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Pfaffian equations and the Cartier operator

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1. Notation

k	$=$	algebraically closed field of characteristic $p > 0$
X	$=$	smooth, irreducible, projective variety over k
K	$=$	function field of X
Ω^i_X	$=$	sheaf of differential forms of degree i on X
Ω^i_K	$=$	i-differential forms on the function field of X, also considered as a constant sheaf on X
$C[n]$	$=$	a complex C shifted in such a way that $C[n]^i = C^{i+n}$
F^nC	$=$	a complex C truncated (brutally) so that $F^nC^i = 0$ for $i < n$ and $F^nC^i = C^i$ for $i \geqslant n$
$\Omega^{\cdot}_X$	$=$	the algebraic de Rham complex of X

A *Pfaffian equation* on X is an invertible subsheaf $L \hookrightarrow \Omega^1_X$ of the sheaf of differential 1-forms on X. It can thus be thought of as a global section of $\Omega^1_X \otimes L^{-1}$ or, by choosing an isomorphism $L^{-1} \simeq \mathcal{O}_X(E) \subset K$, as a meromorphic differential form ω on X. We say that the Pfaffian equation has a *first integral* if there is a non-empty open subset $U \subset X$ and a smooth map $f\colon U \to \mathbf{P}^1$ such that $L_U \simeq f^*\Omega^1_{\mathbf{P}^1}$ as subsheaves of Ω^1_X; that is, if L 'comes from' a rational map to a curve. In this case, f, viewed as a rational function, is called a first integral. Note that as rational differential forms, $\mathrm{d}f \wedge \omega = 0$.

A (reduced and irreducible) subvariety of codimension one $i\colon D \hookrightarrow X$ is said to be a *hypersurface solution* for ω, if $i^*\omega = 0$, as a section of $\Omega_D \otimes L^{-1}$. This is the same as requiring the composed map

$$L \to \Omega^1_X \to i_*\Omega^1_D,$$

to be zero. Note that if ω has a first integral f, then the closure of the irreducible components of $f = h^p$ are solutions, for any rational function h. The reader is referred to [4] for a discussion of relations to classical differential equations.

If D is represented as a Cartier divisor by a collection (f_i, U_i), then D is a solution for ω if and only if $(\mathrm{d}f_i/f_i) \wedge \omega \in \Gamma(U_i, \Omega^2_X \otimes L^{-1})$ for each i.

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The paper [4] studies Pfaffian equations on compact complex manifolds satisfying certain conditions on its Hodge-to-de Rham spectral sequence. The main result there is that for Pfaffian equations on such manifolds, there are infinitely many irreducible hypersurface solutions only in the trivial case when ω admits a first integral. We wish to generalize this to include the case of varieties in positive characteristic. For compact smooth varieties in characteristic zero, Jouanolou's condition is automatically satisfied. However, an hypothesis is necessary over fields of positive characteristic:

THEOREM 1. *Suppose all global one-forms on X are closed and ω does not have a first integral. Then there are only finitely many irreducible hypersurface solutions for ω.*

The proof of this theorem is largely modeled on the complex case studied by Jouanolou. However the existence of non-constant d-closed functions, namely p-powers, keeps the translation from being entirely straightforward. It was thus quite surprising to the author that a pleasant resolution arises from the systematic use of the Cartier operator. That is, this endows the eventual proof with a nature particular to characteristic p. The relation between the solution varieties of a closed differential form and of its Cartier descendants seems to deserve careful study.

This theorem has been used by Vojta (in characteristic zero) [6] to obtain bounds for heights on algebraic points for curves over function fields. It also can be used to bound families of curves on surfaces satisfying certain numerical conditions, as a consequence of Bogomolov's inequality [1]. Although the class of surfaces satisfying Bogomolov's inequality in positive characteristic is still unknown, this paper illustrates that intimately related results can be obtained, provided certain 'ordinarity hypotheses' are made. One can conjecture, then, that the counterexamples to Bogomolov's inequality in positive characteristic arise from a failure of ordinarity.

2. Preliminaries

We will need a few facts about the de Rham cohomology $H^i_{DR}(X) := \mathbf{H}^i(X, \Omega^{\cdot}_X)$ of X (the boldface denotes hypercohomology) as well as the crystalline cohomology $H^i_{cr}(X/W)$ of X with coefficients in the Witt vectors W of k. The latter can be realized as the hypercohomology of the *de Rham–Witt complex* $W\Omega^{\cdot}_X$ [3]. There is a map of complexes

$$W\Omega^{\cdot}_X \to \Omega^{\cdot}_X,$$

which induces a map from the slope spectral sequence

$$H^q(W\Omega^p_X) \Rightarrow H^{p+q}_{cr}(X),$$

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to the Hodge-to-de Rham spectral sequence

$$H^q(\Omega_X^p) \Rightarrow H_{DR}^{p+q}(X).$$

Recall the following facts about the slope spectral sequence ([3] Corollary II.3.3 and Proposition II.3.11)

$$\begin{aligned} E_\infty^{p,0} &= E_1^{p,0} = H^0(W\Omega_X^p), \\ E_\infty^{0,1} &= E_1^{0,1} = H^1(W\mathcal{O}_X). \end{aligned} \tag{$*$}$$

Given an invertible sheaf L on X, we can associate to it a first Chern class $c_1(L)$ in the crystalline cohomology of X by using the map of complexes

$$\mathcal{O}_X^*[-1] \to W\Omega_X^{\cdot},$$

given by the logarithmic derivative $f \mapsto \mathrm{d}\tilde{f}/\tilde{f}$ ($\tilde{f} = (f, 0, 0, \ldots) \in W\mathcal{O}_X$). That is, this map of complexes induces the Chern class map

$$c_1\colon H^1(\mathcal{O}_X^*) \to H_{cr}^2(X).$$

The map of complexes factors through

$$\mathcal{O}_X^*[-1] \to F^1 W\Omega_X^{\cdot} \subset W\Omega_X^{\cdot},$$

and $(*)$ implies, in fact, that

$$\mathbf{H}^2(F^1 W\Omega_X^{\cdot}) \subset H_{cr}^2(X),$$

so that the Chern class can be seen as lying in the first group.

There is also a quotient map $F^1 W\Omega_X^{\cdot}[1] \to W\Omega_X^1$, sending $c_1(L)$ to the Chern–Hodge class $\mathrm{ch}(L) \in E_\infty^{1,1} \subset H^1(W\Omega_X^1)$. Its image $\mathrm{ch}'(L)$ inside $H^1(\Omega_X^1)$ is the usual Chern–Hodge class, which may be interpreted as the class of the Ω_X^1-torsor given by the connections on L. In particular, L admits a connection iff $\mathrm{ch}'(L) = 0$. Recall that if L is associated to the Cartier divisor (f_i, U_i) with transition functions g_{ij}, then a connection is equivalent to the data of regular 1-forms A_i on U_i satisfying $A_i - A_j = (\mathrm{d}f_i/f_i) - (\mathrm{d}f_j/f_j) = \mathrm{d}g_{ij}/g_{ij}$. We shall refer to such a collection also as connection forms for the Cartier divisor, as well as for the invertible sheaf it defines. The relation between $\mathrm{ch}'(L)$ and connections on L follows from a straightforward computation using Cech cocycles for the de Rham complex, and the same computation for $W\Omega_X^{\cdot}$ yields the fact that if $\mathrm{ch}(L) = 0$, then we can find local sections $\tilde{A}_i$ of $W\Omega_X^1$, which satisfy $\tilde{A}_i - \tilde{A}_j = \mathrm{d}\tilde{g}_{ij}/\tilde{g}_{ij}$, and which therefore reduce to connection forms for L. Since $W\Omega_X^{\cdot}$ forms a differential graded algebra, we get $\mathrm{d}(\mathrm{d}\tilde{g}_{ij}/\tilde{g}_{ij}) = 0$, so we see that $\mathrm{d}\tilde{A}_i = \mathrm{d}\tilde{A}_j$. Therefore, the collection $(\mathrm{d}\tilde{A}_i)$

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defines a global section $\tilde{R}$ of $W\Omega^2_X$, which reduces to the usual curvature form R of the connection (A_i). As a special case of $(*)$

$$E_1^{2,0} = E_\infty^{2,0} = H^0(W\Omega^2_X) \subset H^2_{cr}(X),$$

and in the situation just described when $\mathrm{ch}(L) = 0$, another Cech computation show us that $c_1(L) = \tilde{R}$ with respect to this inclusion. In particular, if $c_1(L) = 0 (\Rightarrow \mathrm{ch}(L) = 0)$, then we can find a collection $\tilde{A}_i$ so that $\tilde{R} = 0$. This implies the following

LEMMA 1. *Suppose $c_1(L) = 0$ in crystalline cohomology. Then L admits a connection with vanishing curvature.*

Our preceding remarks amount to the fact that it admits something stronger, namely, an 'integrable de Rham–Witt connection.'

This lemma applies, for example, in the case where L is algebraically equivalent to zero. Note that the de Rham–Witt complex is used to circumvent the possible non-degeneration of the Hodge-to-de Rham spectral sequence.

Since any two connections differ by a global one-form we see that under the hypothesis that all global one-forms are closed, *any* connection for a line bundle with vanishing first Chern class will have zero curvature.

Recall that *the Cartier operator C* ([5], 7.2) fits into an exact sequence

$$\Omega^{i-1}_X \stackrel{d}{\to} Z\Omega^i_X \stackrel{C}{\to} \Omega^i_X \to 0,$$

where $Z\Omega^i_X$ refers to the sheaf of closed differential i-forms. Together with this exact sequence (which includes a theorem of Cartier), it is characterized by

$$C(f^{p-1}\mathrm{d}f) = \mathrm{d}f, \qquad C(f^p\alpha) = fC(\alpha), \qquad C(1) = 1,$$

and

$$C(\alpha \wedge \beta) = C(\alpha) \wedge C(\beta),$$

for any function f and closed forms α and β.

Taking the stalk at the generic point gives an exact sequence of rational differential forms

$$\Omega^{i-1}_K \stackrel{d}{\to} Z\Omega^i_K \stackrel{C}{\to} \Omega_K \to 0.$$

Now when all global forms on X are closed, if (A_i) defines a connection form for a Cartier divisor $D = (f_i)$ with vanishing first Chern class, then as noted above, the A_i are closed so we may apply C to them. The fact that $C(\mathrm{d}f_i/f_i) = \mathrm{d}f_i/f_i$ implies that $C(A_i)$ also defines a connection form for D, and hence, is again closed. Thus all iterations $C^k(A_i)$ are defined, and are all closed.

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We will use the map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & K^* & \longrightarrow & K^*/\mathcal{O}_X^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_X^1 & \longrightarrow & \Omega_K^1 & \longrightarrow & \Omega_K^1/\Omega_X^1 & \longrightarrow & 0 \end{array},$$

where the vertical arrows are induced by the logarithmic derivatives.

LEMMA 2. *Logarithmic differentiation induces an injection*

$$\frac{K^*}{(K^*)^{(p)}O^*} \hookrightarrow \frac{\Omega_K^1}{\Omega_X^1}.$$

Proof. Let f be a rational function such that $\mathrm{d}f/f$ is regular. We will show that f is locally a unit times a p-power.

It suffices to show this in the local ring R of a point. But since R is a unique factorization domain, we may write $f = u\pi_1^{n_1}\cdots\pi_k^{n_k}$ where u is a unit and the π_i's are distinct primes. So $\mathrm{d}f/f = \mathrm{d}u/u + \Sigma_i n_i(\mathrm{d}\pi_i/\pi_i)$. Since $\mathrm{d}f/f$ and $\mathrm{d}u/u$ are regular, we see by localizing at each prime in turn that all the n_i's must be multiples of p. □

Thus, we get an injection

$$\Gamma\left(\frac{K^*}{(K^*)^{(p)}O^*}\right) \hookrightarrow \Gamma\left(\frac{\Omega_K^1}{\Omega_X^1}\right).$$

Denote by Div the group of Weil divisors on X. It is a free abelian group generated by the irreducible Weil divisors and we have Div/p Div $\simeq$ Div $\otimes\, \mathbf{F}_p =$ the $\mathbf{F}_p$ vector space generated by the irreducible Weil divisors.

LEMMA 3. Div/p Div $\hookrightarrow \Gamma(K^*/(K^*)^{(p)}O^*)$.

Proof. We need only prove the corresponding statement for Cartier divisors, that is, that

$$\Gamma\left(\frac{K^*}{\mathcal{O}^*}\right) \Big/ p\Gamma\left(\frac{K^*}{\mathcal{O}^*}\right) \hookrightarrow \Gamma\left(\frac{K^*}{(K^*)^{(p)}O^*}\right),$$

via the canonical map.

Let D be a Cartier divisor which is (for a fine enough covering) locally represented by p-powers: $D = [(f_i^p, U_i)]$. Since $(f_i/f_j)^p = f_i^p/f_j^p$ is a unit on

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the overlaps, so is f_i/f_j. Thus, (f_i, U_i) defines a Cartier divisor D' such that $pD' = D$. □

Composing the two lemmas above, we get an injection

$$\frac{\mathrm{Div}}{p\,\mathrm{Div}} \hookrightarrow \Gamma\left(\frac{\Omega^1_K}{\Omega^1_X}\right).$$

PROPOSITION 1. *The inclusion above induces an injection*

$$\psi\colon \mathit{Div} \otimes_{\mathbf{Z}} k = \left(\frac{\mathit{Div}}{p\,\mathit{Div}}\right) \otimes_{\mathbf{F}_p} k \hookrightarrow \Gamma\left(\frac{\Omega^1_K}{\Omega^1_X}\right).$$

Proof. Let $D_1, \ldots, D_r$ be Weil divisors. We will show by induction on r that a non-trivial k-linear relation between the $\psi(D_i)$ implies a non-trivial $\mathbf{F}_p$-linear relation between the D_i. This will show that if $D_1, \ldots D_r$ are linearly independent mod p, then, for any non-trivial set of coefficients $a_1, \ldots, a_r$ in k, $\psi(\Sigma_i a_i D_i) = \Sigma_i a_i \psi(D_i) \neq 0$. The case $r = 1$ is the injection above. Assume now that

$$\sum_{i=1}^{r} a_i \psi(D_i) = 0,$$

where none of the $\psi(D_i)$ are zero. (Otherwise, we are done, by induction.) Since $\psi(D_1) \neq 0$, D_1 has an irreducible component E with multiplicity prime to p. Let U be an affine open set which intersects E and on which each D_i has a defining equation f_i. Our assumption implies that $\Sigma_{i=1}^r a_i(\mathrm{d}f_i/f_i)$ is regular on U. Now, choose a curve C in U which intersects E tranversally at a simple point $x_0 \in E$–(other components of D_1) and is not contained in the support of any D_i. Such a C exists by Bertini's Theorem. So each f_i restricts to a rational function f_i^0 on C and if we put $m_i = \mathrm{Res}_{x_0}(\mathrm{d}f_i^0/f_i^0)$, then $\Sigma_1^r a_i m_i = 0$. Also, $m_1 \neq 0$ in $\mathbf{F}_p$, by our choice of C, E, and x_0. Thus, we get

$$\sum_{i=2}^{r} a_i \psi(m_1 D_i - m_i D_1) = m_1 \sum_{1}^{r} a_i \psi(D_i) = 0.$$

Hence, by the induction hypothesis there exist $n_i \in \mathbf{F}_p$ not all zero, such that

$$\sum_{2}^{r} n_i(m_1 D_i - m_i D_1) = 0. \quad (\mathrm{mod}\ \ p).$$

But then, since some $n_i m_1 \neq 0$, this gives a non-trivial $\mathbf{F}_p$-linear relation among the D_i. □

Assume, henceforward, that all global one-forms on X are closed.

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Now denote by Div^0 the subgroup of Div whose associated invertible sheaves have vanishing Chern classes in crystalline cohomology and let $D^0 := \mathrm{Im}(\mathrm{Div}^0 \otimes k \to \mathrm{Div} \otimes k)$. The inclusion constructed above

$$\mathrm{Div} \otimes k \hookrightarrow \Gamma\left(\frac{\Omega^1_K}{\Omega^1_X}\right),$$

when composed with the boundary map δ in the exact sequence

$$0 \to \Gamma(\Omega^1_X) \to \Gamma(\Omega^1_K) \to \Gamma\left(\frac{\Omega^1_K}{\Omega^1_X}\right) \xrightarrow{\delta} H^1(\Omega^1_X),$$

gives nothing but the Chern–Hodge class $\mathrm{ch}' \otimes 1_k$. Thus, it is zero on D^0. We use this fact to lift the map above to an inclusion

$$\psi\colon D^0 \hookrightarrow \Gamma\frac{(\Omega^1_K)}{\Gamma(\Omega^1_X)}.$$

Given a divisor $D \in \mathrm{Div}^0$, a rational differential form whose class is $\psi(D)$ can be explicitly described as follows: Choose a Cartier representative (f_i, U_i) for D. We may then find a collection of 1-forms (A_i) defining a connection for D. Then

$$A_i - A_j = \frac{\mathrm{d}f_i}{f_i} - \frac{\mathrm{d}f_j}{f_j} \Rightarrow A_i - \frac{\mathrm{d}f_i}{f_i} = A_j - \frac{\mathrm{d}f_j}{f_j},$$

on the overlaps of the U_i's. Thus, these local forms glue to give a rational one form, α. In this case, since $c_1(D) = 0$, α is closed, as discussed above, so we may apply the Cartier operator to it. But locally, $C(\alpha)$ is just $\mathrm{d}f_i/f_i - C(A_i)$ which is again closed, since the $C(A_i)$ define a connection for the same divisor.

LEMMA 5. *If $D \in \mathrm{Div}^0$, then, given any rational representative α for $\psi(D)$, we may apply the Cartier operator repeatedly to get closed one-forms $C^k(\alpha)$.*

This lemma follows from the preceding discussion and the fact that any two representatives in $\Gamma(\Omega_K)$ for $\psi(D)$ will differ by a global one-form.

If α and β are rational 1-forms and $a, b \in k$, then $C(a\alpha + b\beta) = a^{1/p}C(\alpha) + b^{1/p}C(\beta)$, so the property mentioned in the lemma is closed under k-linear combination, in an obvious sense.

COROLLARY 1. *For any $x \in D^0$, and any representative α for $\psi(x)$, all the $C^k(\alpha)$ are defined and closed.*

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3. Proof of theorem

Denote by N the group of divisors generated by the irreducible solutions of ω. Thus, N is a free abelian subgroup of Div and the number of irreducible solutions is equal to the rank of N or, equivalently, the dimension of $N \otimes \mathbf{F}$. Let $N_0 \subset N \cap \mathrm{Div}^0$ be the subgroup of N consisting of divisors algebraically equivalent to zero.

Note now that N being generated by irreducible divisors, is a saturated subgroup of Div, allowing

$$N \otimes \mathbf{F}_p \subset \mathrm{Div} \otimes \mathbf{F}_p.$$

Also, the exact sequence

$$N_0 \otimes \mathbf{F}_p \to N \otimes \mathbf{F}_p \to \frac{N}{N_0} \otimes \mathbf{F}_p \to 0,$$

together with the finite generation of $N/N_0 \subset NS(X)$ (the Neron–Severi group of X) tells us that it suffices to prove that

$$M := [\mathrm{Im}(N_0 \otimes \mathbf{F}_p \to \mathrm{Div} \otimes \mathbf{F}_p)] \otimes k \subset D^0,$$

is finite-dimensional. We constructed above an injection, still denoted by the same letter

$$\psi\colon M \hookrightarrow \frac{\Gamma(\Omega^1_K)}{\Gamma(\Omega^1_X)}.$$

Composing ψ with wedge product by ω, we get a map

$$h\colon M \to \frac{\Gamma(\Omega^2 \otimes L^{-1})}{\omega \wedge \Gamma(\Omega^1)}.$$

The image lies in the given subspace because elements of M are linear combinations of solutions for ω.

Let x be in the kernel of h and let α be a rational form representing $\psi(x)$. Then there exists a global 1-form γ such that $\alpha \wedge \omega = \gamma \wedge \omega$, or $(\alpha - \gamma) \wedge \omega = 0$. That is, $x \in \mathrm{Ker}\ h$ iff $\psi(x)$ has a representative which is proportional to ω as a rational 1-form.

We now distinguish two distinct cases:

(1) There exists an $x \in M$ and a representative α for $\psi(x)$ such that $\alpha \wedge \omega = 0$ and some iteration $C^k(\alpha)$ has a first integral.

In this case, note that α itself does not have a first integral by assumption, since it is proportional to ω. Write $C^k(\alpha) = f\,\mathrm{d}g$ for rational functions f, g. Then

$$C^{k-1}(\alpha) = f^p g^{p-1}\,\mathrm{d}g + \mathrm{d}x_{k-1},$$

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for some rational function x_{k-1}

$$C^{k-2}(\alpha) = f^{p^2} g^{p(p-1)} g^{p^2-1}\,\mathrm{d}g + x_{k-1}^{p-1}\,\mathrm{d}x_{k-1} + \mathrm{d}x_{k-2},$$

and, continuing down the powers

$$\alpha = f^{p^k} g^{p^k-1}\,\mathrm{d}g + x_{k-1}^{p^{k-1}-1}\,\mathrm{d}x_{k-1} + \cdots + x_1^{p-1}\,\mathrm{d}x_1 + \mathrm{d}x_0.$$

Now, suppose $i\colon V \hookrightarrow X$ is a subvariety such that $i^*\omega = 0$, and hence $i^*\alpha = 0$. Then, by the naturality of the Cartier operator, $C^n(i^*\alpha) = i^*C^n(\alpha) = 0$ for all n. So

$$i^*(f\,\mathrm{d}g) = 0 \;\Rightarrow\; i^*\,\mathrm{d}x_{k-1} = 0 \;\Rightarrow \cdots \Rightarrow\; i^*\mathrm{d}x_0 = 0.$$

That, is all the differential form summands in the formula above for α must vanish. However, since α does not have a first integral, at least two of the summands are generically linearly independent. Therefore, there is a Zariski closed subset $Z \subset X$ such that all V as above not lying in Z has codimension at least two. This proves the theorem in this case.

(2) For any $x \in M$, if α is a representative for $\psi(x)$ such that $\alpha \wedge \omega = 0$, then none of the $C^k(\alpha)$ have first integrals.

For this case, suppose x and y are contained in the kernel of h, and let α and β (respectively) be rational 1-forms representing $\psi(x)$ and $\psi(y)$ such that $\alpha \wedge \omega = 0,\ \ \beta \wedge \omega = 0$.

Then $\alpha = f\beta$ for some rational function f. But since α and β are both closed, this gives us $\mathrm{d}f \wedge \beta = 0$. Thus $\mathrm{d}f = 0$ since β does not have a first integral, and hence, $f = f_1^p$, for some function f_1. So $\alpha = f_1^p\beta \Rightarrow C(\alpha) = f_1 C(\beta)$. But $C(\beta)$ and $C(\alpha)$ are also closed, and also do not have first integrals. Thus f_1 is also a p-power. Continuing in this way, we see that f must be a infinite p-power. So f is a constant and α, β are k-linearly dependent. By injectivity of ψ, this implies that x and y are linearly dependent.

That is, in this second case, dim(Ker h) $\leqslant 1$. So we are done again, because

$$\frac{\Gamma(\Omega_X^2 \otimes L^{-1})}{\omega \wedge \Gamma(\Omega^1)},$$

is finite-dimensional. □

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