonrecurrent regular elliptic function and* $\Omega(f) \neq \emptyset$, *then*

$$\Pi_s^h(J(f)) = +\infty.$$

Proof Fix $\xi \in \Omega(f)$. Since the set

$$\bigcup_{n \geq 0} f^{-n}(\xi)$$

is dense in $J(f)$ and since, by Lemma 18.2.6, $\omega(\mathrm{Crit}(f))$ is nowhere dense in $J(f)$, there exist an integer $s \geq 0$, a real number $\eta > 0$, and a point

$$y \in f^{-s}(\xi) \backslash B_e\left(\bigcup_{n \geq 0} f^n(\mathrm{Crit}(f)), \eta\right).$$

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

Since, by Theorem 17.3.1, $h > 1$, it follows from Lemmas 15.4.1 and 10.4.10 (y may happen to be a critical point of f^s!) that

$$\liminf_{r\to 0} \frac{m_e(B_e(y,r))}{r^h} = 0. \tag{21.4}$$

Consider now a transitive point $z \in J(f)$, i.e., $z \in \mathrm{Tr}(f)$. Then there exists an infinite increasing sequence $n_j = n_j(z) \geq 1$, $j \geq 1$, of positive integers such that

$$\lim_{j\to\infty} |f^{n_j}(z) - y| = 0 \qquad \text{and} \qquad r_j = |f^{n_j}(z) - y| < \eta/7$$

for every $j \geq 1$. By the choice of y and Theorem 17.1.8, for all $j \geq 1$, there exist holomorphic inverse branches

$$f_z^{-n_j} : B_e(f^{n_j}(z), 6r_j) \longrightarrow \mathbb{C}$$

of f^{n_j} sending $f^{n_j}(z)$ to z. So, applying Lemmas 8.3.13 and 10.4.7 with $R = 3r_j$, we conclude from (21.4) that

$$\liminf_{r\to 0} \frac{m_{h,e}(B_e(z,r))}{r^h} = 0.$$

Applying Lemma 20.3.4, we conclude that the same formulas remain true with $m_{h,e}$ replaced by $m_{h,s}$ and $B_e(z,r)$ by $B_s(z,r)$. Therefore, it follows from Theorems 20.3.11 ($m_{h,s}(\mathrm{Tr}(f)) = 1$) and 1.6.4(1) that $\Pi_s^h(J(f)) = +\infty$. We are done. ■

21.3 Packing Measure II

As before, from now on throughout this section, m_h denotes the unique atomless h-conformal measure proven to exist in Theorem 20.3.11. Our aim in this section is to show that, in the absence of parabolic periodic points, the h-dimensional Euclidean packing measure is finite on bounded subsets of $J(f)$ and that $\Pi_s^h(J(f)) < +\infty$. This will complete item (c) of Theorem 21.0.1.

Recall that the numbers $R_l(f)$ and $A_l(f)$ have been defined by (20.28) and (20.29), respectively.

Recall, for the needs of this section, that the sequence $\{Cr_i(f)\}_{i=1}^p$ was defined inductively by (18.33) and the sequence $\{S_i(f)\}_{i=1}^p$ was defined by (18.35), while the number p, here and in what follows in this section, comes from Lemma 18.2.11(c).

Since the number N_f of equivalence classes of the relation $\sim_f$ between critical points of an elliptic function $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is finite, looking at Lemmas 18.2.15 and 17.6.6, the following lemma follows immediately from Lemma 10.4.11.

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

Lemma 21.3.1 *Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent regular elliptic function. Fix $0 \le i \le p-1$. If $C_i^{(l)} > 0$, $0 < R_i^{(l)} \le R_l(f)/3$, and $0 < \sigma \le 1$ are three real numbers such that all points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ are $(r, \sigma, C_i^{(l)})$-h-s.l.e. with respect to the measure $m_{h,e}$, then there exists $\tilde{C}_i^{(l)} > 0$ such that all critical points $c \in Cr_{i+1}(f)$ are $(r, \tilde{\sigma}, \tilde{C}_i^{(l)})$-h-s.l.e. with respect to the measure $m_{h,e}$ for all $0 < r \le A_l(f)^{-1} R_i^{(l)}$, where $\tilde{\sigma}$ was defined in Lemma 10.4.11.*

Let us prove the following.

Lemma 21.3.2 *Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent regular elliptic function. Suppose that $\Omega(f) = \emptyset$. Fix $0 \le i \le p$. Assume that $C_{i,1}^{(l)} > 0$, $R_{i,1}^{(l)} > 0$, and $0 < \sigma \le 1$ are three real numbers such that all critical points $c \in S_i(f)$ are $(r, \sigma, C_{i,1}^{(l)})$-h-s.l.e. with respect to the measure $m_{h,e}$ for all $0 < r \le R_{i,1}^{(l)}$. Then there exist $\tilde{C}_{i,1}^{(l)} > 0$, $\tilde{R}_{i,1}^{(l)} > 0$ such that all points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ are $(r, 8K^3A^2 2^{N_f}\sigma, \tilde{C}_{i,1}^{(l)})$-h-s.l.e. with respect to the measure $m_{h,e}$ for all $0 < r \le \tilde{R}_{i,1}^{(l)}$, where $A > 0$ was defined in* (18.20).

Proof Recall that, by Lemma 20.3.5, the set $\overline{\mathrm{PC}_c^0(f)}$ is f-pseudo-compact. We shall show that this time one can take

$$\tilde{R}_{i,1}^{(l)} := \min\left\{\tau\theta \min\{1, \|f'\|_i^{-1}\}\lambda^{-1}, R_{i,1}^{(l)}, 1\right\} \quad \text{and}$$
$$\tilde{C}_{i,1}^{(l)} := \left(8(KA^2)2^{N_f}\right)^h C_{i,1}^{(l)},$$

where $\|f'\|_i := \|f'\|_{\overline{\mathrm{PC}_c^0(f)_i}}$. Indeed, take $\varepsilon := 4K(KA^2)2^{N_f}$ and then choose $\lambda > 0$ so large that

$$\varepsilon < \lambda \min\left\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\min\{\rho, R_{i,1}^{(l)}/2\}\right\}. \tag{21.5}$$

Consider $0 < r \le \tilde{R}_{i,1}^{(l)}$ and $z \in \overline{\mathrm{PC}_c^0(f)_i}$. If $z \in \mathrm{Crit}_c(J(f))$, then $z \in S_i(f)$ and we are done. Thus, we may assume that $z \notin \mathrm{Crit}_c(J(f))$, then $z \notin \mathrm{Crit}(J(f))$.

Let $s = s(\lambda, \varepsilon, r, z)$. By the definition of ε,

$$4Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-N_f}\varepsilon r|(f^s)'(z)|. \tag{21.6}$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then, by item (20.4) in Proposition 20.2.1, applied with $\eta = K$, we see that the point

$$f^s(z) \text{ is } (Kr|(f^s)'(z)|, \sigma/K^2, W_h(\sigma/K^2))\text{-}h\text{-s.l.e.}$$

Using (21.6), it follows from item (20.16) in Proposition 20.2.2 and Lemma 10.4.8 that the point z is $(r, \sigma, W_h(\sigma/K^2))$-$h$-s.l.e. If either u is not defined or

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

$s \le u(\lambda, r, z)$, then, in view of item (20.15) in Proposition 20.2.2, there exists a critical point $c \in \mathrm{Crit}(f)$ such that

$$|f^s(z) - c| \le \varepsilon r |(f^s)'(z)|.$$

Since $s \le u$, by Proposition 20.2.2 and (21.5), we get that

$$|f^s(z) - c| \le \varepsilon r |(f^s)'(z)| < \min\{\rho, R_{i,1}^{(l)}/2\}. \tag{21.7}$$

Since $z \in \overline{\mathrm{PC}_c^0(f)_i}$, this implies that $c \in S_i(f)$. Therefore, by the assumptions of Lemma 21.3.2 and by (21.7), we conclude that c is $(2\varepsilon r|(f^s)'(z)|, \sigma, C_{i,1}^{(l)})$-$h$-s.l.e. Consequently, in view of Lemma 10.4.4, the point $f^s(z)$ is $(\varepsilon r|(f^s)'(z)|, 2\sigma, 2^h C_{i,1}^{(l)})$-$h$-s.l.e. So, by Lemma 10.4.5, this point is

$$(Kr|(f^s)'(z)|, 2\sigma\varepsilon/K, (2\varepsilon K^{-1})^h C_{i,1}^{(l)}) - h\text{-s.l.e.}$$

Using now (21.6) and item (20.16) in Proposition 20.2.2 along with the fact that $K\varepsilon^{-1} < 1$, we have from Lemma 10.4.8 that the point z is $(r, 2K\varepsilon\sigma, (2\varepsilon K^{-1})^h C_{i,1}^{(l)})$-$h$-s.l.e. The proof is complete. ■

As a fairly straightforward consequence of these two lemmas, we get the following.

Lemma 21.3.3 *If $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a compactly nonrecurrent regular elliptic function, then, with some $R > 0$ and some $G > 0$, each point of $\overline{\mathrm{PC}_c^0(f)}$ (in particular, each point of $\mathrm{Crit}_c(f)$) is $(r, 1/2, G)$-h-s.l.e. with respect to the measure $m_{h,e}$ for every $r \in [0, R]$.*

Proof Since $S_0(f) = \emptyset$, starting with $\sigma > 0$ as small as we wish, it immediately follows from Lemmas 21.3.2, 21.3.1, and 18.2.14 by induction on $i = 0, 1, \ldots, p$ that all the points of $S_i(f)$ and $\overline{\mathrm{PC}_c^0(f)_i}$ are $(r, 1/2, G)$-h-s.l.e. with the same $G, R > 0$ and all $r \in [0, R]$. We are done. ■

This lemma and Lemma 20.3.8, taken together, yield the following.

Lemma 21.3.4 *If $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a compactly nonrecurrent regular elliptic function, then every point of the set $\mathrm{Crit}(J(f)) \cup f^{-1}(\infty)$ is h-s.l.e. with respect to the measure $m_{h,e}$ with $\sigma \in (0,1)$ arbitrary.*

Fix $c \in \mathrm{Crit}_\infty(f)$. Since $\lim_{n\to\infty} f^n(c) = \infty$, there exists an integer $k \ge 1$ such that $q_{b_n} \le q_c$ (where $b_n \in f^{-1}(\infty)$, defined in (18.48), is near $f^n(c)$, and q_c was defined in (18.49)) and

$$|f^n(c)| > \max\{1, 2\mathrm{Dist}_e(0, f(\mathrm{Crit}(f)))\} \tag{21.8}$$

for all $n \geq k$. Put

$$a := f^k(c) \tag{21.9}$$

(we may need, in the course of the proof, $k \geq 1$ to be bigger). We shall prove the following.

Lemma 21.3.5 *If $f: \mathbb{C} \to \widehat{\mathbb{C}}$ is a compactly nonrecurrent regular elliptic function and $c \in \mathrm{Crit}_\infty(f)$, then there exists a constant $C_1 \geq 1$ such that*

$$m_{h,e}(B_e(a,r)) \geq C_1^{-1} r^h$$

for all radii $r > 0$ small enough, where a is defined by (21.9).

Proof Put

$$q := q_c.$$

In view of (21.8) and Theorem 17.1.8, for every $n \geq 1$, there is a well-defined holomorphic inverse branch

$$f_n^{-1}\colon B_e\left(f^n(a), \frac{1}{2}|f^n(a)|\right) \longrightarrow \mathbb{C}$$

of f sending $f^n(a)$ to $f^{n-1}(a)$. Let $b_n \in f^{-1}(\infty)$ be the unique pole (assuming that $k \neq 1$ is large enough) such that

$$|f^n(a) - b_n| \leq \delta(f^{-1}(\infty)) \preceq 1,$$

where $\delta(f^{-1}(\infty))$ comes from (18.47). Then, by Theorem 8.3.8,

$$\begin{aligned}
f_n^{-1}\left(B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right)\right) &\subset B_e\left(f^{n-1}(a), \frac{K}{4}|f^n(a)||f'(f^{n-1}(a))|^{-1}\right) \\
&\subseteq B_e\left(f^{n-1}(a), C|f^n(a)||f^n(a)|^{-\frac{q+1}{q}}\right) \\
&= B_e\left(f^{n-1}(a), C|f^n(a)|^{-\frac{1}{q}}\right) \\
&\subseteq B_e\left(f^{n-1}(a), \frac{1}{4}|f^{n-1}(a)|\right),
\end{aligned}$$

where $C \in (0,+\infty)0$ is a constant and the last inclusion was written assuming that

$$|f^{n-1}(a)| \geq 4c|f^n(a)|^{-\frac{1}{q}},$$

which we can assume to hold for all $n \geq k$ if k is large enough. So, the composition

$$f_a^{-n} := f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1}\colon B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right) \longrightarrow \mathbb{C},$$

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

sending $f^n(a)$ to a, is well defined and forms a holomorphic branch of f^{-n}. Take $0 < r < 8K/|a|$ and let $n+1 \geq 1$ be the least integer such that

$$r|(f^{n+1})'(a)| \geq \frac{K}{8}|f^{n+1}(a)|.$$

Such an integer exists since $|f'(z)| \asymp |f(z)|^{\frac{q_b+1}{q_b}}$ if z is near a pole b. By definition $n \geq 0$ and since $r < 8K/|a|$, we have that

$$r|(f^n)'(a)| < \frac{K}{8}|(f^n)(a)|.$$

Then, by Theorem 8.3.8, we have that

$$B_e(a,r) \supset f_a^{-n}\big(B_e(f^n(a)), K^{-1}|(f^n)'(a)|\big). \tag{21.10}$$

Now we consider three cases determined by the value of $r|(f^n)'(a)|$.

Case 1. $\delta(f^{-1}(\infty)) \leq r|(f^n)'(a)| < \frac{K}{8}|f^n(a)|$.

In view of (21.8) and Theorem 8.3.8 along with almost conformality of the measure $m_{h,e}$, we get that

$$\begin{aligned} m_{h,e}(B(a,r)) &\geq K^{-h}|(f^n)'(a)|^{-h} m_{h,e}(B_e(f^n(a), 4r|(f^n)'(a)|)) \\ &\succeq K^{-h}|(f^n)'(a)|^{-h}(4r|(f^n)'(a)|)^2 \\ &\succeq |(f^n)'(a)|^{-h}(4r|(f^n)'(a)|)^h \\ &= 4^h r^h \end{aligned} \tag{21.11}$$

and we are done in this case.

Case 2. $|f^n(a) - b_n| \leq 32A^{\frac{q_{\min}+1}{q_{\min}}} r|(f^n)'(a)| < 32A^{\frac{q_{\min}+1}{q_{\min}}}\delta(f^{-1}(\infty))$, *where* A *was defined in* (18.20).

It then follows from Lemma 20.3.8 that

$$m_{h,e}\Big(B_e(f^n(a), K^{-1}|(f^n)'(a)|)\Big) \succeq (K^{-1}|(f^n)'(a)|)^h.$$

Thus,

$$m_{h,e}(B_e(a,r)) \geq K^{-h}|(f^n)'(a)|^{-h}(K^{-1}|(f^n)'(a)|)^h \asymp r^h.$$

And we are done in this case.

It remains to consider the following.

Case 3. $r|(f^n)'(a)| < \frac{1}{8}KA^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a) - b_n|$.

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

But then

$$\begin{aligned} r|(f^{n+1})'(a)| &= r|(f^n)'(a)||f'(f^n(a))| \\ &< \frac{K}{8}A^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a)-b_n|(A|f^{n+1}(z)|)^{\frac{q_n+1}{q_n}} \\ &\le \frac{K}{8}A^{-\frac{q_{\min}+1}{q_{\min}}}A^{\frac{1}{q_n}+1}|f^{n+1}(a)| \\ &\le \frac{K}{8}|f^{n+1}(a)| \end{aligned}$$

contrary to the definition of n. So this case is ruled out and Lemma 21.3.5 is proved. ■

We are ready to prove the following.

Theorem 21.3.6 *Let $f:\mathbb{C}\longrightarrow\widehat{\mathbb{C}}$ be a compactly nonrecurrent regular elliptic function. If $\Omega(f)=\emptyset$, then the h-dimensional packing measure Π_e^h of every bounded Borel subset of $J(f)$ is finite and $\Pi_s^h(J(f))<+\infty$.*

Proof Consider an arbitrary point

$$z\in J(f)\backslash\bigcup_{n=0}^{\infty}f^{-n}(\{\infty\}\cup\mathrm{Crit}(f))$$

and a radius $r\in(0,\gamma(a\xi)^{-1})$, where $\gamma>0$ was defined in (18.22) while a and ξ come from Lemma 20.2.3. Let $s\ge 0$ be associated with the point z and the radius r/ξ according to Lemma 20.2.3. It follows from this lemma and Theorem 17.1.8 that there exists

$$f_z^{-s}:B_e\big(f^s(z),2\xi r|(f^s)'(z)|\big)\longrightarrow\mathbb{C},$$

a unique holomorphic branch of f^{-s} sending $f^s(z)$ to z. Therefore, if Case (a) of this lemma holds, then we get from Lemmas 8.3.13 and 10.4.7 that

$$\begin{aligned} m_{h,e}(B_e(z,r)) &\ge K^{-h}|(f^s)'(z)|^{-h}m_{h,e}\Big(B_e\big(f^s(z),K^{-1}r|(f^s)'(z)|\big)\Big) \\ &\succeq K^{-h}|(f^s)'(z)|^{-h}(K^{-1}r|(f^s)'(z)|)^2 \\ &\asymp r^h(r|(f^s)'(z)|)^{2-h} \\ &\succeq r^h. \end{aligned} \tag{21.12}$$

If Case (b) of Lemma 20.2.3 holds, then, applying this lemma along with Lemma 21.3.4 (with $\sigma\le K^{-1}\xi$), we get that

$$\begin{aligned} m_{h,e}(B_e(z,r)) &\ge K^{-h}|(f^s)'(z)|^{-h}m_{h,e}(B_e(f^s(z),K^{-1}r|(f^s)'(z)|)) \\ &\succeq K^{-4}|(f^s)'(z)|^{-h}(K^{-1}r|(f^s)'(z)|)^h \\ &\asymp r^h. \end{aligned}$$

Combining this and (21.12) completes the proof of the first part because of Theorem 1.6.4(a). Since $\Pi_e^h(A) = \Pi_e^h(A+w)$ for every $\omega \in \Lambda$ and since $\frac{d\Pi_s^h}{d\Pi_e^h}(z) = (1+|z|^2)^{-h}$, we get with $R = 4\mathrm{diam}(\mathcal{R})$, where $\mathcal{R}$ is an arbitrary fixed fundamental parallelogram, that

$$\begin{aligned}
\Pi_s^h(J(f)) &= \sum_{k=0}^{\infty} \Pi_s^h(A(0;2^kR,2^{k+1}R)) + \Pi_s^h(B_e(0,R)) \\
&\preceq \Pi_e^h(B_e(0,R)) + \sum_{k=0}^{\infty} \Pi_e^h(A(0;2^kR,2^{k+1}R))R^{-2h}4^{-hk} \\
&\preceq \Pi_e^h(B_e(0,R)) + \sum_{k=0}^{\infty}(2^kR)^2R^{-2h}4^{-hk} \\
&= \Pi_e^h(B_e(0,R)) + R^{(2(1-h))}\sum_{k=0}^{\infty}4^{(1-h)k} < +\infty
\end{aligned}$$

since $h > 1$. We are done. ■

Proposition 21.3.7 *Let* $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *be a compactly nonrecurrent regular elliptic function. If* $\mathrm{HD}(J(f)) = 2$*, then* $J(f) = \mathbb{C}$*.*

Proof Since Π_e^2 and S, the two-dimensional Lebesgue measure on $\mathbb{C}$, coincide up to a multiplicative constant, it follows from (already proved) Theorem 21.0.1(b) that if $h = 2$, then $S(J(f)) > 0$. So, in order to prove our proposition, it suffices to show that if $J(f) \subsetneq \mathbb{C}$, then $S(J(f)) = 0$. So, suppose that $J(f) \neq \mathbb{C}$. We want to show that

$$S(J(f)\backslash \mathrm{Sing}^-(f))) = 0.$$

For any integer $l \geq 1$, let the set Z_l have exactly the same meaning as in the proof of Theorem 20.3.11. Since $J(f)$ is a Λ_f-invariant nowhere dense subset of $\mathbb{C}$, there exists $\varepsilon > 0$ such that, for every $y \in \mathbb{C}$, there exists $y_\varepsilon \in B_e\left(y, \frac{1}{2l}\right)$ such that

$$B_e(y_\varepsilon, \varepsilon) \subseteq B_e\left(y, \frac{1}{2l}\right)\backslash J(f). \tag{21.13}$$

Keep the notation from the proof of Theorem 20.3.11. Fix an arbitrary point $z \in Z_l$. By Theorem 8.3.8, the $\frac{1}{4}$-Koebe Theorem (Theorem 8.3.3), and (21.13), we have that

$$\begin{aligned}
f_z^{-n_k}(B_e(f^{n_k(z)}(z),\varepsilon)) &\subseteq f_z^{-n_k}\big(B_e\big(f^{n_k(z)}(z),(2l)^{-1}\big)\backslash J(f)\big) \\
&\subseteq B_e\big(z, K|(f^{n_k})'(z)|^{-1}(2l)^{-1}\big)\backslash J(f)
\end{aligned}$$

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press

and

$$f_z^{-n_k}(B_e(f^{n_k}(z)_\varepsilon, \varepsilon)) \supset B_e\left(f_z^{-n_k}(f^{n_k}(z)_\varepsilon), \frac{1}{4}\varepsilon|(f^{n_k(z)})'(z)|^{-1}\right).$$

Therefore, we see that

$$\frac{S\big(B_e\big(z, K|(f^{n_k})'(z)|^{-1}(2l)^{-1}\big)\backslash J(f)\big)}{S\big(B_e\big(z, K|(f^{n_k})'(z)|^{-1}(2l)^{-1}\big)\big)} \geq \left(\frac{\varepsilon l}{2K}\right)^2 > 0.$$

So, z is not a Lebesgue density point for the set Z_l; therefore, $S(Z_l) = 0$. Hence,

$$S(J(f)) = S\big(J(f)\backslash \mathrm{Sing}^-(f)\big) = S\left(\bigcup_{l=1}^{\infty} Z_l\right) = \sum_{l=1}^{\infty} S(Z_l) = 0.$$

The proof of Theorem 21.3.6 is complete. ■

Theorem 21.0.1 is now a logical consequence of Section 21.1, Proposition 21.2.1, Lemma 21.2.2, and Theorem 21.3.6.

https://doi.org/10.1017/9781009215985.016 Published online by Cambridge University Press