Can. J. Math., Vol. XLII, No. 3, 1990, pp. 561–573

TENSOR PRODUCTS OF UNITARY SUPER-VIRASORO MODULES WITH CENTRAL CHARGE $\frac{7}{10}$

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ABSTRACT. The two Virasoro superalgebras, known as the Neveu-Schwarz algebra and the Ramond algebra, each have two unitary irreducible lowest weight modules with central charge $\frac{7}{10}$. In this paper, I show how tensor products of these modules decompose into finite direct sums of irreducible modules with central charge $\frac{7}{5}$.

1. Introduction. Let $\mathcal{V}$ denote the *Virasoro algebra*; that is, the complex Lie algebra with basis $\{L_m : m \in \mathbf{Z}\} \cup \{c\}$ and commutation relations

$$[L_m L_n] = (n-m)L_{m+n} - \delta_{m+n,0}\tfrac{1}{12}(m^3-m)c,$$
$$[L_m c] = 0.$$

We consider two $\mathbf{Z}/2\mathbf{Z}$-graded extensions of $\mathcal{V}$; that is, Lie superalgebras $\mathcal{A} := \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)}$ where $\mathcal{A}^{(0)} := \mathcal{V}$ and $\mathcal{A}^{(1)}$ is defined as follows: for $\mathcal{A} = \mathcal{NS}$ (the *Neveu-Schwarz algebra*), $\mathcal{A}^{(1)}$ has basis $\{G_m : m \in \mathbf{Z} + \frac{1}{2}\}$, and for $\mathcal{A} = \mathcal{R}$ (the *Ramond algebra*), $\mathcal{A}^{(1)}$ has basis $\{G_m : m \in \mathbf{Z}\}$. In each case the additional (anti)commutation relations are

$$[L_m G_n] = (\tfrac{1}{2}m - n)G_{m+n}, \quad [cG_m] = 0,$$
$$[G_m G_n] = 2L_{m+n} - \delta_{m+n,0}\tfrac{1}{3}(m^2 - \tfrac{1}{4})c.$$

We have $[\mathcal{A}^{(\alpha)}\mathcal{A}^{(\beta)}] \subseteq \mathcal{A}^{(\alpha+\beta)}$, $\alpha, \beta \in \mathbf{Z}/2\mathbf{Z}$; in the enveloping associative superalgebra $\mathcal{U}(\mathcal{A})$ we have $[xy] = xy - (-1)^{\alpha\beta}yx$ for $x \in \mathcal{A}^{(\alpha)}, y \in \mathcal{A}^{(\beta)}$.

We write $\mathcal{A}_0$ for the subalgebra spanned by L_0 and c (and G_0 if $\mathcal{A} = \mathcal{R}$), and $\mathcal{A}_\pm$ for the subalgebra of $\mathcal{A}$ spanned by the L_m and G_n with $\pm m, \pm n > 0$. If $\mathcal{A} = \mathcal{NS}$, let $Z := \frac{1}{2}\mathbf{Z}$ and if $\mathcal{A} = \mathcal{R}$, let $Z := \mathbf{Z}$; then Z is the set of eigenvalues for the adjoint action of L_0 on $\mathcal{A}$.

By an $\mathcal{A}$-module we mean a $\mathbf{Z}/2\mathbf{Z}$-graded vector space $\mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(1)}$ on which $\mathcal{A}$ acts such that $[xy]v = x(yv) - (-1)^{\alpha\beta}y(xv)$ for $x \in \mathcal{A}^{(\alpha)}, y \in \mathcal{A}^{(\beta)}, v \in \mathcal{M}$, and $\mathcal{A}^{(\alpha)}\mathcal{M}^{(\beta)} \subseteq \mathcal{M}^{(\alpha+\beta)}$, for $\alpha, \beta \in \mathbf{Z}/2\mathbf{Z}$. The nonzero elements of $\mathcal{M}^{(0)}$ are *even*; those of $\mathcal{M}^{(1)}$ are *odd*. An element $v \in \mathcal{M}$ is *homogeneous* if $v \in \mathcal{M}^{(0)}$ or $v \in \mathcal{M}^{(1)}$. Let $\mathcal{M}$ be an $\mathcal{A}$-module with *lowest weight* $h \in \mathbf{C}$; that is,

$$\mathcal{M} = \bigoplus_{n \in Z} \mathcal{M}_{h+n}, \quad \mathcal{M}_{h+n} = \{x \in \mathcal{M} : L_0 x = (h+n)x\},$$

Received June 30, 1989.

1980 Mathematics Subject Classification (1985 Revision): 17B65, 17A70, 81E40.

This research was done at Yale University and was partially supported by NSF grant DMS-88-06371.

where $\mathcal{M}_{h+n} = \{0\}$ for $n < 0$, $\dim \mathcal{M}_h \geqq 1$, and $\dim \mathcal{M}_{h+n} < \infty$ for $n \in Z, n \geqq 0$. If $\mathcal{M}$ is generated by an eigenvector for c, then c acts on $\mathcal{M}$ as multiplication by a scalar $z \in \mathbf{C}$, the *central charge* of $\mathcal{M}$. From now on we assume that all modules $\mathcal{M}$ satisfy these conditions.

An important class of lowest weight modules are the *Verma modules*. Given $\mathcal{A}$ and $z, h \in \mathbf{C}$, we first define $\mathbf{Z}/2\mathbf{Z}$-graded vector spaces $\mathbf{C}_{z,h}$ and $\bar{\mathbf{C}}_{z,h}$, which always have dimension 1 or 2. If $\mathcal{A} = \mathcal{N}S$, or if $\mathcal{A} = \mathcal{R}$ and $h = -\frac{1}{24}z$, then $\mathbf{C}_{z,h} := \mathbf{C}x$ where x is even, and $\bar{\mathbf{C}}_{z,h} := \mathbf{C}x$ where x is odd. If $\mathcal{A} = \mathcal{R}$ and $h \neq -\frac{1}{24}z$, then $\mathbf{C}_{z,h} := \mathbf{C}x \oplus \mathbf{C}G_0x$ where x is even. We regard each $\mathbf{C}_{z,h}$ and $\bar{\mathbf{C}}_{z,h}$ as an irreducible $\mathcal{A}_- \oplus \mathcal{A}_0$-module by defining $\mathcal{A}_-x := 0$, $L_0x := hx$, $cx := zx$, and if $\mathcal{A} = \mathcal{R}$ and $h = -z/24$ we also define $G_0x := 0$. Now define

$$V(z,h) := \mathcal{U}(\mathcal{A}) \otimes_{\mathcal{U}(\mathcal{A}_- \oplus \mathcal{A}_0)} \mathbf{C}_{z,h}, \quad \text{and}$$
$$\bar{V}(z,h) := \mathcal{U}(\mathcal{A}) \otimes_{\mathcal{U}(\mathcal{A}_- \oplus \mathcal{A}_0)} \bar{\mathbf{C}}_{z,h},$$

respectively the *even* and *odd* Verma modules over $\mathcal{A}$ with central charge z and lowest weight h. For all $z, h \in \mathbf{C}$, $V(z,h)$ (respectively $\bar{V}(z,h)$) has a unique maximal submodule $M(z,h)$ (respectively $\bar{M}(z,h)$), and therefore a unique irreducible quotient $L(z,h)$ (respectively $\bar{L}(z,h)$), the *even* (respectively *odd*) irreducible lowest weight $\mathcal{A}$-module with central charge z and lowest weight h. The evenness or oddness of a module is its *parity*. For more detail, see [**9**]. The distinction between even and odd Verma modules is a special case of the parity change functor, for which see [**8**; pp. 156–7].

Let σ be the conjugate-linear anti-automorphism of $\mathcal{A}$ of order 2 given by $\sigma(L_m) = L_{-m}$, $\sigma(c) = c$, and $\sigma(G_n) = G_{-n}$. A lowest weight $\mathcal{A}$-module $\mathcal{M}$ is *unitary* if it admits a positive definite Hermitian form $(,)$ such that $(ax, y) = (x, \sigma(a)y)$ for all $a \in \mathcal{A}$, $x, y \in \mathcal{M}$. (Taking $a = L_0$ one sees that $\mathcal{M}_{h+n} \perp \mathcal{M}_{h+n'}$ for $n \neq n'$.) Let $\mathcal{M}$ be an $\mathcal{A}$-module with lowest weight $h \in \mathbf{C}$. Since $\dim \mathcal{M}_{h+n} < \infty$ for $n \in Z$ we can define

$$\chi(\mathcal{M}) := q^h \sum_{n \in Z} (\dim \mathcal{M}_{h+n}) q^n,$$

the *character* of $\mathcal{M}$, where the sum is a formal power series in q with exponents in Z. We need to distinguish the even and odd subspaces; thus we define, for $\alpha \in \mathbf{Z}/2\mathbf{Z}$,

$$\chi^{(\alpha)}(\mathcal{M}) = q^h \sum_{n \in Z} \left(\dim \mathcal{M}_{h+n}^{(\alpha)}\right) q^n,$$

where $\chi^{(0)}(\mathcal{M})$ is the *even character* of $\mathcal{M}$ and $\chi^{(1)}(\mathcal{M})$ is the *odd character* of $\mathcal{M}$. We have $\chi(\mathcal{M}) = \chi^{(0)}(\mathcal{M}) + \chi^{(1)}(\mathcal{M})$.

The following results can be proved by standard methods (see [**6**], [**8**], [**10**] for basic information on infinite dimensional Lie algebras and superalgebras). Details can be found in [**1**].

https://doi.org/10.4153/CJM-1990-029-7 Published online by Cambridge University Press

THEOREM 1.1. *If $\mathcal{M}$ is a unitary lowest weight $\mathcal{A}$-module, then $\mathcal{M}$ is completely reducible.*

COROLLARY 1.2. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be unitary lowest weight $\mathcal{A}$-modules. Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is a direct sum of unitary irreducible lowest weight $\mathcal{A}$-modules.*

THEOREM 1.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be unitary lowest weight $\mathcal{A}$-modules. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same central charge, and that $\chi^{(\alpha)}(\mathcal{M}_1) = \chi^{(\alpha)}(\mathcal{M}_2)$ for each $\alpha \in \mathbf{Z}/2\mathbf{Z}$. Then $\mathcal{M}_1 \cong \mathcal{M}_2$.*

COROLLARY 1.4. *Suppose that $\mathcal{M}$ is a unitary lowest weight $\mathcal{A}$-module, that $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots$ are unitary irreducible lowest weight $\mathcal{A}$-modules, and that all of these modules have the same central charge $z \in \mathbf{C}$. Suppose that $\chi^{(\alpha)}(\mathcal{M}) = \sum_{i=0}^{\infty} \chi^{(\alpha)}(\mathcal{N}_i)$, for both $\alpha \in \mathbf{Z}/2\mathbf{Z}$. Then $\mathcal{M} \cong \bigoplus_{i=0}^{\infty} \mathcal{N}_i$.*

The significance of these results is as follows. If we know which $\mathcal{A}$-modules are unitary (see Theorem 2.1), and the character formulas for these modules (see Theorem 2.2), then purely combinatorial manipulation of characters suffices to establish certain isomorphisms of unitary $\mathcal{A}$-modules, in particular to decompose certain tensor products of unitary $\mathcal{A}$-modules. The next section gives an application of this principle.

2. Main results. We first recall two basic theorems in the representation theory of $\mathcal{NS}$ and $\mathcal{R}$.

THEOREM 2.1. *The $\mathcal{A}$-modules $L^{\mathcal{A}}(z,h)$ and $\bar{L}^{\mathcal{A}}(z,h)$ are unitary if and only if either for $m, r, s \in \mathbf{Z}$, $m \geqq 2$, we have*

$$z = z_m := \frac{3}{2}\left[1 - \frac{8}{m(m+2)}\right],$$
$$h = h_{r,s}^{(m)} :=$$
$$\begin{cases} \dfrac{[(m+2)r - ms]^2 - 4}{8m(m+2)}, & 1 \leqq s \leqq r \leqq m-1,\ r-s \in 2\mathbf{Z}, \\ \textit{when } \mathcal{A} = \mathcal{NS}; \textit{ and} & \\ \dfrac{[(m+2)r - ms]^2 - 4}{8m(m+2)} + \dfrac{1}{16}, & 1 \leqq s \leqq r+1 \leqq m,\ r-s \in 2\mathbf{Z}+1, \\ \textit{when } \mathcal{A} = \mathcal{R}, & \end{cases}$$

or $z \geqq \frac{3}{2}$, $h \geqq 0$.

Proof. The necessity of this condition is outlined in [**4**]; the sufficiency is shown in [**5**] or [**7**]. QED

The unitary irreducible modules with $z < \frac{3}{2}$ are known as the *discrete series*. For $\mathcal{A} = \mathcal{R}$ the only unitary modules $L^{\mathcal{R}}(z,h)$ and $\bar{L}^{\mathcal{R}}(z,h)$ such that $h = -z/24$ are $L^{\mathcal{R}}(0,0)$ and $\bar{L}^{\mathcal{R}}(0,0)$, the even and odd trivial one-dimensional modules. Thus we do not need to distinguish between even and odd unitary irreducible $\mathcal{R}$-modules except in this case.

The characters $\chi^{\mathcal{A}}(z,h)$ of the unitary modules $L^{\mathcal{A}}(z,h)$ are stated in [**5**] using unpublished results of the second author; however, in that paper, formula 4.10 is missing a factor of 2 on the right-hand side, and the term $2m(m+1)n$ in formulas 4.12a,b should read $2m(m+2)n$. An outline of the proof of these character formulas is given in [**11**].

THEOREM 2.2. *We have*

$$\chi^{\mathcal{A}}(z_m, h_{r,s}^{(m)}) = \begin{cases} \chi(m,r,s)\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})/(1-q^n), & \mathcal{A} = \mathcal{NS} \\ 2\chi(m,r,s)\prod_{n=1}^{\infty}(1+q^n)/(1-q^n), & \mathcal{A} = \mathcal{R}, \end{cases}$$

where

$$\chi(m,r,s)(q) := \sum_{n\in\mathbf{Z}}(q^{\gamma(n)} - q^{\delta(n)}),$$
$$\gamma(n) = \gamma_{r,s}^{(m)}(n) := \tfrac{1}{2}m(m+2)n^2 + \tfrac{1}{2}((m+2)r - ms)n + h_{r,s}^{(m)},$$
$$\delta(n) = \delta_{r,s}^{(m)}(n) := \tfrac{1}{2}m(m+2)n^2 + \tfrac{1}{2}((m+2)r + ms)n + \tfrac{1}{2}rs + h_{r,s}^{(m)}.$$

We write $\bar{\chi}^{\mathcal{A}}(z,h)$ for the character of $\bar{L}^{\mathcal{A}}(z,h)$ when $\mathcal{A} = \mathcal{NS}$, or $\mathcal{A} = \mathcal{R}$ and $z = h = 0$. Note that we always have $\chi(z,h) = \bar{\chi}(z,h)$, since $\bar{\chi}^{(\alpha)}(z,h) = \chi^{(1-\alpha)}(z,h)$ for $\alpha \in \mathbf{Z}/2\mathbf{Z}$.

To define the tensor product of two modules over a Lie superalgebra, we need a $\mathbf{Z}/2\mathbf{Z}$-graded extension of the action of a Lie algebra on a tensor product. Thus for $\mathcal{A} = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)}$ and two $\mathcal{A}$-modules $\mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(1)}$ and $\mathcal{N} = \mathcal{N}^{(0)} \oplus \mathcal{N}^{(1)}$, we define

$$x(u\otimes v) := (xu)\otimes v + (-1)^{\alpha\beta}u\otimes(xv),$$

where $x \in \mathcal{A}^{(\alpha)}, u \in \mathcal{M}^{(\beta)}$, and $v \in \mathcal{N}$. (The parity of $v \in \mathcal{N}$ is irrelevant.) This definition gives $\mathcal{M} \otimes \mathcal{N}$ the structure of an $\mathcal{A}$-module.

If we take $m = 2$ in Theorem 2.1 we obtain only the trivial representations with $z = h = 0$. For $m = 3$ we obtain $z = z_3 = \frac{7}{10}$ and the lowest weights $h \in \{0, \frac{1}{10}\}$ ($(r,s) = (1,1)$ or $(2,2)$) for $\mathcal{NS}$, and $h \in \{\frac{3}{80}, \frac{7}{16}\}$ ($(r,s) = (1,2)$ or $(2,1)$) for $\mathcal{R}$. For $m = 4$ we have $z \geqq 1$. Notice that $\frac{7}{10} + \frac{7}{10} = \frac{7}{5}$ and that $\frac{7}{5} = z_{10}$, the central charge corresponding to $m = 10$. Thus we have two nonzero discrete series central charges for which the sum is also in the discrete series, and this occurs only when both central charges are $\frac{7}{10}$. (This contrasts with the situation for the Virasoro algebra, for which see [**2**].) The corresponding tensor products are

$$\begin{array}{ll} \mathcal{L}^{\mathcal{NS}}(\frac{7}{10},0)\otimes\mathcal{L}^{\mathcal{NS}}(\frac{7}{10},0), & \mathcal{L}^{\mathcal{R}}(\frac{7}{10},\frac{3}{80})\otimes\mathcal{L}^{\mathcal{R}}(\frac{7}{10},\frac{3}{80}), \\ \mathcal{L}^{\mathcal{NS}}(\frac{7}{10},0)\otimes\mathcal{L}^{\mathcal{NS}}(\frac{7}{10},\frac{1}{10}), & \mathcal{L}^{\mathcal{R}}(\frac{7}{10},\frac{3}{80})\otimes\mathcal{L}^{\mathcal{R}}(\frac{7}{10},\frac{7}{16}), \\ \mathcal{L}^{\mathcal{NS}}(\frac{7}{10},\frac{1}{10})\otimes\mathcal{L}^{\mathcal{NS}}(\frac{7}{10},\frac{1}{10}), & \mathcal{L}^{\mathcal{R}}(\frac{7}{10},\frac{7}{16})\otimes\mathcal{L}^{\mathcal{R}}(\frac{7}{10},\frac{7}{16}). \end{array}$$

TABLE 2.3. ($\mathcal{NS}$)

(1, 1)	0	(6, 2)	$\frac{45}{16}$	(8, 6)	$\frac{323}{240}$
(2, 2)	$\frac{1}{80}$	(6, 4)	$\frac{17}{16}$	(8, 8)	$\frac{21}{80}$
(3, 1)	$\frac{7}{10}$	(6, 6)	$\frac{7}{48}$	(9, 1)	10
(3, 3)	$\frac{1}{30}$	(7, 1)	$\frac{57}{10}$	(9, 3)	$\frac{19}{3}$
(4, 2)	$\frac{13}{16}$	(7, 3)	$\frac{91}{30}$	(9, 5)	$\frac{7}{2}$
(4, 4)	$\frac{1}{16}$	(7, 5)	$\frac{6}{5}$	(9, 7)	$\frac{3}{2}$
(5, 1)	$\frac{13}{5}$	(7, 7)	$\frac{1}{5}$	(9, 9)	$\frac{1}{3}$
(5, 3)	$\frac{14}{15}$	(8, 2)	$\frac{481}{80}$		
(5, 5)	$\frac{1}{10}$	(8, 4)	$\frac{261}{80}$		

TABLE 2.4. ($\mathcal{R}$)

(1, 2)	$\frac{1}{8}$	(5, 4)	$\frac{19}{40}$	(8, 3)	$\frac{1103}{240}$
(2, 1)	$\frac{21}{80}$	(5, 6)	$\frac{7}{120}$	(8, 5)	$\frac{181}{80}$
(2, 3)	$\frac{23}{140}$	(6, 1)	$\frac{65}{16}$	(8, 7)	$\frac{61}{80}$
(3, 2)	$\frac{13}{40}$	(6, 3)	$\frac{91}{48}$	(9, 2)	$\frac{65}{8}$
(3, 4)	$\frac{3}{40}$	(6, 5)	$\frac{9}{16}$	(9, 4)	$\frac{39}{8}$
(4, 1)	$\frac{25}{16}$	(7, 2)	$\frac{173}{40}$	(9, 6)	$\frac{59}{24}$
(4, 3)	$\frac{19}{48}$	(7, 4)	$\frac{83}{40}$	(9, 8)	$\frac{7}{8}$
(4, 5)	$\frac{1}{16}$	(7, 6)	$\frac{79}{120}$		
(5, 2)	$\frac{69}{40}$	(8, 1)	$\frac{621}{80}$		

Since each of these tensor products has finite dimensional weight spaces, and since there is only a finite number of discrete series modules for either $\mathcal{N}S$ or $\mathcal{R}$ with $z = \frac{7}{5}$, each of these tensor products must decompose into a finite direct sum of modules with $z = \frac{7}{5}$, by Corollary 1.2.

In a tensor product of $\mathcal{N}S$-modules, we can replace $L^{\mathcal{N}}S(\frac{7}{10}, h)$ by $\bar{L}^{\mathcal{N}}S(\frac{7}{10}, h)$. However, the tensor product will decompose in the same way, except for the parity of the irreducible summands. Thus $L \otimes \bar{L}$ and $\bar{L} \otimes L$ have parity opposite to that of $L \otimes L$, whereas $\bar{L} \otimes \bar{L}$ has the same parity as $L \otimes L$.

The lowest weights of the discrete series modules with $z = \frac{7}{5}$ for $\mathcal{N}S$ and $\mathcal{R}$ are listed in Tables 2.3 and 2.4; there are 25 in each case.

All the weights of a tensor product of modules differ from the lowest weight by an element of Z. Thus for a module $L^{\mathcal{A}}(\frac{7}{5}, h)$ to occur in the decomposition of $L^{\mathcal{A}}(\frac{7}{10}, h_1) \otimes L^{\mathcal{A}}(\frac{7}{10}, h_2)$ we must have $h - (h_1 + h_2) \in \frac{1}{2}\mathbf{Z}$ for $\mathcal{N}S$, and $h - (h_1 + h_2) \in \mathbf{Z}$ for $\mathcal{R}$. This cuts down the number of possibilities considerably. For each tensor product, the following list gives the lowest weights that can occur in the decomposition, with notation $(r,s) \longmapsto h = h^{10}_{r,s}$.

$$\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10},0)\otimes\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10},0),\quad h_1+h_2=0:$$
$$(1,1)\longmapsto h=0,\quad (9,7)\longmapsto h=\tfrac{3}{2},\quad (9,5)\longmapsto h=\tfrac{7}{2},\quad (9,1)\longmapsto h=10.$$

—

$$\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10},0)\otimes\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10}),\quad h_1+h_2=\tfrac{1}{10}:$$
$$(5,5)\longmapsto h=\tfrac{1}{10},\quad (5,1)\longmapsto h=\tfrac{13}{5}=\tfrac{1}{10}+\tfrac{5}{2}.$$

—

$$\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10})\otimes\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10}),\quad h_1+h_2=\tfrac{1}{5}:$$
$$(7,7)\longmapsto h=\tfrac{1}{5},\quad (3,1)\longmapsto h=\tfrac{7}{10}=\tfrac{1}{5}+\tfrac{1}{2},\quad (7,5)\longmapsto h=\tfrac{6}{5}=\tfrac{1}{5}+1,$$
$$(7,1)\longmapsto h=\tfrac{57}{10}=\tfrac{1}{5}+\tfrac{11}{2}.$$

—

$$\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80})\otimes\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80}),\quad h_1+h_2=\tfrac{3}{40}:$$
$$(3,4)\longmapsto h=\tfrac{3}{40},\quad (7,4)\longmapsto h=\tfrac{83}{40}=\tfrac{3}{40}+2.$$

—

$$\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80})\otimes\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16}),\quad h_1+h_2=\tfrac{19}{40}:$$
$$(5,4)\longmapsto h=\tfrac{19}{40}.$$

—

$$\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16})\otimes\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16}),\quad h_1+h_2=\tfrac{7}{8}:$$
$$(9,8)\longmapsto h=\tfrac{7}{8},\quad (9,4)\longmapsto h=\tfrac{39}{8}=\tfrac{7}{8}+4.$$

To decompose each of these tensor products, by Corollary 1.4 it suffices to decompose the character of the tensor product, which is the product of the characters of the factors, into a sum of characters of unitary modules with $z=\frac{7}{5}$. Since there are only finitely many possible summands, and the lowest power of q in the character of $L^{\mathcal{A}}(\frac{7}{5},h)$ is q^h, the highest power of q in the character of the tensor product that we need consider is $q^{h_{\max}}$, where $h_{\max}$ is the maximal lowest weight that can occur in the decomposition according to the above list. Every h in the list is $\leqq 10$, but for clarity in what follows we will see all powers of q less than 20.

We now list, for each tensor product, the character of the tensor product, the characters of the allowed summands, and the decomposition. All the power series listed omit one product factor in the character formula, that is,

$$\prod_{n=1}^{\infty}\frac{1+q^{n-\frac{1}{2}}}{1-q^n}\quad\text{for } \mathcal{NS},\qquad 2\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}\ \text{for } \mathcal{R}.$$

For $\mathcal{NS}$, since we are assuming that both tensor factors are even, a lowest weight vector of weight h in an irreducible summand of the tensor product will

https://doi.org/10.4153/CJM-1990-029-7 Published online by Cambridge University Press

generate an even (respectively odd) irreducible submodule when $h-(h_1+h_2) \in \mathbf{Z}$ (respectively $h-(h_1+h_2) \in \mathbf{Z}+\frac{1}{2}$). (This is the reason for the distinction between even and odd Verma modules.)

$$\chi^{\mathcal{NS}}(\tfrac{7}{10},0)^2 = 1 - q^{\frac{1}{2}} + q^{\frac{3}{2}} + q^{\frac{7}{2}} - q^4 - q^7 + q^{10} - q^{\frac{29}{2}} - q^{\frac{31}{2}} + \cdots$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{5},0) = 1 - q^{\frac{1}{2}} + \cdots \qquad \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{3}{2}) = q^{\frac{3}{2}}(1 - q^{\frac{5}{2}} + \cdots)$$
$$\bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{7}{2}) = q^{\frac{7}{2}}(1 - q^{\frac{7}{2}} + \cdots) \qquad \chi^{\mathcal{NS}}(\tfrac{7}{5},10) = q^{10}(1 - q^{\frac{9}{2}} - q^{\frac{11}{2}} + \cdots)$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{10},0)^2 = \chi^{\mathcal{NS}}(\tfrac{7}{5},0) + \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{3}{2}) + \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{7}{2}) + \chi^{\mathcal{NS}}(\tfrac{7}{5},10)$$

—

$$\chi^{\mathcal{NS}}(\tfrac{7}{10},0)\chi^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10}) = q^{\frac{1}{10}}(1 + q^{\frac{5}{2}} - q^5 - q^{\frac{25}{2}} - q^{\frac{35}{2}} + \cdots)$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{1}{10}) = q^{\frac{1}{10}}(1 - q^{\frac{25}{2}} - q^{\frac{35}{2}} + \cdots) \qquad \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{15}{5}) = q^{\frac{1}{10}}q^{\frac{5}{2}}(1 - q^{\frac{5}{2}} + \cdots)$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{10},0)\chi^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10}) = \chi^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{1}{10}) + \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{15}{5})$$

—

$$\chi^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10})^2 = q^{\frac{1}{5}}(1 + q^{\frac{1}{2}} + q - q^2 + q^{\frac{11}{2}} - q^{\frac{15}{2}} - q^9 - q^{\frac{23}{2}} - q^{\frac{37}{2}} + \cdots)$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{1}{5}) = q^{\frac{1}{5}}(1 - q^{\frac{15}{2}} + \cdots) \qquad \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{7}{10}) = q^{\frac{1}{5}}q^{\frac{1}{2}}(1 - q^{\frac{3}{2}} + \cdots)$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{6}{5}) = q^{\frac{1}{5}}q(1 - q^{\frac{21}{2}} - q^{\frac{35}{2}} + \cdots) \qquad \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{57}{10}) = q^{\frac{1}{5}}q^{\frac{11}{2}}(1 - q^{\frac{7}{2}} + \cdots)$$
$$\chi^{\mathcal{NS}}(\tfrac{7}{10},\tfrac{1}{10})^2 = \chi^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{1}{5}) + \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{7}{10}) + \chi^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{6}{5}) + \bar{\chi}^{\mathcal{NS}}(\tfrac{7}{5},\tfrac{57}{10})$$

—

$$\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80})^2 = 2q^{\frac{3}{40}}(1 + q^2 - q^6 - q^{14} - q^{16} + \cdots)$$
$$\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{3}{40}) = q^{\frac{3}{40}}(1 - q^6 + \cdots) \qquad \chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{83}{40}) = q^{\frac{3}{40}}q^2(1 - q^{12} - q^{14} + \cdots)$$
$$\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80})^2 = 2\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{3}{40}) + 2\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{83}{40})$$

—

$$\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80})\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16}) = 2q^{\frac{19}{40}}(1 - q^{10} + \cdots)$$
$$\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{19}{40}) = q^{\frac{19}{40}}(1 - q^{10} + \cdots)$$
$$\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{3}{80})\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16}) = 2\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{19}{40})$$

—

$$\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16})^2 = 2q^{\frac{7}{8}}(1 - q^2 + q^4 - q^8 + \cdots)$$
$$\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{7}{8}) = q^{\frac{7}{8}}(1 - q^2 + \cdots) \qquad \chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{39}{8}) = q^{\frac{7}{8}}q^4(1 - q^4 + \cdots)$$
$$\chi^{\mathcal{R}}(\tfrac{7}{10},\tfrac{7}{16})^2 = 2\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{7}{8}) + 2\chi^{\mathcal{R}}(\tfrac{7}{5},\tfrac{39}{8})$$

From these character decompositions and Corollary 1.4 we deduce the following result.

THEOREM 2.5A. *The tensor product $L(\frac{7}{10},h_1) \otimes L(\frac{7}{10},h_2)$ of unitary irreducible lowest weight modules for $\mathcal{NS}$ or $\mathcal{R}$ decomposes as follows. For $\mathcal{NS}$, $L(\frac{7}{5},h)$*

(respectively $\bar{L}(\frac{7}{5}, h)$*) occurs if and only if* $h - (h_1 + h_2) \in \mathbf{Z}$ *(respectively* $h - (h_1 + h_2) \in \mathbf{Z} + \frac{1}{2}$*), and its multiplicity is 1. For* $\mathcal{R}$, $L(\frac{7}{5}, h)$ *occurs if and only if* $h - (h_1 + h_2) \in \mathbf{Z}$*, and its multiplicity is 2.*

The symmetric and skew-symmetric submodules in the tensor square of a module $\mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(1)}$ over a Lie superalgebra are defined as follows. Let

$$S^2(\mathcal{M}) := \text{span}\{e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes o_1 + o_1 \otimes e_3 + o_2 \otimes o_3 - o_3 \otimes o_2 \mid e_i \in \mathcal{M}^{(0)}, o_j \in \mathcal{M}^{(1)}\}, \text{ and}$$

$$\Lambda^2(\mathcal{M}) := \text{span}\{e_1 \otimes e_2 - e_2 \otimes e_1 + e_3 \otimes o_1 - o_1 \otimes e_3 + o_2 \otimes o_3 + o_3 \otimes o_2 \mid e_i \in \mathcal{M}^{(0)}, o_j \in \mathcal{M}^{(1)}\}.$$

The characters of $S^2(\mathcal{M})$ and $\Lambda^2(\mathcal{M})$ can be expressed in terms of the character of $\mathcal{M}$: for $\chi(\mathcal{M})(q) = \chi^{(0)}(\mathcal{M})(q) + \chi^{(1)}(\mathcal{M})(q)$ we obtain

$$\chi(S^2(\mathcal{M}))(q) = \tfrac{1}{2}\left(\chi^{(0)}(\mathcal{M})(q)^2 + \chi^{(0)}(\mathcal{M})(q^2)\right) + \chi^{(0)}(\mathcal{M})(q)\chi^{(1)}(\mathcal{M})(q) + \tfrac{1}{2}\left(\chi^{(1)}(\mathcal{M})(q)^2 - \chi^{(1)}(\mathcal{M})(q^2)\right),$$

and $\chi(\Lambda^2(\mathcal{M}))(q) = \chi(\mathcal{M})(q)^2 - \chi(S^2(\mathcal{M}))(q)$.

Calculations with this formula give

$$\chi(S^2(\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, 0))) = 1 - q^{\frac{1}{2}} + q^{\frac{7}{2}} - q^7 + \cdots,$$

$$\chi(S^2(\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, \tfrac{1}{10}))) = q^{\frac{1}{5}}(1 + q^{\frac{11}{2}} - q^{\frac{15}{2}} - q^9 + \cdots).$$

For $\mathcal{R}$, given any quotient $\mathcal{M}$ of a Verma module $V^{\mathcal{R}}(z, h)$ with $h > -z/24$, we have $\chi^{(0)}(\mathcal{M}) = \chi^{(1)}(\mathcal{M}) = \frac{1}{2}\chi(\mathcal{M})$, since given any homogeneous vector v in the kernel of the homomorphism from $V^{\mathcal{R}}(z, h)$ onto $\mathcal{M}$, the vector $G_0 v$ of opposite parity will also be in the kernel (since the kernel is a submodule). Thus we see that

$$\chi(S^2(\mathcal{M})) = \chi(\Lambda^2(\mathcal{M})) = \tfrac{1}{2}\chi(\mathcal{M})^2.$$

Summarizing, we have the following result.

THEOREM 2.5B. *The symmetric and skew-symmetric submodules of the tensor squares of the modules* $L(\frac{7}{10}, h)$ *for* $\mathcal{NS}$ *and* $\mathcal{R}$ *are*

$$S^2(\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, 0)) \cong \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{5}, 0) \oplus \bar{\mathcal{L}}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{7}{2}),$$
$$\Lambda^2(\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, 0)) \cong \bar{\mathcal{L}}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{3}{2}) \oplus \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{5}, 10),$$
$$S^2(\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, \tfrac{1}{10})) \cong \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{1}{5}) \oplus \bar{\mathcal{L}}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{57}{10}),$$
$$\Lambda^2(\mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, \tfrac{1}{10})) \cong \bar{\mathcal{L}}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{7}{10}) \oplus \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{6}{5}),$$
$$S^2(\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10}, \tfrac{3}{80})) \cong \Lambda^2(\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10}, \tfrac{3}{80})) \cong \mathcal{L}^{\mathcal{R}}(\tfrac{7}{5}, \tfrac{3}{40}) \oplus \mathcal{L}^{\mathcal{R}}(\tfrac{7}{5}, \tfrac{83}{40}),$$
$$S^2(\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10}, \tfrac{7}{16})) \cong \Lambda^2(\mathcal{L}^{\mathcal{R}}(\tfrac{7}{10}, \tfrac{7}{16})) \cong \mathcal{L}^{\mathcal{R}}(\tfrac{7}{5}, \tfrac{7}{8}) \oplus \mathcal{L}^{\mathcal{R}}(\tfrac{7}{5}, \tfrac{39}{8}).$$

3. Characters as infinite products, and an alternative proof. Theorem 2.5a,b can also be proved using power series identities, as were the results of [**2**]. We first recall the quintuple product identity.

THEOREM 3.1. *For $z \neq 0$ and $|x| < 1$ we have*

$$\text{(a)} \quad \prod_{n=1}^{\infty}(1-x^n)(1-x^n z)(1-x^{n-1}z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2}) = \sum_{n\in\mathbf{Z}} x^{\frac{3}{2}n^2+\frac{1}{2}n}(z^{3n}-z^{-3n-1}).$$

or equivalently,

$$\text{(b)} \quad \prod_{n=1}^{\infty}(1-x^n)(1+x^n z)(1+x^{n-1}z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2}) = \sum_{n\in\mathbf{Z}} (-1)^n x^{\frac{1}{2}n(3n-1)} z^{-3n}(1+x^n z^{-1}).$$

Proof. See [**3**] for (a); (b) is an easy corollary. This identity is the denominator identity for the affine Lie algebra $A_2^{(2)}$; see [**6**, exercises 10.9, 12.1]. QED

For certain values of m, r and s the expressions $\chi(m,r,s)$ of Theorem 2.2 can be written as infinite products.

THEOREM 3.2. *If $3|m$ and $r = m/3$ or $r = 2m/3$, or if $3|(m+2)$ and $s = (m+2)/3$ or $s = 2(m+2)/3$, then for suitable x and z, $\chi(m,r,s)$ equals the product side of Theorem 3.1a (up to a power of q).*

Proof. First consider the case $3|m$, $r = m/3$. Then

$$\begin{aligned} q^{-h^{(m)}_{m/3,s}}\chi(m, \tfrac{1}{3}m, s) &= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+\frac{1}{2}(\frac{1}{3}m(m+2)-ms)n} - q^{\frac{1}{2}m(m+2)n^2+\frac{1}{2}(\frac{1}{3}m(m+2)+ms)n+\frac{1}{6}ms} \\ &= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+\frac{1}{6}m(m+2)n}\left(q^{-\frac{1}{2}msn} - q^{\frac{1}{2}msn+\frac{1}{6}ms}\right), \end{aligned}$$

which matches the sum side of Theorem 3.1a for $x = q^{\frac{1}{3}m(m+2)}$ and $z = q^{-\frac{1}{6}ms}$.

Second, consider the case $3|m$, $r = 2m/3$. Then

$$\begin{aligned} q^{-h^{(m)}_{2m/3,s}}\chi(m, \tfrac{2}{3}m, s) = &\sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+\frac{1}{2}(\frac{2}{3}m(m+2)-ms)n} \\ &- \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+\frac{1}{2}(\frac{2}{3}m(m+2)+ms)n+\frac{1}{3}ms}. \end{aligned}$$

Now replace n by $-n$ in the first sum, and n by $-n-1$ in the second sum, to obtain

$$
\begin{aligned}
&q^{-h^{(m)}_{2m/3,s}}\chi(m, \tfrac{2}{3}m, s)\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2 - \frac{1}{2}(\frac{2}{3}m(m+2)-ms)n}\\
&- \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)(n^2+2n+1)-\frac{1}{2}(\frac{2}{3}m(m+2)+ms)(n+1)+\frac{1}{3}ms}\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+(-\frac{1}{3}m(m+2)+\frac{1}{2}ms)n}\\
&- \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+(m(m+2)-\frac{1}{3}m(m+2)-\frac{1}{2}ms)n+(\frac{1}{2}m(m+2)-\frac{1}{3}m(m+2)-\frac{1}{2}ms+\frac{1}{3}ms)}\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+(-\frac{1}{3}m(m+2)+\frac{1}{2}ms)n}\\
&- \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+(\frac{2}{3}m(m+2)-\frac{1}{2}ms)n+(\frac{1}{6}m(m+2)-\frac{1}{6}ms)}\\
&= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}m(m+2)n^2+\frac{1}{6}m(m+2)n} \times\\
&\left(q^{(-\frac{1}{2}m(m+2)+\frac{1}{2}ms)} - q^{(\frac{1}{2}m(m+2)-\frac{1}{2}ms)n+\frac{1}{3}(\frac{1}{2}m(m+2)-\frac{1}{2}ms)}\right),
\end{aligned}
$$

which is the sum side of Theorem 3.1a for $x = q^{\frac{1}{3}m(m+2)}$ and $z = q^{-\frac{1}{6}m(m+2)+\frac{1}{6}ms}$.

The calculations for $3|(m+2)$ are similar. If $s = (m+2)/3$, replace n by $-n$ in the first sum, and then take $x = q^{\frac{1}{3}m(m+2)}$, and $z = q^{-\frac{1}{6}(m+2)r}$. If $s = 2(m+2)/3$, replace n by $-n-1$ in the second sum, and then take $x = q^{\frac{1}{3}m(m+2)}$ and $z = q^{-\frac{1}{6}(m+2)(m-r)}$. QED

Second Proof of Theorem 2.5A, B. We prove only

$$
\begin{aligned}
(3.3)\qquad \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, 0) \otimes \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{10}, 0) &\cong \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{5}, 0) \oplus \bar{\mathcal{L}}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{3}{2})\\
&\oplus \bar{\mathcal{L}}^{\mathcal{NS}}(\tfrac{7}{5}, \tfrac{7}{2}) \oplus \mathcal{L}^{\mathcal{NS}}(\tfrac{7}{5}, 10), 0
\end{aligned}
$$

as an example. By Corollary 1.4 it suffices to show that the even and odd characters of the two sides of (3.3) are equal. However, for $\mathcal{NS}$, the even (respectively odd) character is the subseries of the character consisting of powers of q differing from the lowest weight by an element of $\mathbf{Z}$ (respectively an element of $\mathbf{Z}+\frac{1}{2}$). Hence it suffices to show that the characters of the two sides of (3.3) are equal.

We obtain $z = \frac{7}{10}$, $h = 0$, when $m = 3$, $r = s = 1$. By Theorem 3.2 we have

$$
\chi(3,1,1) = \sum_{n\in\mathbf{Z}} q^{\frac{15}{2}n^2+\frac{5}{2}n}\left(q^{-\frac{3}{2}n} - q^{\frac{3}{2}n+\frac{1}{2}}\right),
$$

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and hence by Theorem 3.1a, with $x = q^5$, $z = q^{-\frac{1}{2}}$, we obtain

$$\chi(3,1,1) = \prod_{n=1}^{\infty}(1-q^{5n})(1-q^{5n-\frac{1}{2}})(1-q^{5n-\frac{9}{2}})(1-q^{10n-6})(1-q^{10n-4}).$$

To avoid fractions in the exponents, define $p := q^{\frac{1}{2}}$. Thus the character of the left side of (3.3) is

$$(3.4)\qquad \chi(3,1,1)^2 \prod_{n=1}^{\infty}\left(\frac{1+p^{2n+1}}{1-p^{2n}}\right)^2$$
$$= \prod_{n=1}^{\infty}(1-p^{10n})^2(1-p^{10n-1})^2(1-p^{10n-9})^2(1-p^{20n-12})^2$$
$$\times\,(1-p^{20n-8})^2 \prod_{n=1}^{\infty}\left(\frac{1+p^{2n+1}}{1-p^{2n}}\right)^2.$$

By Theorem 2.2 we have

$$\chi(10,1,1) = \sum_{n\in\mathbf{Z}} p^{120n^2+2n} - p^{120n^2+22n+1}, \quad (z,h) = (\tfrac{7}{5},0),$$
$$\chi(10,9,7) = p^3\sum_{n\in\mathbf{Z}} p^{120n^2+38n} - p^{120n^2+178n+63}, \quad (z,h) = (\tfrac{7}{5},\tfrac{3}{2}),$$
$$\chi(10,9,5) = p^7\sum_{n\in\mathbf{Z}} p^{120n^2+58n} - p^{120n^2+158n+45}, \quad (z,h) = (\tfrac{7}{5},\tfrac{7}{2}),$$
$$\chi(10,9,1) = p^{20}\sum_{n\in\mathbf{Z}} p^{120n^2+98n} - p^{120n^2+118n+9}, \quad (z,h) = (\tfrac{7}{5},10),$$

Now let $f_1(n) := 30n^2+n$, $f_2(n) := 30n^2+19n+3$, $f_3(n) := 30n^2+29n+7$, and $f_4(n) := 30n^2+49n+20$. Then $f_1(2n) = 120n^2+2n$, $f_1(-2n-1) = 120n^2+118n+29$, $f_2(2n) = 120n^2+38n+3$, $f_2(2n+1) = 120n^2+158n+52$, $f_3(2n) = 120n^2+58n+7$, $f_3(2n+1) = 120n^2+178n+66$, $f_4(2n) = 120n^2+98n+20$, and $f_4(-2n-1) = 120n^2+22n+1$. Thus

$$(3.5)\qquad \chi(10,1,1)+\chi(10,9,7)+\chi(10,9,5)+\chi(10,9,1)$$
$$= \sum_{n\in\mathbf{Z}}(-1)^n(p^{30n^2+n}+p^{30n^2+19n+3}+p^{30n^2+29n+7}+p^{30n^2+49n+20}).$$

Let $g_1(n) := \frac{15}{2}n^2+\frac{1}{2}n$ and $g_2(n) := \frac{15}{2}n^2+\frac{19}{2}n+3$. Then $g_1(2n) = 30n^2+n$, $g_1(-2n-1) = 30n^2+29n+7$, $g_2(2n) = 30n^2+19n+3$ and $g_2(2n+1) = 30n^2+49n+20$. Thus the right side of (3.5) can be written as

$$(3.6)\qquad \sum_{n\in\mathbf{Z}}(-1)^n(p^{g_1(2n)}+p^{g_1(-2n-1)}+p^{g_2(2n)}+p^{g_2(2n+1)}).$$

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For $k = 2n$ or $-2n-1$, we have $\frac{1}{2}k(k+1) \equiv n \pmod 2$; and for $k = 2n$ or $2n+1$, we have $\frac{1}{2}k(k-1) \equiv n \pmod 2$. Thus (3.6) reduces to

$$(3.7) \qquad \sum_{k\in\mathbf{Z}} (-1)^{\frac{1}{2}k(k+1)} p^{g_1(k)} + (-1)^{\frac{1}{2}k(k-1)} p^{g_2(k)}.$$

Now since $g_1(k)+k-\frac{1}{2}k(k+1) \equiv 0 \pmod 2$ and $g_2(k)+k+1-\frac{1}{2}k(k-1) \equiv 0 \pmod 2$, (3.7) becomes

$$(3.8) \qquad \sum_{k\in\mathbf{Z}} (-1)^k (-p)^{g_1(k)} + \sum_{k\in\mathbf{Z}} (-1)^{k+1} (-p)^{g_2(k)}.$$

Replace k by $-k-1$ in the second sum to get

$$\begin{aligned} &\sum_{k\in\mathbf{Z}} (-1)^k (-p)^{\frac{15}{2}k^2+\frac{1}{2}k} + \sum_{k\in\mathbf{Z}} (-1)^k (-p)^{\frac{15}{2}k^2+\frac{11}{2}k+1} \\ &= \sum_{k\in\mathbf{Z}} (-1)^k (-p)^{\frac{15}{2}k^2-\frac{5}{2}k} (-p)^{3k} \left(1+(-p)^{5k+1}\right). \end{aligned}$$

Now by Theorem 3.1b with $x = (-p)^5$ and $z = (-p)^{-1}$, this last expression equals

$$\begin{aligned} &\prod_{n=1}^{\infty} (1+(-1)^{n+1}p^{5n})(1+(-1)^{n+1}p^{5n-1})(1+(-1)^n p^{5n-4}) \\ &\times (1+p^{10n-7})(1+p^{10n-3}) \\ &= \prod_{n=1}^{\infty} (1+p^{10n-5})(1-p^{10n})(1+p^{10n-6})(1-p^{10n-1}) \\ &\times (1-p^{10n-9})(1+p^{10n-4})(1+p^{10n-7})(1+p^{10n-3}). \end{aligned}$$

Thus the character of the right side of (3.3) is

$$\begin{aligned} (3.9) \qquad &\left(\chi(10,1,1)+\chi(10,9,7)+\chi(10,9,5)+\chi(10,9,1)\right) \prod_{n=1}^{\infty} \frac{1+p^{2n-1}}{1-p^{2n}} \\ &= \prod_{n\neq 0,\pm1,\pm2\,(10)} (1+p^n) \prod_{n=0,\pm1\,(10)} (1-p^n) \prod_{n=1}^{\infty} \frac{1+p^{2n-1}}{1-p^{2n}}. \end{aligned}$$

We must show that the right-hand sides of (3.4) and (3.9) are equal. We can immediately cancel

$$\prod_{n=0,\pm1\,(10)} (1-p^n) \prod_{n=1}^{\infty} \frac{1+p^{2n-1}}{1-p^{2n}}.$$

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Thus we must show

$$\prod_{n=0,\pm 1\,(10)} (1-p^n) \prod_{n=1}^{\infty} (1-p^{20n-12})^2 (1-p^{20n-8})^2 \prod_{n=1}^{\infty} \frac{1+p^{2n-1}}{1-p^{2n}} = \prod_{n\neq 0,\pm 1,\pm 2\,(10)} (1+p^n).$$

Take the left side of this equation, and split the factors which repeat mod 20 into factors which repeat mod 10:

$$\prod_{n=0,\pm 1\,(10)} (1-p^n) \prod_{n=1}^{\infty} \frac{1+p^{2n-1}}{1-p^{2n}} \times \prod_{n=1}^{\infty} (1-p^{10n-6})^2 (1+p^{10n+6})^2 (1-p^{10n-4})^2 (1-p^{10n+4})^2.$$

Expanding the first product mod 10 and rearranging gives:

$$\prod_{n=1}^{\infty} (1+p^{10n-6})^2 (1+p^{10n-4})^2 (1+p^{2n-1}) \times \prod_{n=1}^{\infty} \frac{(1-p^{10n-9})(1-p^{10n-6})^2 (1-p^{10n-4})^2 (1-p^{10n-1})(1-p^{10n})}{(1-p^{2n})}.$$

Expand the factors which repeat mod 2 and cancel where possible:

$$\prod_{n=1}^{\infty} (1+p^{10n-9})(1+p^{10n-7})(1+p^{10n-6})^2 (1+p^{10n-5}) \times (1+p^{10n-4})^2 (1+p^{10n-3})(1+p^{10n-1}) \times \prod_{n=1}^{\infty} \frac{(1-p^{10n-9})(1-p^{10n-6})(1-p^{10n-4})(1-p^{10n-1})}{(1-p^{10n-8})(1-p^{10n-2})}.$$

Separating the factors we are looking for (as the first product) gives

$$\prod_{n\neq 0,\pm 1,\pm 2\,(10)} (1+p^n) \prod_{n=1}^{\infty} (1+p^{10n-9})(1+p^{10n-6})(1+p^{10n-4})(1+p^{10n-1}) \times \prod_{n=1}^{\infty} \frac{(1-p^{10n-9})(1-p^{10n-6})(1-p^{10n-4})(1-p^{10n-1})}{(1-p^{10n-8})(1-p^{10n-2})}.$$

Now combine the factors $(1+p^{10n-a})(1-p^{10n-a})$ for $a = 9, 6, 4$ and 1:

$$\prod_{n\neq 0,\pm 1,\pm 2\,(10)} (1+p^n) \prod_{n=1}^{\infty} \frac{(1-p^{20n-18})(1-p^{20n-12})(1-p^{20n-8})(1-p^{20n-2})}{(1-p^{10n-8})(1-p^{10n-2})},$$

https://doi.org/10.4153/CJM-1990-029-7 Published online by Cambridge University Press

and notice that the second product is 1. This completes the proof. QED

4. Acknowledgements. I thank G. B. Seligman and L. Crane for helpful conversations, and the Graduate School of Yale University for financial support. I also thank the referee for helping to clarify the exposition.

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