DYNAMIC CATASTROPHES IN STARS

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The invariance of the equations of linear stability of a physical system under a change of sign of the dependent variables implies that we cannot foresee, once an instability is detected, in which direction the system will evolve. In the context of radial dynamical stability this means that the linear analysis does not allow us to discriminate between a subsequent contraction or expansion. To fix up the arrow of the evolution beyond the onset of the instability, a nonlinear analysis is required.

Recently mathematicians have developed a broad framework currently known as "catastrophe theory" (CT) which can be typified as a systematically precise equivalent to the analysis of a singular point of a vector field. Therefore we are entitled to expect that CT offers an ideal context for a nonlinear description of stellar stability. To take full advantage of CT however, we have to meet two conditions for our system : (a) it is a "gradient system", i.e. stable equilibrium states correspond to the minima of a potential ; (b) the total number of "effective" dependent variables ("behaviour variables" in Zeeman's 1977 terminology) must not exceed 2. At first sight these requirements seem so stringent that one might rightly wonder whether Nature has any such systems in store. To our surprise the problem of radial dynamical stability can actually be reformulated to obey both requirements. Under conservation of entropy the equilibrium states of a star satisfy a minimum energy theorem. The total energy which plays the part of the potential of CT (condition a) is given in standard notations by

$$E\,\{r\} = \int_0^M dm\,\{U\,[V\,(r,\dot{r}),S(m),X(m)] - Gm/r\} \qquad 1$$

(V is here the specific volume, X stands for the composition profiles and $\dot{r}$ is the derivative of the radial position with respect to mass). The explicit form of V in terms of r and $\dot{r}$ flows from the conservation of mass. Any distribution r(m) that renders the functional E{r} minimum specifies a dynamically stable equilibrium state. Since r(m) stands in fact for an infinity of behaviour variables, namely the radial location at any mass point m, one has to devise a method to drastically reduce this number of variables to $\leqslant 2$ in order to conform to condition (b).

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This can be performed if we focus on the neighbourhood of a neutral model.

To this end we consider a set of data D_o (total mass, fixed entropy distribution, fixed composition profile and given thermodynamics $U = U(V,S,X)$). Let $r_o(m)$ be the corresponding equilibrium model for which (1) takes on the minimum value $E_o\{r_o\}$. We next alter slightly the initial data, to obtain new models characterized by distributions $r(m)$. Without loss of generality the latter can be related to $r_o(m)$ by an expression of the form

$$r(m) = r_o(m)(1 + \eta(m)\delta z) \ , \delta z = (R - R_o)/R_o \ , \ \eta(M) = 1, \qquad 2$$

where δz is the relative excess of the new radius with respect to that of the reference model and $\eta(m)$ is a function of m. On substitution of (2) into the energy functional (1) we generate the following Taylor expansion in z :

$$E\ \{r_o(1 + \eta\delta z)\} - E_o\ \{r_o\} = E_1\delta z + \ldots + E_N\ \delta z^N + \ldots, \qquad 3$$

where the coefficients E_n, $n = 1,2,\ldots,N$ are functionals of r_o and η. Note at this stage that the quantities $\eta(m)$ and δz still make up an infinite number of behaviour variables.

The reference model being neutral, the coefficients E_n, $n = 1,2,\ldots$ of the RHS of (3) must vanish up to $n = 3$ at least. In fact E_1 has to vanish identically in η to satisfy the stationarity condition of the energy ; this fixes $r_o(m)$. One can convince oneself that $E_2 = 0$ is equivalent to the equation of the radial neutral eigenfunctions which provides $\eta(m)$. Moreover as a consequence of the nondegeneracy of the eigenvalues of this Sturm-Liouville type problem, $\eta(m)$ is necessarily unique. The same nondegeneracy argument can be used to show that E_3 does not vanish together with E_1 and E_2 (with one prominent exception, namely a star with $\Gamma_1 = 4/3$ all over the configuration (cf. Demaret et al., 1978)). For the typical reference model expansion (3) thus reduces to

$$V(\delta z) \equiv \{r_o(1 + \eta\delta z)\} - E_o\{r_o\} = E_3\ \delta z^3 + O(\delta z^4)\ , \qquad 4$$

where r_o and $\eta(m)$ are now uniquely fixed so that E_3 becomes a constant. In (4) just a single behaviour variable survives, namely δz, so that we now do satisfy condition (b).

To complete our analysis it remains to compare the potential (4) to the list of "catastrophes" classified in CT. It is seen to coincide with the simplest representative of this list, the "fold", $v(x) = \pm x^3$. Under an arbitrary perturbation this potential is deformed into $v(x,\gamma) = \pm x^3 + \gamma x$, which represents the universal unfolding of the fold catastrophe. It is characterized by a single parameter ("control parameter").

The message of CT is then twofold : i) Under any slight alteration of the reference model, in particular under natural evolution, the new models distribute over a branch of a parabola in a $\delta z - \gamma$ plane, where γ parametrizes the modification ; the vertex of the parabola coincides with the neutral model. In other words, the most contrived local behaviour of a linear series $R - \gamma$ of hydrostatic models is given by folds ; critical points of higher complexity (multiple points,...) are not allowed to occur. ii) Since the star seeks a minimum of energy the arrow of the evolution at the fold catastrophe is fully determined by the sign of the easily calculable integral E_3 : if $E_3 > (<) \, 0$ at the onset of the instability, the star will undergo a contraction (expansion) (fig. 1). A particularly instructive situation arises if Γ_1 differs sufficiently little from 4/3, say $\Gamma_1 = 4/3 + O(\varepsilon)$, the correction depending on the local physics. Then the functional E_3 reduces to

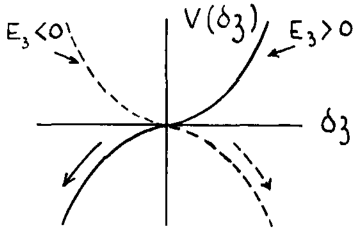

fig. 1

$$E_3 = 3/2 \int_0^M dm \; PV \left\{ \Gamma_1 - 4/3 + 4 \left(\frac{\partial \ln \Gamma_1}{\partial \ln V} \right)_{S,X} \right\} + O(\varepsilon^2) \quad , \qquad 5$$

so that a sufficient condition for contraction (expansion) obtains if the curly bracket is positive (negative). For massive stars which are essentially convective, the thermodynamic derivative in (5) can be replaced by the local derivative in the model. If then the $O(\varepsilon)$ term entering Γ_1 is roughly linear in the density, we notice that this criterion implies that if Γ_1 decreases outwards, the star must contract (expand) once it becomes unstable.

Elementary CT thus nicely captures the outcome of radial dynamical instabilities. Since requirement (a) remains true for nonradial dynamical motions as well, one may ask whether the same mathematical framework applies for nonradial stability. The typical neutral model occurs now when the entropy profile becomes flat in one zone of the star. But such a model displays an infinite number of neutral g-modes ; this property prevents us from eliminating the infinity of behaviour variables. Hence we cannot satisfy requirement (b). Standard CT is thus unable to cope with the unfolding of this neutral model, simply because the archetypal patterns it gives rise to are too numerous : in fact there is a continuous infinity of distinct régimes beyond the limit of stability.

The breakdown of CT in the field of nonradial dynamical stability - i.e. in the characterization of the onset of stellar convection - is to be traced to its pervading assumption of smoothness. To efficiently deal with convection the latter premiss is to be dispensed with. A novel mathematical frame in which smoothness is superseded by infinite irregularity is indeed offered by fractal theory (Mandelbrot, 1977). The latter enables one to encompass unwieldy structures such as "strange attractors" and "chaos", currently encountered in elementary hydrodynamic models (Ruelle and Takens, 1971). Attempts to tame convection via fractals have already been sketched by Frisch (1978).

Successful applications of CT to other direct stellar stability considerations have been made by Casti (1974 ; collapse), Poston and Stewart (1978 ; bifurcation of Maclaurin spheroids), Barbaro et al. (1979 ; Lynden-Bell's gravothermal catastrophe). The CT formalism is however of much wider applicability. In particular questions connected with the unfolding of multiple eigenvalues receive an elegant solution in this context (Arnold, 1976 ; Poston and Stewart, 1978). The recently stressed phenomenon of "avoided crossing" of g-modes (Aizenman et al.,

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1977 ; Christensen-Dalsgaard, 1979) emerges from CT as a mere effect of the structural instability of the crossing : What requires an explanation is not the avoided crossing but the possible occurrence of actual crossings ! The latter always stem from a special invariance group ("symmetry") of the underlying stellar model, as flows directly from an application of the von Neumann-Wigner (1929) perturbation scheme. This point was already explicited in Perdang (1969) for the avoided crossing observed in compressible cylinders, which precisely degenerates into an actual crossing for the incompressible model (Ostriker, 1964). In the framework of secular stability CT predicts that if a crossing of two real secular eigenvalues occurs for a given model, then arbitrarily nearby models must exist which have complex roots.

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