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GENERALIZED FOURIER INTEGRAL OPERATOR METHODS FOR HYPERBOLIC EQUATIONS WITH SINGULARITIES

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Abstract This paper addresses linear hyperbolic partial differential equations and pseudodifferential equations with strongly singular coefficients and data, modelled as members of algebras of generalized functions. We employ the recently developed theory of generalized Fourier integral operators to construct parametrices for the solutions and to describe propagation of singularities in this setting. As required tools, the construction of generalized solutions to eikonal and transport equations is given and results on the microlocal regularity of the kernels of generalized Fourier integral operators are obtained.

Keywords: hyperbolic equations; Colombeau generalized functions; microlocal analysis

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1. Introduction

This paper is one of a series aiming at developing microlocal analysis in Colombeau algebras of generalized functions [**20**, **21**, **23**, **25**, **26**, **37**, **42**, **46**]. While these algebras were originally introduced for handling nonlinear partial differential equations with distributional data, they have turned out, in the past decade, to be equally useful for treating linear hyperbolic equations with strongly singular coefficients and data. Consider the Cauchy problem for the linear hyperbolic system

$$\left.\begin{aligned} \partial_t u = \sum_{j=1}^{m} A_j(x,t)\partial_j u + B(x,t)u + F(x,t), \quad (x,t) \in \mathbb{R}^{m+1}, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^m. \end{aligned}\right\} \tag{1.1}$$

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In the case where the initial data g or the driving term F are distributions and the coefficient matrices are, for example, discontinuous, the system formally contains multiplicative products of distributions. Simple examples [**38**] show that such systems may fail to have solutions in the sense of distributions (referred to as classical solutions in the following). For second-order strictly hyperbolic equations, the minimal regularity of the principal coefficients is integral log-Lipschitz regularity in time for admitting unique classical solutions [**7**, **10**]. For transport equations, one may go down to bounded variation (BV)-coefficients and measure data (see the recent surveys [**2**, **29**]). Differentiating the equation and trying to establish an equation for the higher derivatives of the formal solution already brings one beyond the admissible non-regularity, as will, for example, letting B be a delta potential or letting F be a space-time white noise.

The Colombeau algebra $\mathcal{G}(\mathbb{R}^{m+1})$ is a differential algebra containing the space of distributions $\mathcal{D}'(\mathbb{R}^{m+1})$ as a subspace, and thus it provides a framework in which all operations arising in (1.1) are meaningful. Indeed, existence and uniqueness of solutions to (1.1) in the Colombeau algebra (as well as in its dual) have been proved under various conditions: when A_j and B are real-valued $n \times n$ matrices with entries in $\mathcal{G}(\mathbb{R}^{m+1})$, for A_j symmetric [**35**, **40**]; for symmetric hyperbolic systems of pseudodifferential operators with Colombeau symbols [**32**, **46**]; for strictly hyperbolic systems of pseudodifferential operators [**24**].

In the classical setting, Fourier integral operators (FIOs) and microlocal methods are used to study the dependence of the solution on the initial data, in particular, the propagation of singularities. Indeed, it is the aim of this paper to derive a generalized FIO representation of the Colombeau solution and to predict its (generalized) wavefront set. In the Colombeau framework, pseudodifferential operators and their application to microlocal regularity and hypoellipticity were developed in [**20**, **23**, **25**, **32**, **37**, **42**]. Fourier integral operators with generalized phase and amplitude were developed in [**21**, **26**]. In the present paper we initiate the application of generalized FIOs to hyperbolic equations with Colombeau coefficients. We demonstrate how to construct FIO parametrices for transport equations and scalar hyperbolic pseudodifferential equations. Furthermore, we determine the generalized wavefront sets of the kernels of generalized FIOs, as well as the propagation of the generalized wavefront sets through application of a generalized FIO. Finally, we study the generalized Hamiltonian flow and prove various results on the propagation of singularities.

In short, it is the aim of this paper to solve generalized strictly hyperbolic problems, which might be generated by singular coefficients and data, by means of FIO techniques placed in the Colombeau context. Generalized pseudodifferential and Fourier integral operators, as well as the microlocal tools in the Colombeau framework that have been developed in past years, provide new and powerful tools and techniques that were not available at the time of the first work on hyperbolic generalized systems and equations in [**32**, **40**, **43**–**45**]. On the other hand, the application of FIOs to generalized strictly hyperbolic problems is also very natural in the Colombeau context; by means of diagonalization methods these techniques will be applicable not only to scalar equations but also to hyperbolic systems with Colombeau coefficients. This, and the singularity structure in the case of systems and of higher-order equations, is the subject of ongoing research and will be published elsewhere.

To put our work into context, we comment briefly on existing classical approaches to hyperbolic equations with coefficients of low regularity. Systems with discontinuous coefficients were already considered in the 1950s and 1960s, using jump conditions, energy methods and regularization [**11**, **27**, **38**, **39**]. Second-order hyperbolic equations with Lipschitz coefficients were the subject of research in the 1970s and 1980s, culminating in the aforementioned results on equations with log-Lipschitz and integral log-Lipschitz properties and beyond [**7**, **9**, **10**]. For coefficients depending on both time and space see [**6**], and for the variational approach see [**12**]. For the general theory of pseudodifferential and paradifferential operators with symbols of low regularity we refer the reader to the monographs [**31**, **49**]. Another avenue of investigation was the case of transport equations with discontinuous coefficients and/or measures as initial data. Renormalized solutions were introduced in [**13**], BV-vector fields were studied in [**1**, **8**], and measure-theoretic concepts were exploited in [**5**, **47**]. An exploration into products of distributions can be found in [**33**]. In summary, this line of research has aimed at pushing the regularity to its lower bounds for classical (distributional) solutions.

We comment on some technical aspects of the theory. Elements of the Colombeau algebra $\mathcal{G}(\mathbb{R}^{m+1})$ are equivalence classes of nets $(u_\varepsilon)_\varepsilon$ of $\mathcal{C}^\infty$-functions satisfying asymptotic bounds of order ε^{-N} in terms of the local L^∞-seminorms of all derivatives as $\varepsilon \to 0$. The generalized wavefront set is also defined by means of decay in the dual variable and uniform asymptotic bounds as $\varepsilon \to 0$. The existence theory is based on energy estimates and Gronwall's inequality and requires more restrictive asymptotic bounds of the type $|\log \varepsilon|$ in certain places; propagation of regularity may require additional so-called slow-scale estimates. The necessity of such stronger bounds has been argued in [**25**, **34**, **44**]. When the coefficients are discontinuous functions or distributions, corresponding members of the Colombeau algebra satisfying the stronger bounds can always be constructed by suitable regularization [**43**]. Note that the choice of the scale involved in the regularization of the coefficients is crucial in determining the Colombeau well-posedness of the corresponding Cauchy problem, whereas no regularity is required on the distributional coefficients themselves as long as the problem is (generalized) strictly hyperbolic. For a precise study of the necessary type of scales we refer the reader to [**32**] for pseudodifferential equations and to [**40**] for systems of differential equations.

As mentioned above there are hyperbolic equations with coefficients of low regularity, or even with discontinuities, for which classical solutions exist. In these cases, as demonstrated by many examples (see [**33**, **40**, **43**, **44**] and the recent [**24**]), the generalized Colombeau solution is, as a rule, associated with the classical solution (i.e. the representing nets of smooth functions have the classical solution as their distributional limit).

We know that admitting distributional coefficients in hyperbolic equations may lead to infinite propagation speed [**40**]. To avoid this phenomenon, one may assume that the coefficients are constant for large $|x|$, as is often done in the classical case. Matters are simplified further by assuming that the coefficients are compactly supported in x. This still allows one to model and study singularities as strong as desired (in the non-constant regime), but facilitates the application of Sobolev estimates.

It may also be useful to employ Colombeau spaces with asymptotic estimates based on Sobolev spaces. It was shown in [**32**] that the first-order scalar hyperbolic pseudodifferential problem

$$\partial_t u - \mathrm{i}a(t, x, D_x)u = f, \quad u(0) = g \tag{1.2}$$

with a real-valued principal part has a unique solution in the Colombeau algebra $\mathcal{G}_{2,2}([0, T] \times \mathbb{R}^n)$ based on H^∞. Assuming that the symbol a is compactly supported in x, the Sobolev embedding theorem also yields existence and uniqueness of a Colombeau solution in $\mathcal{G}([0, T] \times \mathbb{R}^n)$.

Proofs of the aforementioned existence and uniqueness results are based on *a priori* L^2-estimates in the representing nets of smooth functions. These methods have been successful regarding the well-posedness of the Cauchy problems under consideration, but do not provide deep qualitative information on the solution. This already applies to the classical case of hyperbolic systems with $\mathcal{C}^\infty$-coefficients and was one of the main motivations for the constructive FIO approach introduced by Duistermaat and Hörmander [**14, 15, 30**], which we extend here to the Colombeau framework. For a survey on local and global regularity of Fourier integral operators on L^p spaces, as well as on Colombeau spaces, we refer the reader to [**48**].

Note that our solutions are global in time. Few results of local solvability are available at the moment in the Colombeau context; we quote [**3**] for nonlinear first-order systems solved in terms of generalized germs, [**16**] for ordinary differential equations, [**22**] for differential operators of constant strength, [**17**] for the Hamilton–Jacobi equation and [**36**] for wave equations on non-smooth space-times. The key idea in this kind of research is to find suitable assumptions about coefficients and scales to ensure the existence of an ε-independent time interval. By means of this asymptotic approach we believe we will be able to obtain further results of local existence and uniqueness in the Colombeau setting, for instance, a generalization of [**9**] to wave equations with less than log-Lipschitz coefficients.

The paper has the following structure. In § 2, we recall required notions from the Colombeau theory of generalized functions, in particular, microlocal tools, generalized symbols and phase functions and generalized FIOs. In § 3, we look at scalar first-order hyperbolic partial differential equations with Colombeau coefficients. We show how the generalized solution can be represented by means of a generalized FIO. This includes the construction of Colombeau solutions to the eikonal equations and to the transport equation for the generalized symbol. In § 4, we treat the case of first-order hyperbolic pseudodifferential equations whose principal part depends only on time. This has the advantage that generalized solutions to the eikonal equation can be given explicitly (the more intricate t- and x-dependent case, which requires additional asymptotic bounds on the coefficients, as follows already from [**40**], is postponed to work in preparation on hyperbolic problems with singularities). We solve an infinite system of transport equations yielding the asymptotic expansion of the generalized symbol, and construct an FIO parametrix of the generalized solution. In § 5, we address microlocal properties of the

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kernel of a generalized FIO, as well as its action on Colombeau generalized functions. This allows one to compute the generalized wavefront set (in space and time) of solutions to first-order hyperbolic partial differential equations with non-smooth coefficients depending on time. Furthermore, the spatial wavefront set of solutions to first-order partial differential equations at fixed time is computed, when the coefficients are generalized functions depending on space and time, in the case where the Hamiltonian flow has a limit. We show that the generalized wavefront set is invariant under the Hamiltonian flow for time-dependent coefficients. The paper concludes with some explicit examples involving jump discontinuities and delta functions in the coefficients.

2. Basic notions

2.1. Basic notions of Colombeau theory

This section gives some background on the Colombeau techniques used in the rest of this paper. As main sources, we refer the reader to [**19**, **23**, **25**, **28**].

2.1.1. *Nets of complex numbers*

A net $(u_\varepsilon)_\varepsilon$ in $\mathbb{C}^{(0,1]}$ is said to be *strictly non-zero* if there exist $r > 0$ and $\eta \in (0, 1]$ such that $|u_\varepsilon| \geqslant \varepsilon^r$ for all $\varepsilon \in (0, \eta]$. For several reasons relating to regularity issues we will make use of the concept of a *slow-scale net* (ssn). A slow-scale net is a net $(r_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]}$ such that

$$|r_\varepsilon|^q = O(\varepsilon^{-1}) \text{ for all } q \geqslant 0 \quad \text{as } \varepsilon \to 0.$$

A net $(u_\varepsilon)_\varepsilon$ in $\mathbb{C}^{(0,1]}$ is said to be *slow-scale strictly non-zero* if there exist a slow-scale net $(s_\varepsilon)_\varepsilon$ and $\eta \in (0, 1]$ such that $|u_\varepsilon| \geqslant 1/s_\varepsilon$ for all $\varepsilon \in (0, \eta]$.

2.1.2. $\tilde{\mathbb{C}}$-*modules of generalized functions based on a locally convex topological vector space*

The most common algebras of generalized functions of Colombeau type, as well as the spaces of generalized symbols we deal with, are introduced by referring to the following general models.

Let E be a locally convex topological vector space topologized through the family of seminorms $\{p_i\}_{i\in I}$. The elements of

$$\begin{aligned}
\mathcal{M}_E &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \colon \forall i \in I,\ \exists N \in \mathbb{N},\ p_i(u_\varepsilon) = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},\\
\mathcal{M}^{\mathrm{sc}}_E &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \colon \forall i \in I,\ \exists (\omega_\varepsilon)_\varepsilon \mathrm{ssn},\ p_i(u_\varepsilon) = O(\omega_\varepsilon) \text{ as } \varepsilon \to 0\},\\
\mathcal{M}^{\infty}_E &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \colon \exists N \in \mathbb{N},\ \forall i \in I,\ p_i(u_\varepsilon) = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},\\
\mathcal{N}_E &:= \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \colon \forall i \in I,\ \forall q \in \mathbb{N},\ p_i(u_\varepsilon) = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}
\end{aligned}$$

are called E-moderate, E-moderate of slow-scale type, E-regular and E-negligible, respectively. We define the space of *generalized functions based on* E as the factor space $\mathcal{G}_E := \mathcal{M}_E/\mathcal{N}_E$.

The ring of *complex generalized numbers*, denoted by $\tilde{\mathbb{C}}$, is obtained by taking $E = \mathbb{C}$. Note that $\tilde{\mathbb{C}}$ is not a field since, by [**28**, Theorem 1.2.38], only the elements that are strictly non-zero (i.e. the elements that have a representative strictly non-zero) are invertible, and vice versa. Note that all the representatives of $u \in \tilde{\mathbb{C}}$ are strictly non-zero once we know that there exists at least one that is strictly non-zero. When u has a representative that is slow-scale strictly non-zero, we say that it is *slow-scale invertible*.

For any locally convex topological vector space E the space $\mathcal{G}_E$ has the structure of a $\tilde{\mathbb{C}}$-module. The $\mathbb{C}$-module $\mathcal{G}^{\mathrm{sc}}_E := \mathcal{M}^{\mathrm{sc}}_E/\mathcal{N}_E$ of *slow-scale regular generalized functions* and the $\tilde{\mathbb{C}}$-module $\mathcal{G}^{\infty}_E := \mathcal{M}^{\infty}_E/\mathcal{N}_E$ of *regular generalized functions* are subspaces of $\mathcal{G}_E$, whose role is to describe different notions of regularity. We use the notation $u = [(u_\varepsilon)_\varepsilon]$ for the class u of $(u_\varepsilon)_\varepsilon$ in $\mathcal{G}_E$. This is the usual way, adopted in the paper, to denote an equivalence class.

2.1.3. *The most common Colombeau algebras*

The Colombeau algebra $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$ (see [**28**]) can be obtained as a $\tilde{\mathbb{C}}$-module of $\mathcal{G}_E$ type by choosing $E = \mathcal{C}^\infty(\Omega)$. From a structural point of view, $\Omega \to \mathcal{G}(\Omega)$ is a fine sheaf of differential algebras on $\mathbb{R}^n$. $\mathcal{G}_{\mathrm{c}}(\Omega)$ denotes the Colombeau algebra of generalized functions with compact support.

Regularity theory in the Colombeau context, as initiated in [**44**], is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of all elements u of $\mathcal{G}(\Omega)$ having a representative $(u_\varepsilon)_\varepsilon$ belonging to the set

$$\mathcal{E}^\infty_M(\Omega) := \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}[\Omega] \colon \forall K \Subset \Omega,\ \exists N \in \mathbb{N}\ \forall \alpha \in \mathbb{N}^n, \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\Big\}.$$

The $\mathcal{G}^\infty$-singular support of $u \in \mathcal{G}(\Omega)$ ($\operatorname{sing\,supp}_{\mathcal{G}^\infty} u$) is defined as the complement of the set of points x such that $u|_V \in \mathcal{G}^\infty(V)$ for some open neighbourhood V of x.

The intersection $\mathcal{G}^\infty(\Omega) \cap \mathcal{G}_{\mathrm{c}}(\Omega)$ will be denoted by $\mathcal{G}^\infty_{\mathrm{c}}(\Omega)$. In the course of the paper, due to issues relating to the Fourier transform, we make use of the Colombeau algebra $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n) = \mathcal{G}_{\mathscr{S}(\mathbb{R}^n)}$ of generalized functions based on $\mathscr{S}(\mathbb{R}^n)$ and of the corresponding regular version $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$. Finally, we recall that the Colombeau algebra $\mathcal{G}_\tau(\mathbb{R}^n)$ of tempered generalized functions is defined as $\mathcal{E}_\tau(\mathbb{R}^n)/\mathcal{N}_\tau(\mathbb{R}^n)$, where $\mathcal{E}_\tau(\mathbb{R}^n)$ is the space

$$\Big\{(u_\varepsilon)_\varepsilon \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^n)^{(0,1]} \colon \forall \alpha \in \mathbb{N}^n,\ \exists N \in \mathbb{N}, \sup_{x\in\mathbb{R}^n} (1+|x|)^{-N} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\Big\}$$

of τ-moderate nets and $\mathcal{N}_\tau(\mathbb{R}^n)$ is the space

$$\Big\{(u_\varepsilon)_\varepsilon \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^n)^{(0,1]} \colon \forall \alpha \in \mathbb{N}^n,\ \exists N \in \mathbb{N},\ \forall q \in \mathbb{N}, \sup_{x\in\mathbb{R}^n} (1+|x|)^{-N} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\Big\}$$

of τ-negligible nets. The subalgebra $\mathcal{G}^\infty_\tau(\mathbb{R}^n)$ of regular and tempered generalized functions is the quotient $\mathcal{E}^\infty_\tau(\mathbb{R}^n)/\mathcal{N}_\tau(\mathbb{R}^n)$, where $\mathcal{E}^\infty_\tau(\mathbb{R}^n)$ is the set of all $(u_\varepsilon)_\varepsilon \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^n)^{(0,1]}$ satisfying the condition that

$$\exists N \in \mathbb{N},\ \forall \alpha \in \mathbb{N}^n,\ \exists M \in \mathbb{N}, \quad \sup_{x\in\mathbb{R}^n} (1+|x|)^{-M}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}).$$

2.1.4. *Special types of Colombeau generalized functions in* $\mathcal{G}(\mathbb{R}^n)$

An element of the Colombeau algebra is *real valued* if it is represented by a net of real-valued smooth functions. We recall that $u \in \mathcal{G}(\mathbb{R}^n)$ is

- *of logarithmic type* if there exists a representative $(u_\varepsilon)_\varepsilon$ with the property that, for all $K \Subset \mathbb{R}^n$,
$$\sup_{x\in K} |u_\varepsilon(x)| = O(\log(\varepsilon^{-1})) \quad \text{as } \varepsilon \to 0;$$

- *of slow-scale logarithmic type* if there exists a representative $(u_\varepsilon)_\varepsilon$ with the property that, for all $K \Subset \mathbb{R}^n$, there exists a slow-scale net $(\mu_\varepsilon)_\varepsilon$ such that
$$\sup_{x\in K} |u_\varepsilon(x)| = O(\log(\mu_\varepsilon)) \quad \text{as } \varepsilon \to 0;$$

- *slow-scale regular* if there exists a representative $(u_\varepsilon)_\varepsilon$ with the property that, for all $K \Subset \mathbb{R}^n$ and all $\alpha \in \mathbb{N}^n$, there exists a slow-scale net $(\mu_\varepsilon)_\varepsilon$ such that
$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\mu_\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Clearly, the previous properties hold for all representatives of u once they are known to be valid for one.

2.1.5. *Topological dual of a Colombeau algebra*

A topological theory of Colombeau algebras was developed in [**18**, **19**]. The duality theory for $\tilde{\mathbb{C}}$-modules presented in [**19**] in the framework of topological and locally convex topological $\tilde{\mathbb{C}}$-modules has provided the theoretical tools for dealing with the topological duals of the Colombeau algebras $\mathcal{G}_{\mathrm{c}}(\Omega)$ and $\mathcal{G}(\Omega)$. We recall that $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ and $\mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}})$ denote the space of all $\tilde{\mathbb{C}}$-linear and continuous functionals on $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathrm{c}}(\Omega)$, respectively. For the choice of topologies given in [**18**] one has the following chains of continuous embeddings:

$$\mathcal{G}^\infty(\Omega) \subseteq \mathcal{G}(\Omega) \subseteq \mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}}), \tag{2.1}$$

$$\mathcal{G}^\infty_{\mathrm{c}}(\Omega) \subseteq \mathcal{G}_{\mathrm{c}}(\Omega) \subseteq \mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}}), \tag{2.2}$$

$$\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}}) \subseteq \mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}}). \tag{2.3}$$

In (2.1) and (2.2) the inclusion in the dual is given via integration ($u \to (v \to \int_\Omega u(x)v(x)\,\mathrm{d}x)$) (for definitions and properties of the integral of Colombeau generalized functions see [**28**]), while the embedding in (2.3) is determined by the inclusion

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$\mathcal{G}_{\mathrm{c}}(\Omega) \subseteq \mathcal{G}(\Omega)$. Since $\Omega \to \mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}})$ is a sheaf, we can define the *support of a functional* T (denoted by $\operatorname{supp} T$). In analogy with distribution theory, we have that $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ can be identified with the set of functionals in $\mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}})$ having compact support.

For questions relating to regularity theory and microlocal analysis, particular attention is given to those functionals in $\mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}})$ and $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ that have a 'basic' structure. More precisely, we say that $T \in \mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}})$ is basic if there exists a net $(T_\varepsilon)_\varepsilon \in \mathcal{D}'(\Omega)^{(0,1]}$ fulfilling the condition that, for all $K \Subset \Omega$, there exist $j \in \mathbb{N}$, $c > 0$, $N \in \mathbb{N}$ and $\eta \in (0,1]$ such that,

$$\forall f \in \mathcal{D}_K(\Omega),\ \forall \varepsilon \in (0,\eta], \quad |T_\varepsilon(f)| \leqslant c\varepsilon^{-N} \sup_{x \in K,\, |\alpha| \leqslant j} |\partial^\alpha f(x)|$$

and $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$ for all $u \in \mathcal{G}_{\mathrm{c}}(\Omega)$.

In the same way, a functional $T \in \mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ is said to be basic if there exists a net $(T_\varepsilon)_\varepsilon \in \mathcal{E}'(\Omega)^{(0,1]}$ such that there exist $K \Subset \Omega$, $j \in \mathbb{N}$, $c > 0$, $N \in \mathbb{N}$ and $\eta \in (0,1]$ with the property that,

$$\forall f \in \mathcal{C}^\infty(\Omega),\ \forall \varepsilon \in (0,\eta], \quad |T_\varepsilon(f)| \leqslant c\varepsilon^{-N} \sup_{x \in K,\, |\alpha| \leqslant j} |\partial^\alpha f(x)|$$

and $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$ for all $u \in \mathcal{G}(\Omega)$. The sets of basic functionals on $\mathcal{G}_{\mathrm{c}}(\Omega)$ and $\mathcal{G}(\Omega)$ are denoted by $\mathcal{L}_{\mathrm{b}}(\mathcal{G}_{\mathrm{c}}(\Omega), \tilde{\mathbb{C}})$ and $\mathcal{L}_{\mathrm{b}}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$, respectively.

2.1.6. *The Fourier transform on $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$, $\mathcal{L}(\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$*

The Fourier transform on $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ is defined by the corresponding transformation at the level of representatives as

$$\mathcal{F}\colon \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n) \to \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)\colon u \to [(\hat{u}_\varepsilon)_\varepsilon].$$

$\mathcal{F}$ is a $\tilde{\mathbb{C}}$-linear continuous map from $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ into itself, which extends to the dual in a natural way. More precisely, we define the Fourier transform of $T \in \mathcal{L}(\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n), \tilde{\mathbb{C}})$ as the functional in $\mathcal{L}(\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n), \tilde{\mathbb{C}})$ given by

$$\mathcal{F}(T)(u) = T(\mathcal{F}u).$$

As shown in [**20**, Remark 1.5], $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ is embedded in $\mathcal{L}(\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n), \tilde{\mathbb{C}})$ by means of the map

$$\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}}) \to \mathcal{L}(\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n), \tilde{\mathbb{C}})\colon T \to (u \to T((u_\varepsilon|_\Omega)_\varepsilon + \mathcal{N}(\Omega))).$$

In particular, when T is a basic functional in $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$, we have from [**20**, Proposition 1.6, Remark 1.7] that the Fourier transform of T is the tempered generalized function obtained as the action of $T(y)$ on $\mathrm{e}^{-\mathrm{i}y\xi}$, i.e. $\mathcal{F}(T) = T(\mathrm{e}^{-\mathrm{i}\cdot\xi}) = (T_\varepsilon(\mathrm{e}^{-\mathrm{i}\cdot\xi}))_\varepsilon + \mathcal{N}_\tau(\mathbb{R}^n)$. More precisely, $\mathcal{F}(T)$ belongs to $\mathcal{G}^\infty_\tau(\mathbb{R}^n)$.

2.1.7. *Microlocal analysis in the Colombeau context: the $\mathcal{G}^\infty$-wavefront set for generalized functions and functionals*

For an introduction to microlocal analysis in the Colombeau context we refer the reader to [**20**, **23**]. Here we only recall those microlocal concepts that we will employ in the final section of the paper. The $\mathcal{G}^\infty$-wavefront set of $u \in \mathcal{G}_\mathrm{c}(\Omega)$ ($\mathrm{WF}^\infty_{\mathcal{G}} u$) is defined as the complement of the set of points $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ fulfilling the property that there exist a representative $(u_\varepsilon)_\varepsilon$ of u, a cut-off function $\varphi \in \mathcal{C}^\infty_\mathrm{c}(\Omega)$ with $\varphi(x_0) = 1$, a conic neighbourhood Γ of ξ_0 and a number N such that, for all $l \in \mathbb{R}$,

$$\sup_{\xi \in \Gamma} \langle \xi \rangle^l |\widehat{\varphi u_\varepsilon}(\xi)| = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

By construction, $\pi_\Omega \mathrm{WF}_{\mathcal{G}^\infty} u = \operatorname{sing\,supp}_{\mathcal{G}^\infty} u$. In addition, [**23**, Theorem 3.11] shows that $\mathrm{WF}_{\mathcal{G}^\infty}$ coincides with the set

$$\mathrm{W}_{\mathrm{cl},\mathcal{G}^\infty}(u) := \bigcap_{Au \in \mathcal{G}^\infty(\Omega)} \mathrm{Char}(A),$$

where the intersection is taken over all the standard properly supported pseudodifferential operators $A \in \Psi^0(\Omega)$ such that $Au \in \mathcal{G}^\infty(\Omega)$. The adjective 'standard' refers to symbols that do not depend on the parameter ε, but belong to the usual Hörmander classes. This pseudodifferential characterization provides the blueprint for extending the notion of wavefront set from $\mathcal{G}(\Omega)$ to the dual $\mathcal{L}(\mathcal{G}_\mathrm{c}(\Omega), \tilde{\mathbb{C}})$. More precisely, for $T \in \mathcal{L}(\mathcal{G}_\mathrm{c}(\Omega), \tilde{\mathbb{C}})$ we define

$$\mathrm{WF}_{\mathcal{G}^\infty}(T) = \mathrm{W}_{\mathrm{cl},\mathcal{G}^\infty}(T) := \bigcap_{AT \in \mathcal{G}^\infty(\Omega)} \mathrm{Char}(A), \tag{2.4}$$

where the intersection is taken over all the standard properly supported pseudodifferential operators $A \in \Psi^0(\Omega)$ such that $AT \in \mathcal{G}^\infty(\Omega)$. It follows that $\pi_\Omega(\mathrm{WF}_{\mathcal{G}^\infty} T) = \operatorname{sing\,supp}_{\mathcal{G}^\infty} T$.

A useful characterization of $\mathrm{WF}_{\mathcal{G}^\infty}(T)$, in terms of estimates at the Fourier transform level, is valid when T is basic. It involves the sets of generalized functions $\mathcal{G}^\infty_{\mathscr{S},0}(\Gamma)$, where Γ is a conic subset of $\mathbb{R}^n \setminus 0$, of all tempered generalized functions u having a representative $(u_\varepsilon)_\varepsilon$ fulfilling the condition that

$$\exists N \in \mathbb{N},\ \forall l \in \mathbb{R}, \quad \sup_{\xi \in \Gamma} \langle \xi \rangle^l |u_\varepsilon(\xi)| = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

Let $T \in \mathcal{L}(\mathcal{G}_\mathrm{c}(\Omega), \tilde{\mathbb{C}})$. Theorem 3.13 in [**20**] shows that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} T$ if and only if there exists a conic neighbourhood Γ of ξ_0 and a cut-off function $\varphi \in \mathcal{C}^\infty_\mathrm{c}(\Omega)$ with $\varphi(x_0) = 1$ such that $\mathcal{F}(\varphi T) \in \mathcal{G}^\infty_{\mathscr{S},0}(\Gamma)$.

2.2. Generalized Fourier integral operators: generalized symbols and phase functions

In this subsection we collect some basic notions concerning generalized pseudodifferential and Fourier integral operators. For a detailed presentation we refer the reader to [**21**, **26**].

2.2.1. *Generalized symbols*

Let Ω be an open subset of $\mathbb{R}^n$. By a *generalized symbol* of order m we mean an element of the space $\mathcal{G}_E$ based on $E = S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$. Analogously, $\mathcal{G}^{\rm sc}_E$, with $E = S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$, is the space of slow-scale regular generalized symbols. We say that a generalized symbol a of order m is *regular* if it has a representative $(a_\varepsilon)_\varepsilon$ fulfilling the condition that,

$$\forall K \Subset \Omega,\ \exists N \in \mathbb{N},\ \forall j \in \mathbb{N},\quad |a_\varepsilon|^{(m)}_{K,j} = \sup_{|\alpha|+|\beta|\leqslant j} |a_\varepsilon|^{(m)}_{K,\alpha,\beta} = O(\varepsilon^{-N}) \quad \text{as } \varepsilon \to 0.$$

A notion of asymptotic expansion for generalized symbols was introduced in [**21**, § 2.5] and employed in developing a complete symbolic calculus for generalized pseudodifferential operators (see [**21**, **25**]). Finally, we recall that the *conic support* of a generalized symbol a of order m is the complement of the set of points $(x_0, \xi_0) \in \Omega \times \mathbb{R}^p$ such that there exist a relatively compact open neighbourhood U of x_0, a conic open neighbourhood Γ of ξ_0 and a representative $(a_\varepsilon)_\varepsilon$ of a satisfying the condition that,

$$\forall \alpha \in \mathbb{N}^p,\ \forall \beta \in \mathbb{N}^n,\ \forall q \in \mathbb{N},\quad \sup_{x\in U,\,\xi\in\Gamma} \langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|} |\partial^\alpha_\xi \partial^\beta_x a_\varepsilon(x,\xi)| = O(\varepsilon^q) \quad \text{as } \varepsilon \to 0. \tag{2.5}$$

By definition, $\operatorname{cone\,supp} a$ is a closed conic subset of $\Omega \times \mathbb{R}^p$.

2.2.2. *Generalized phase functions*

A *phase function* $\phi(y, \xi)$ on $\Omega \times \mathbb{R}^p$ is a smooth function on $\Omega \times \mathbb{R}^p \setminus 0$, real valued and positively homogeneous of degree 1 in ξ such that $\nabla_{y,\xi}\phi(y, \xi) \neq 0$ for all $y \in \Omega$ and $\xi \in \mathbb{R}^p \setminus 0$. We denote the set of all phase functions on $\Omega \times \mathbb{R}^p$ by $\Phi(\Omega \times \mathbb{R}^p)$ and the set of all nets in $\Phi(\Omega \times \mathbb{R}^p)^{(0,1]}$ by $\Phi[\Omega \times \mathbb{R}^p]$. We recall that $S^1_{\rm hg}(\Omega \times \mathbb{R}^p \setminus 0)$ is the space of symbols on $\Omega \times \mathbb{R}^p \setminus 0$ homogeneous of order 1 in ξ, i.e.

$$\sup_{x\in K,\,\xi\in\mathbb{R}^p\setminus 0} |\xi|^{-1+\alpha} |\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| < \infty$$

for all $K \Subset \Omega$, for all $\alpha \in \mathbb{N}^p$ and for $\beta \in \mathbb{N}^n$.

Definition 2.1. An element of $\mathcal{M}_\Phi(\Omega \times \mathbb{R}^p)$ is a net $(\phi_\varepsilon)_\varepsilon \in \Phi[\Omega \times \mathbb{R}^p]$ satisfying the following conditions:

(i) $(\phi_\varepsilon)_\varepsilon \in \mathcal{M}_{S^1_{\rm hg}(\Omega\times\mathbb{R}^p\setminus 0)}$,

(ii) for all $K \Subset \Omega$ the net

$$\left(\inf_{y\in K,\,\xi\in\mathbb{R}^p\setminus 0} \left| \nabla_{y,\xi}\phi_\varepsilon\left(y, \frac{\xi}{|\xi|}\right) \right|^2 \right)_\varepsilon$$

is strictly non-zero.

We introduce the equivalence relation $\sim$ on $\mathcal{M}_\Phi(\Omega \times \mathbb{R}^p)$ as follows: $(\phi_\varepsilon)_\varepsilon \sim (\omega_\varepsilon)_\varepsilon$ if and only if $(\phi_\varepsilon - \omega_\varepsilon) \in \mathcal{N}_{S^1_{\rm hg}(\Omega\times\mathbb{R}^p\setminus 0)}$. The elements of the factor space

$$\tilde{\Phi}(\Omega \times \mathbb{R}^p) := \mathcal{M}_\Phi(\Omega \times \mathbb{R}^p)/\sim$$

will be called *generalized phase functions*.

Finally, we say that $\phi \in \tilde{\Phi}(\Omega \times \mathbb{R}^p)$ is a *slow-scale generalized phase function* if it has a representative $(\phi_\varepsilon)_\varepsilon \in \mathcal{M}^{\mathrm{sc}}_{S^1_{\mathrm{hg}}(\Omega\times\mathbb{R}^p\setminus 0)}$ such that the net in (ii) is slow-scale strictly non-zero.

In the following, Ω' is an open subset of $\mathbb{R}^{n'}$. We denote by $\Phi[\Omega';\Omega\times\mathbb{R}^p]$ the set of all nets $(\phi_\varepsilon)_{\varepsilon\in(0,1]}$ of continuous functions on $\Omega'\times\Omega\times\mathbb{R}^p$ that are smooth on $\Omega'\times\Omega\times\mathbb{R}^p\setminus\{0\}$ and are defined such that $(\phi_\varepsilon(x,\cdot,\cdot))_\varepsilon \in \Phi[\Omega\times\mathbb{R}^p]$ for all $x\in\Omega'$.

Definition 2.2. An element of $\mathcal{M}_\Phi(\Omega';\Omega\times\mathbb{R}^p)$ is a net $(\phi_\varepsilon)_\varepsilon \in \Phi[\Omega';\Omega\times\mathbb{R}^p]$ satisfying the following conditions:

(i) $(\phi_\varepsilon)_\varepsilon \in \mathcal{M}_{S^1_{\mathrm{hg}}(\Omega'\times\Omega\times\mathbb{R}^p\setminus 0)}$,

(ii) for all $K'\Subset\Omega'$ and $K\Subset\Omega$ the net

$$\left(\inf_{x\in K',\,y\in K,\,\xi\in\mathbb{R}^p\setminus 0}\left|\nabla_{y,\xi}\phi_\varepsilon\left(x,y,\frac{\xi}{|\xi|}\right)\right|^2\right)_\varepsilon \tag{2.6}$$

is strictly non-zero.

We introduce the equivalence relation $\sim$ on $\mathcal{M}_\Phi(\Omega';\Omega\times\mathbb{R}^p)$ as follows: $(\phi_\varepsilon)_\varepsilon \sim (\omega_\varepsilon)_\varepsilon$ if and only if $(\phi_\varepsilon-\omega_\varepsilon)_\varepsilon \in \mathcal{N}_{S^1_{\mathrm{hg}}(\Omega'\times\Omega\times\mathbb{R}^p\setminus 0)}$. The elements of the factor space

$$\tilde{\Phi}(\Omega';\Omega\times\mathbb{R}^p) := \mathcal{M}_\Phi(\Omega';\Omega\times\mathbb{R}^p)/\sim$$

are called *generalized phase functions with respect to the variables in* $\Omega\times\mathbb{R}^p$. If $\phi\in\tilde{\Phi}(\Omega';\Omega\times\mathbb{R}^p)$ has a representative

$$(\phi_\varepsilon)_\varepsilon \in \mathcal{M}^{\mathrm{sc}}_{S^1_{\mathrm{hg}}(\Omega'\times\Omega\times\mathbb{R}^p\setminus 0)}$$

such that the net in (2.6) is slow-scale strictly non-zero, then it is called *slow-scale generalized phase function with respect to the variables in* $\Omega\times\mathbb{R}^p$.

2.2.3. *Slow-scale critical points*

Definition 2.3. Let $\phi\in\tilde{\Phi}(\Omega\times\mathbb{R}^p)$. We define $C^{\mathrm{sc}}_\phi \subseteq \Omega\times\mathbb{R}^p\setminus 0$ as the complement of the set of all $(x_0,\xi_0)\in\Omega\times\mathbb{R}^p\setminus 0$ with the property that there exist a relatively compact open neighbourhood $U(x_0)$ of x_0 and a conic open neighbourhood $\Gamma(\xi_0)\subseteq\mathbb{R}^p\setminus 0$ of ξ_0 such that the generalized function $|\nabla_\xi\phi(\cdot,\cdot)|^2$ is slow-scale invertible on $U(x_0)\times\Gamma(\xi_0)$. We set $\pi_\Omega(C^{\mathrm{sc}}_\phi) = S^{\mathrm{sc}}_\phi$ and $R^{\mathrm{sc}}_\phi = (S^{\mathrm{sc}}_\phi)^{\mathrm{c}}$.

By construction, C^{sc}_ϕ is a conic closed subset of $\Omega\times\mathbb{R}^p\setminus 0$ and $R^{\mathrm{sc}}_\phi\subseteq R_\phi\subseteq\Omega$ is open. It is routine to check that the region C^{sc}_ϕ coincides with the classical one $\{(x,\xi)\in\Omega\times\mathbb{R}^p\setminus 0\colon \nabla_\xi\phi(x,\xi)=0\}$ when ϕ is a standard phase function independent of ε.

2.2.4. *Generalized Fourier integral operators*

Let $\phi\in\tilde{\Phi}(\Omega';\Omega\times\mathbb{R}^p)$, let $a\in\mathcal{G}_{S^m_{\rho,\delta}(\Omega'\times\Omega\times\mathbb{R}^p)}$ and let $u\in\mathcal{G}_{\mathrm{c}}(\Omega)$. The generalized oscillatory integral

$$I_\phi(a)(u)(x) = \int_{\Omega\times\mathbb{R}^p} \mathrm{e}^{\mathrm{i}\phi(x,y,\xi)} a(x,y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\!\!\bar{}\,\xi$$

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defines a generalized function in $\mathcal{G}(\Omega')$ and the map

$$A\colon \mathcal{G}_{\mathrm{c}}(\Omega) \to \mathcal{G}(\Omega')\colon u \to I_\phi(a)(u) \tag{2.7}$$

is continuous. The operator A defined in (2.7) is called a *generalized Fourier integral operator*, with amplitude a and phase function $\phi \in \tilde{\Phi}(\Omega'; \Omega \times \mathbb{R}^p)$. From [**21**, Theorem 4.6] we have that, when phase function and amplitude are both slow-scale regular, the corresponding generalized Fourier integral operator maps $\mathcal{G}_{\mathrm{c}}^\infty(\Omega)$ continuously into $\mathcal{G}^\infty(\Omega')$. If $a \in \mathcal{G}^{\mathrm{sc}}_{S^{-\infty}(\Omega'\times\Omega\times\mathbb{R}^p)}$, then A maps $\mathcal{G}_{\mathrm{c}}(\Omega)$ into $\mathcal{G}^\infty(\Omega')$. Pseudodifferential operators are special Fourier integral operators with $\phi(x, y, \xi) = (x - y)\xi$ in (2.7).

2.2.5. *Composition of a generalized Fourier integral operator with a generalized pseudodifferential operator*

We conclude this survey of generalized Fourier integral operators by studying the composition with a generalized pseudodifferential operator. First, we focus on Fourier integral operators of the form

$$F_\omega(b)(u)(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\omega(x,\eta)} b(x, \eta)\hat{u}(\eta)\,\text{đ}\eta,$$

where $\omega \in \mathcal{G}_{S^1_{\mathrm{hg}}(\Omega'\times\mathbb{R}^n\setminus 0)}$, $b \in \mathcal{G}_{S^m(\Omega'\times\mathbb{R}^n)}$ and $u \in \mathcal{G}_{\mathrm{c}}(\Omega)$. Note that $\phi(x, y, \eta) := \omega(x, \eta) - y\eta$ is a well-defined generalized phase function belonging to $\tilde{\Phi}(\Omega'; \Omega \times \mathbb{R}^n)$ and $I_\phi(b) = F_\omega(b)$. Theorem 5.11 in [**21**] provides the composition formula stated below.

2.2.6. *Composition formula for* $a(x, D)F_\omega(b)$

Let $\omega \in \mathcal{G}^{\mathrm{sc}}_{S^1_{\mathrm{hg}}(\Omega\times\mathbb{R}^n\setminus 0)}$ have a representative $(\omega_\varepsilon)_\varepsilon$ such that $\nabla_x\omega_\varepsilon \neq 0$ for all $\varepsilon \in (0, 1]$ and such that, for all $K \Subset \Omega$,

$$\left(\inf_{x\in K,\, \eta\in\mathbb{R}^n\setminus 0} \left|\nabla_x\omega_\varepsilon\left(x, \frac{\eta}{|\eta|}\right)\right|\right)_\varepsilon$$

is slow-scale strictly non-zero. Let $a \in \mathcal{G}^{\mathrm{sc}}_{S^m(\Omega\times\mathbb{R}^n)}$ and let $b \in \mathcal{G}^{\mathrm{sc}}_{S^l(\Omega\times\mathbb{R}^n\setminus 0)}$ with $\mathrm{supp}_x\, b \Subset \Omega$. Then, the operator $a(x, D)F_\omega(b)$ has the following properties.

(i) It maps $\mathcal{G}_{\mathrm{c}}^\infty(\Omega)$ into $\mathcal{G}^\infty(\Omega)$.

(ii) It is of the form

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\omega(x,\eta)} h(x, \eta)\hat{u}(\eta)\,\text{đ}\eta + r(x, D)u,$$

where $h \in \mathcal{G}^{\mathrm{sc}}_{S^{l+m}(\Omega\times\mathbb{R}^n\setminus 0)}$ has the asymptotic expansion given by the symbols

$$h_\alpha(x, \eta) = \frac{\partial_\xi^\alpha a(x, \nabla_x\omega(x, \eta))}{\alpha!} D_z^\alpha(\mathrm{e}^{\mathrm{i}\bar{\omega}(z,x,\eta)} b(z, \eta))\Big|_{z=x}, \quad \alpha \in \mathbb{N}^n,$$

with $\bar{\omega}(z, x, \eta) := \omega(z, \eta) - \omega(x, \eta) - \langle\nabla_x\omega(x, \eta), z - x\rangle$, and r is slow-scale regular and of order $-\infty$.

3. Transport equations with generalized coefficients

In this section, we are concerned with the Cauchy problem for the first-order hyperbolic equation

$$D_t u = \sum_{j=1}^{n} a_{1,j}(t,x) D_j u + a_0(t,x) u, \quad u(0,\cdot) = u_0, \tag{3.1}$$

where $D_j = D_{x_j}$, the coefficients $a_{1,j}$ are real-valued Colombeau generalized functions in $\mathcal{G}(\mathbb{R}^{n+1})$, $a_0 \in \mathcal{G}(\mathbb{R}^{n+1})$ and $u_0 \in \mathcal{G}_c(\mathbb{R}^n)$. The following theorem combines the well-posedness results of [**40**] with the more recent investigations of [**32**].

Theorem 3.1. *Let the coefficients $a_{1,j}$, $j = 1, \ldots, n$, and a_0 be Colombeau generalized functions in $\mathcal{G}(\mathbb{R}^{n+1})$ compactly supported in x. Assume that the coefficients $a_{1,j}$ are real valued and that $\partial_k a_{1,j}$ and a_0 are of logarithmic type $(k, j = 1, \ldots, n)$. We then have the following.*

(i) *For each $u_0 \in \mathcal{G}(\mathbb{R}^n)$, (3.1) has a unique solution $u \in \mathcal{G}(\mathbb{R}^{n+1})$.*

(ii) *If, in addition, $u_0 \in \mathcal{G}_c(\mathbb{R}^n)$, then the solution u is compactly supported in x.*

(iii) *If, in addition, a_0 and u_0 are real valued, then the solution u is a real-valued generalized function.*

(iv) *If the coefficients $a_{1,j}$ and a_0 are slow-scale regular and $\partial_k a_{1,j}$ $(k, j = 1, \ldots, n)$ and a_0 are of slow-scale logarithmic type, then for each $u_0 \in \mathcal{G}^\infty(\mathbb{R}^n)$ the unique solution $u \in \mathcal{G}(\mathbb{R}^{n+1})$ of (3.1) belongs to $\mathcal{G}^\infty(\mathbb{R}^{n+1})$.*

(v) *Under hypothesis (iv), if the initial data u_0 are slow-scale regular, then u is slow-scale regular as well.*

The aim of this section is to prove that the solution u of (3.1) can be written as the action of a generalized Fourier integral operator $F_\phi(b)$ on the initial data $u_0 \in \mathcal{G}_c(\mathbb{R}^n)$. This requires us to determine the phase function ϕ and the symbol b.

3.1. The generalized phase function and the characteristic curves

The generalized phase function ϕ is the solution of the eikonal equation determined by the principal part of the operator

$$D_t - \sum_{j=1}^{n} a_{1,j}(t,x) D_j - a_0(t,x)$$

under the initial condition $\phi(0,x,\eta) = x\eta$. Thus, one has to solve the linear Cauchy problem

$$\partial_t \phi(t,x,\eta) = \sum_{j=1}^{n} a_{1,j}(t,x) \partial_j \phi(t,x,\eta), \quad \phi(0,x,\eta) = x\eta. \tag{3.2}$$

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Under the hypotheses of Theorem 3.1 on the coefficients $a_{1,j}$, we already know that there exists a unique solution $\phi \in \mathcal{G}(\mathbb{R}^{n+2})$. More precisely, it has the form

$$\phi(t,x,\eta) = \sum_{h=1}^{n} \omega_h(t,x)\eta_h,$$

where ω_h, $h = 1,\dots,n$, are solutions of the Cauchy problems

$$\partial_t \omega_h(t,x) = \sum_{j=1}^{n} a_{j,1}(t,x)\partial_j \omega_h(t,x), \quad \omega_h(0,x) = x_h. \tag{3.3}$$

In the following proposition we describe the properties of the generalized functions ω_h more specifically. The solutions $\gamma_1,\dots,\gamma_n$ of the initial-value problem

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\gamma_h(x,t,s) &= -a_{1,h}(s,\gamma_1(x,t,s),\gamma_2(x,t,s),\dots,\gamma_n(x,t,s)),\\ \gamma_h(x,t,t) &= x_j, \quad h = 1,\dots,n, \end{aligned}\right\} \tag{3.4}$$

are the components of the *characteristic curve* $\gamma = (\gamma_1,\dots,\gamma_n)$ associated with the differential operator $\sum_{j=1}^{n} a_{1,j}(t,x)D_j$. Note that, from [**28**, Theorem 1.5.2 and Remark 1.5.3], this initial-value problem is well-posed in $\mathcal{G}(\mathbb{R}^{n+2})$ when $a_{1,j}$ is compactly supported in x and has first-order x-derivatives of logarithmic type.

We make the following assumptions.

(h_1) The coefficients $a_{1,j}$ are real-valued generalized functions in $\mathcal{G}(\mathbb{R}^{n+1})$, compactly supported with respect to x, with $\partial_k a_{1,j}$ of logarithmic type ($k,j = 1,\dots,n$).

(h_2) The coefficients $a_{1,j}$ are real-valued slow-scale regular generalized functions in $\mathcal{G}(\mathbb{R}^{n+1})$, compactly supported with respect to x, with $\partial_k a_{1,j}$ of slow-scale logarithmic type ($k,j = 1,\dots,n$).

Note that the compact support property is not essential and is introduced here to keep the presentation simple.

Proposition 3.2.

(i) *Under hypothesis* (h_1), *there exists a unique real-valued solution* $\omega_h(t,x) \in \mathcal{G}(\mathbb{R}^{n+1})$ *of (3.3);* $\omega_h(t,x)$ *is the hth component of the characteristic curve* $\gamma(x,t,0)$.

(ii) *Under hypothesis* (h_2), *the solution* ω_h *is slow-scale regular.*

Proof. From parts (i) and (iii) of Theorem 3.1 it is clear that there exists a unique real-valued Colombeau solution $\omega_h \in \mathcal{G}(\mathbb{R}^{n+1})$. Since the initial data x_h are smooth and, therefore, slow-scale regular we have, from parts (iv) and (v) of Theorem 3.1, that ω_h is slow-scale regular under (h_2) on $a_{1,j}$ and $\partial_k a_{1,j}$, $k,j = 1,\dots,n$. It remains to prove that $\omega_h(t,x) = \gamma_h(x,t,0)$. This comes from the fact that ω_h is constant along the characteristic curves $\gamma(x,t,s)$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}\omega_{h,\varepsilon}(s,\gamma_{1,\varepsilon}(x,t,s),\gamma_{2,\varepsilon}(x,t,s),\dots,\gamma_{n,\varepsilon}(x,t,s)) = 0,$$

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working at the level of representatives. Hence,

$$\omega_{h,\varepsilon}(t,\gamma_{1,\varepsilon}(x,t,t),\dots,\gamma_{n,\varepsilon}(x,t,t)) = \omega_{h,\varepsilon}(0,\gamma_{1,\varepsilon}(x,t,0),\dots,\gamma_{n,\varepsilon}(x,t,0))$$

for each $t \in \mathbb{R}$. This implies that $\omega_h(t,x) = \gamma_h(x,t,0)$ in $\mathcal{G}(\mathbb{R}^{n+1})$. □

Summarizing, we can state the following proposition.

Proposition 3.3. *If* (h_1) *holds, then the generalized phase function*

$$\phi(t,x,\eta) = \sum_{h=1}^{n} \omega_h(t,x)\eta_h = \sum_{h=1}^{n} \gamma_h(x,t,0)\eta_h$$

solves the eikonal Cauchy problem (3.2).

If (h_2) *holds, then* ϕ *is a slow-scale generalized phase function.*

Remark 3.4. The Colombeau generalized function ϕ is actually a generalized symbol homogeneous of order 1 in η. With a certain abuse of language, we employ the expression *generalized phase function*, previously used to refer to $\phi'(t,x,y,\eta) = \phi(t,x,\eta) - y\eta$. Indeed, for ϕ' one has the typical invertibility condition on the gradient, i.e.

$$|\nabla\phi'(t,x,y,\eta)| = |(\nabla_t\phi(t,x,\eta), \nabla_x\phi(t,x,\eta), -\eta, \nabla_\eta\phi(t,x,\eta) - y)| \geqslant 1$$

for all η, with $|\eta| = 1$.

3.2. The transport equation for the generalized symbol

To compute the generalized symbol b, we need to solve (3.1) with initial condition 1 at $t = 0$. More precisely, we need to solve

$$D_t b = \sum_{j=1}^{n} a_{1,j}(t,x)D_j b + a_0(t,x)b, \quad b(0,\cdot) = 1. \tag{3.5}$$

We introduce the following set of hypotheses on a_0.

(i_1) a_0 is a generalized function in $\mathcal{G}(\mathbb{R}^{n+1})$, compactly supported in x and of logarithmic type.

(i_2) a_0 is a slow-scale regular generalized function in $\mathcal{G}(\mathbb{R}^{n+1})$, compactly supported in x with zero-derivative of slow-scale logarithmic type.

Using Theorem 3.1 and integrating along the characteristics, we have the following existence and uniqueness result.

Theorem 3.5.

(i) *Under hypothesis* (h_1) *on the coefficients* $a_{1,j}$ *and hypothesis* (i_1) *on* a_0, *there exists a unique solution* $b \in \mathcal{G}(\mathbb{R}^{n+1})$ *of (3.5).*

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(ii) *Under hypothesis* (h_2) *on the coefficients* $a_{1,j}$ *and hypothesis* (i_2) *on* a_0*, the solution* $b \in \mathcal{G}(\mathbb{R}^{n+1})$ *of (3.5) is slow-scale regular.*

(iii) *Furthermore,* $b(t,x) = \mathrm{e}^{\mathrm{i}\beta(t,x)}$*, with*

$$\beta(t,x) = \int_0^t a_0(s, \gamma_1(x,t,s), \dots, \gamma_n(x,t,s))\,\mathrm{d}s.$$

3.3. Generalized FIO formula

A combination of Proposition 3.3 and Theorem 3.5 yields the following FIO formula.

Proposition 3.6.

(i) *Under hypotheses* (h_1) *and* (i_1) *on the coefficients* $a_{1,j}$ *and* a_0*, respectively, the solution* $u \in \mathcal{G}(\mathbb{R}^{n+1})$ *of (3.1) can be written as*

$$u(t,x) = F_\phi(b)(u_0)(t,x) := \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(t,x,\eta)} b(t,x)\hat{u}_0(\eta)\,\text{đ}\eta, \tag{3.6}$$

where the generalized phase function ϕ *is defined in Proposition 3.3 and* $b \in \mathcal{G}(\mathbb{R}^{n+1})$ *is defined in Theorem 3.5.*

(ii) *Under hypotheses* (h_2) *and* (i_2)*, (3.6) holds with* ϕ *and* b *slow-scale regular.*

3.4. The non-homogeneous Cauchy problem

We conclude this section by finding a solution formula for the non-homogeneous Cauchy problem

$$D_t u = \sum_{j=1}^n a_{1,j}(t,x) D_j u + a_0(t,x)u + f(t,x), \quad u(0,\cdot) = u_0, \tag{3.7}$$

where $f \in \mathcal{G}(\mathbb{R}^{n+1})$ is compactly supported with respect to x. Note that the Fourier integral operator $F_\phi(b)$ of Proposition 3.6, solving the homogeneous problem, is given by

$$F_\phi(b)(u_0)(t,x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(t,x,\eta)} b(t,x,\eta)\hat{u}_0(\eta)\,\text{đ}\eta = b(t,x)u_0(\gamma(x,t,0)).$$

This defines, for each $t \in \mathbb{R}$, a map

$$U(t) = F_\phi(b)(t) \colon \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n) \to \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n) \colon u_0 \to F_\phi(b)(u_0)(t,\cdot)$$

such that $U(0) = I$ and

$$U(t)^{-1} = \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n) \to \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n) \colon v \to \frac{1}{b(t,\gamma(x,0,t))} v(\gamma(x,0,t)).$$

Theorem 3.7.

(i) *Under hypotheses* (h_1) *and* (i_1) *on the coefficients* $a_{1,j}$ *and* a_0*, respectively, the solution* $u \in \mathcal{G}(\mathbb{R}^{n+1})$ *of (3.7) can be written as*

$$u(t,x) = F_\phi(b)(t)\biggl(u_0 + \mathrm{i}\int_0^t \frac{1}{b(\tau,\gamma(\cdot,0,\tau))} f(\tau,\gamma(\cdot,0,\tau))\,\mathrm{d}\tau\biggr)(x), \tag{3.8}$$

where the generalized phase function ϕ *is defined in Proposition 3.3 and* $b \in \mathcal{G}(\mathbb{R}^{n+1})$ *is defined in Theorem 3.5.*

(ii) *Under hypotheses* (h_2) *and* (i_2)*, (3.8) holds with* ϕ *and* b *slow-scale regular.*

4. First-order hyperbolic pseudodifferential equations with generalized symbols

We now consider hyperbolic first-order pseudodifferential equations of the type

$$D_t u = a_1(t, D_x)u + a_0(t, x, D_x)u, \tag{4.1}$$

where a_1 and a_0 are generalized symbols of order 1 and 0, respectively, with a_1 real valued and independent of x. As mentioned in § 1, we restrict ourselves to t-dependent principal parts. We begin by collecting what is known about (4.1) in the Colombeau context. The following theorem is due to Hörmann (see [**32**]). The well-posedness of the Cauchy problem

$$D_t u = a_1(t, D_x)u + a_0(t, x, D_x)u + f, \quad u(0,\cdot) = u_0 \tag{4.2}$$

is intended in the Colombeau algebra $\mathcal{G}_{H^\infty((-T,T)\times\mathbb{R}^n)}$ based on $H^\infty((-T,T)\times\mathbb{R}^n)$. Here, we use the notation $\mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n)$ introduced in [**4**]. The choice of this setting is motivated by a uniqueness issue; the solution u to (4.2) fails to be unique in the usual Colombeau algebra $\mathcal{G}([-T,T]\times\mathbb{R}^n)$, whereas it is uniquely determined in $\mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n)$. Finally, with the expressions *generalized symbol* and *slow-scale regular generalized symbol* we refer to the elements of the spaces $\mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$ and $\mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$, respectively. Note that by the notation $S^m(\mathbb{R}^{2n})$ we denote symbols satisfying uniform estimates with respect to $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, i.e.

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} \langle\xi\rangle^{-m+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| < \infty.$$

In the following, (k,h) denotes the η-derivatives and x-derivatives of a symbol up to order k and h, respectively.

Theorem 4.1. *Let* a_1 *be a real-valued generalized symbol of order* 1*, let* a_0 *be a generalized symbol of order* 0*, let* $f \in \mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n)$ *and let* $u_0 \in \mathcal{G}_{2,2}(\mathbb{R}^n)$*.*

(i) *If there exist representatives* $(a_{1,\varepsilon})_\varepsilon$ *and* $(a_{0,\varepsilon})_\varepsilon$ *of* a_1 *and* a_0*, respectively, such that* $(a_{1,\varepsilon})_\varepsilon$ *is of log type up to order* $(k_n, l_n{+}1)$*, with* $k_n = 3(\lfloor n/2\rfloor{+}1)$ *and* $l_n = 2(n{+}2)$*, and* $(a_{0,\varepsilon})_\varepsilon$ *is of log type up to order* (k_n', j_n')*, with* $k_n' = l_n' = \lfloor n/2\rfloor + 1$*, then (4.2) is well-posed in* $\mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n)$*.*

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(ii) *If, for large* $|x|$, *the net* $(a_{0,\varepsilon})_\varepsilon$ *does not depend on* t, *then one can set* $k_n = 1$, $l_n = n+2$, $k'_n = 0$, $l'_n = n+1$ *in (i).*

(iii) *If* a_1 *and* a_0 *are slow-scale regular generalized symbols and the log-type conditions in (i) on* $(a_{1,\varepsilon})_\varepsilon$ *and* $(a_{0,\varepsilon})_\varepsilon$ *are replaced by slow-scale log-type assumptions, then, for each* $f \in \mathcal{G}^\infty_{2,2}((-T,T)\times\mathbb{R}^n)$ *and* $u_0 \in \mathcal{G}^\infty_{2,2}(\mathbb{R}^n)$, *the unique solution* u *to (4.2) belongs to* $\mathcal{G}^\infty_{2,2}((-T,T)\times\mathbb{R}^n)$.

Our aim now is to find an FIO formula by which to express the solution u. More precisely, we will construct a generalized FIO parametrix for the hyperbolic Cauchy problem

$$D_t u = a_1(t, D_x)u + a_0(t, x, D_x)u, \quad u(0,\cdot) = u_0 \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n). \tag{4.3}$$

Under suitable moderateness assumptions, we get well-posedness and $\mathcal{G}^\infty$-regularity.

4.1. The generalized phase function and the eikonal equation

We begin by determining the generalized phase function $\phi(t,x,\eta)$, i.e. by solving the following eikonal equation.

Proposition 4.2. *The eikonal equation*

$$\partial_t\phi(t,x,\eta) = a_1(t, \nabla_x\phi(t,x,\eta)), \quad \phi(0,x,\eta) = x\eta, \tag{4.4}$$

has the solution

$$\phi(t,x,\eta) = x\eta + \int_0^t a_1(s,\eta)\,\mathrm{d}s$$

in $\mathcal{G}_{\mathcal{C}^\infty([-T,T],S^1(\mathbb{R}^{2n}))}$ *if* $a_1 \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^1(\mathbb{R}^{2n}))}$ *and in* $\mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^1(\mathbb{R}^{2n}))}$ *if* $a_1 \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^1(\mathbb{R}^{2n}))}$.

Remark 4.3. (i) Note that $\phi(t,x,\eta)$ is a symbol of order 1 with $\nabla_x\phi(t,x,\eta) = \eta$, but non-homogeneous with respect to η. This is not an obstacle in defining the Fourier integral operator

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(t,x,\eta)} b(t,x,\eta)\hat{v}(\eta)\,\mathrm{đ}\eta = \int_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}(\phi(t,x,\eta)-y\eta)} b(t,x,\eta)v(y)\,\mathrm{d}y\,\mathrm{đ}\eta$$

for $v \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$ and a generalized symbol b. Indeed, to give a meaning to the oscillatory integral on the right-hand side (by means of an operator L_ϕ to be used in the integration by parts), it is sufficient that the bound from below

$$|\nabla_x\phi'_\varepsilon(t,x,\eta)|^2 + |\nabla_y\phi'_\varepsilon(t,x,\eta)|^2 + |\eta|^2|\nabla_\eta\phi'_\varepsilon(t,x,\eta)|^2 \geqslant \lambda_\varepsilon|\eta|^2 \tag{4.5}$$

holds for some strictly non-zero net λ_ε and for $\phi'(t,x,y,\eta) = \phi(t,x,\eta) - y\eta$ uniformly in $t \in [-T,T]$, $(x,y,\eta) \in \mathbb{R}^{3n}$ and $\varepsilon \in (0,1]$. Condition (4.5) is trivially satisfied by the phase function in Proposition (4.2). Indeed, for

$$\phi'(t,x,y,\eta) = (x-y)\eta + \int_0^t a_1(s,\eta)\,\mathrm{d}s$$

one has that $|\nabla_x\phi'_\varepsilon(t,x,\eta)|^2 = |\eta|^2$.

(ii) Easy computations at the level of representatives show that the Fourier integral operator with phase function

$$\phi(t,x,\eta) = x\eta + \int_0^t a_1(s,\eta)\,\mathrm{d}s$$

and symbol

$$b(t,x,\eta) \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$$

maps $\mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$ (or $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$) into $\mathcal{G}_{\mathcal{C}^\infty([-T,T],\mathscr{S}(\mathbb{R}^n))}$. Recalling the embedding $\mathcal{G}_{\mathrm{c}}(\mathbb{R}n) \subseteq \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n) \subseteq \mathcal{G}_{2,2}(\mathbb{R}^n)$ (see [**19**]) we have that

$$F_\phi(b)\colon \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n) \to \mathcal{G}_{\mathcal{C}^\infty([-T,T],\mathscr{S}(\mathbb{R}^n))} \to \mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n)$$

and

$$a_1(t,D_x)F_\phi(b) + a_0(t,x,D_x)F_\phi(b)\colon \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n) \to \mathcal{G}_{\mathcal{C}^\infty([-T,T],\mathscr{S}(\mathbb{R}^n))} \to \mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n).$$

In addition, if we work with slow-scale generalized phase functions and symbols, then the previous mapping properties hold between $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$ and $\mathcal{G}^\infty_{2,2}((-T,T)\times\mathbb{R}^n)$.

Theorem 5.10 in [**21**] concerning the composition of a generalized pseudodifferential operator with a generalized Fourier integral operator can be easily adapted to Fourier integral operators with a phase function, as above. This means that under the slow-scale assumptions of Theorem 4.1 (iii), for any $b \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$ and $v \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$, we have that

$$a_1(t,D_x)F_\phi(b)v + a_0(t,x,D_x)F_\phi(b)v = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(t,x,\eta)} h(t,x,\eta)\hat{v}(\eta)\,\mathrm{đ}\eta + r(t,x,D_x)v, \quad (4.6)$$

where h and r are generalized symbols of order $m+1$ and $-\infty$, respectively, $v \in \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ and the equality is intended in $\mathcal{G}_{2,2}((-T,T)\times\mathbb{R}^n)$. If we take slow-scale regular phase functions and symbols, then h and r are slow-scale regular as well and $r(t,x,D_x)v \in \mathcal{G}^\infty_{2,2}((-T,T)\times\mathbb{R}^n)$. The symbol h has the following asymptotic expansion (as defined in [**21**]):

$$\begin{aligned} h(t,x,\eta) \sim \sum_{\alpha\in\mathbb{N}^n} h_\alpha(t,x,\eta) &= \sum_{\alpha\in\mathbb{N}^n} \frac{\partial^\alpha_\eta(a_1(t,\eta)+a_0(t,x,\eta))}{\alpha!} D^\alpha_z(\mathrm{e}^{\mathrm{i}\overline{\phi(t,z,x,\eta)}} b(t,z,\eta))\Big|_{z=x} \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{\partial^\alpha_\eta(a_1(t,\eta)+a_0(t,x,\eta))}{\alpha!} D^\alpha_x b(t,x,\eta). \end{aligned}$$

4.2. The transport equations and the generalized symbol

We proceed in the construction of a generalized FIO parametrix for (4.3) by looking for a symbol $b \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^0(\mathbb{R}^{2n}))}$ given by the asymptotic expansion

$$\sum_{k\in\mathbb{N}} b_k, \quad b_k \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^{-k}(\mathbb{R}^{2n}))}.$$

The symbols b_k will solve some specific transport equations and will determine b such that

$$D_t F_\phi(b) - a_1(t, D_x)F_\phi(b) - a_0(t, x, D_x)F_\phi(b)$$

is a $\mathcal{G}^\infty_{2,2}$-regularizing operator. We begin by observing that, by composition formula (4.6),

$$a_1(t, D_x)F_\phi(b)v + a_0(t, x, D_x)F_\phi(b)v = F_\phi(h)u + r(t, x, D_x)v, \quad v \in \mathcal{G}_c(\mathbb{R}^n),$$

where r has order $-\infty$ and is slow-scale regular when a_1 and a_0 are also slow-scale regular. In other words, we can write that

$$D_t F_\phi(b)v - a_1(t, D_x)F_\phi(b)v - a_0(t, x, D_x)F_\phi(b)v = D_t F_\phi(b)v - F_\phi(h)v - r(t, x, D_x)v.$$

Note that, taking a suitable cut-off function ψ, we can write the action of $r(t, x, D_x)$ on v as the integral operator

$$\int_{\mathbb{R}^n} \psi(y) k_r(t, x, y) v(y) \, \mathrm{d}y,$$

with $\psi(y)k_r(t, x, y) \in \mathcal{G}^\infty_{2,2}((-T, T) \times \mathbb{R}^{2n})$ rapidly decreasing with respect to x and y. Then, the generalized function

$$\int_{\mathbb{R}^n} \psi(y) k_r(t, x, y) v(y) \, \mathrm{d}y$$

belongs to $\mathcal{G}^\infty_{2,2}((-T, T) \times \mathbb{R}^n)$. We recall that ϕ solves the eikonal equation (4.4) of the previous subsection. Hence, by making use of the asymptotic expansion of h and b written above and by collecting the terms with the same order, we obtain the transport equations

$$\left.\begin{aligned}
D_t b_0 &= \sum_{|\alpha|=1} \frac{\partial^\alpha_\eta a_1(t,\eta)}{\alpha!} D^\alpha_x b_0 + a_0 b_0,\\
D_t b_{-1} &= \sum_{|\alpha|=1} \frac{\partial^\alpha_\eta a_1(t,\eta)}{\alpha!} D^\alpha_x b_{-1} + a_0 b_{-1}\\
&\qquad + \sum_{|\alpha|=1} \frac{\partial^\alpha_\eta a_0(t,x,\eta)}{\alpha!} D^\alpha_x b_0 + \sum_{|\alpha|=2} \frac{\partial^\alpha_\eta a_1(t,\eta)}{\alpha!} D^\alpha_x b_0,\\
&\ \vdots\\
D_t b_{-k} &= \sum_{|\alpha|=1} \frac{\partial^\alpha_\eta a_1(t,\eta)}{\alpha!} D^\alpha_x b_{-k} + a_0 b_{-k} + f_{-k},
\end{aligned}\right\} \tag{4.7}$$

where $f_{-k} \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^{-k}(\mathbb{R}^{2n}))}$. From Proposition 3.6 and Theorem 3.7 we deduce the following statement on transport equations with generalized symbols as coefficients and initial data.

Theorem 4.4. *The Cauchy problem*

$$D_t s = \sum_{|\alpha|=1} \frac{\partial^\alpha_\eta a_1(t,\eta)}{\alpha!} D^\alpha_x s + a_0(t, x, \eta)s + f, \tag{4.8}$$

$$s(0, \cdot, \cdot) = s_0, \tag{4.9}$$

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has a solution $s \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$ *where*

(i) $m \in \mathbb{R}$, $f \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$ *and* $s_0 \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$,

(ii) $a_1 \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^1(\mathbb{R}^{2n}))}$ *and* $a_0 \in \mathcal{G}_{\mathcal{C}^\infty([-T,T],S^0(\mathbb{R}^{2n}))}$,

(iii) a_0 *is of log type.*

If we replace $\mathcal{G}$ *with* $\mathcal{G}^{\mathrm{sc}}$ *in (i)–(iii) and the assumption of log type with slow-scale log type, then* $s \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$.

Proof. By applying Theorem 3.7 to (4.8) we have the solution formula

$$s(t,x,\eta) = b(t,x,\eta)\biggl(s_0(t,\gamma(t,x,\eta),\eta) + \mathrm{i}\int_0^t \frac{f(\tau,\gamma(t,x,\eta,\tau),\eta)}{b(\tau,\gamma(t,x,\eta,\tau),\eta)}\,\mathrm{d}\tau\biggr),$$

where, for $i = 1,\dots,n$,

$$\begin{aligned}\gamma_i(t,x,\eta) &= x_i + \int_0^t \partial_{\eta_i} a_1(\sigma,\eta)\,\mathrm{d}\sigma,\\ \gamma_i(t,x,\eta,\tau) &= x_i + \int_0^t \partial_{\eta_i} a_1(\sigma,\eta)\,\mathrm{d}\sigma - \int_0^\tau \partial_{\eta_i} a_1(\sigma,\eta)\,\mathrm{d}\sigma,\\ b(t,x,\eta) &= \exp\biggl(\mathrm{i}\int_0^t a_0(s,\gamma(t,x,\eta,s),\eta)\,\mathrm{d}s\biggr).\end{aligned}$$

One can easily check that $b(t,x,\eta)$ and $s_0(t,\gamma_1(t,x,\eta),\dots,\gamma_2(t,x,\eta),\eta)$ are elements of the generalized symbol spaces $\mathcal{G}_{\mathcal{C}^\infty([-T,T],S^0(\mathbb{R}^{2n}))}$ and $\mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$, respectively, and that

$$\begin{aligned}&\int_0^t \frac{f(\tau,\gamma(t,x,\eta,\tau),\eta)}{b(\tau,\gamma(t,x,\eta,\tau),\eta)}\,\mathrm{d}\tau\\ &\qquad = \int_0^t \exp\biggl(-\mathrm{i}\int_0^\tau a_0(s,\gamma(t,x,\eta,s),\eta)\,\mathrm{d}s\biggr) f(\tau,\gamma(t,x,\eta,\tau),\eta)\,\mathrm{d}\tau\end{aligned}$$

belongs to $\mathcal{G}_{\mathcal{C}^\infty([-T,T],S^m(\mathbb{R}^{2n}))}$. From this, it follows that $s(t,x,\eta)$ is a generalized symbol of order m satisfying the initial condition $s(0,x,\eta) = s_0(x,\eta)$. □

We now go back to (4.7). With the help of Theorem 4.4 we solve the equations in (4.7).

Proposition 4.5. *If* a_1 *and* a_0 *are slow-scale regular generalized symbols of order* 1 *and* 0, *respectively, satisfying the assumptions of Theorem 4.1 (iii), then there exist symbols* $b_{-j} \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^{-j}(\mathbb{R}^{2n}))}$, $j \in \mathbb{N}$, *which solve transport equations (4.7) and such that* $b_0(0,x,\eta) = 1$ *and* $b_{-j}(0,x,\eta) = 0$ *for all* $j > 0$.

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4.3. Construction of a generalized FIO parametrix and solution formula for the Cauchy problem

We now have all the tools for constructing a generalized FIO parametrix for (4.3). Combining Theorem 4.1 with Proposition 4.2 and Proposition 4.5 we obtain the following statement.

Theorem 4.6. *Let $u_0 \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$ and let a_1 and a_0 fulfil the assumptions of Theorem 4.1 (iii) on a_1 and a_0. Then, there exists a generalized Fourier integral operator $F_\phi(b)$ with slow-scale phase function and symbol b of order 0 such that*

$$D_t F_\phi(b)u_0 = a_1(t, D_x)F_\phi(b)u_0 + a_0(t, x, D_x)F_\phi(b)u_0 + r(t, x, D_x)u_0,$$
$$F_\phi(b)u_0(0, \cdot) = u_0 + r_0(t, D_x)u_0,$$

where $r \in \mathcal{G}^{\mathrm{sc}}_{\mathcal{C}^\infty([-T,T],S^{-\infty}(\mathbb{R}^{2n}))}$ and $r_0 \in \mathcal{G}^{\mathrm{sc}}_{S^{-\infty}(\mathbb{R}^{2n})}$.

Corollary 4.7. *Let the hypotheses of Theorem 4.1 (iii) be satisfied. Then, the solution $u \in \mathcal{G}_{2,2}((-T, T) \times \mathbb{R}^n)$ of the Cauchy problem*

$$D_t u = a_1(t, D_x)u + a_0(t, x, D_x)u, \quad u(0, \cdot) = u_0 \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n),$$

is equal to $F_\phi(b)u_0$ modulo $\mathcal{G}^\infty_{2,2}((-T, T) \times \mathbb{R}^n)$.

Proof. From Theorem 4.6 we have that $v = F_\phi(b)u_0 - u \in \mathcal{G}_{2,2}((-T, T) \times \mathbb{R}^n)$ solves the Cauchy problem

$$D_t v = a_1(t, D_x)v + a_0(t, x, D_x)v + r(t, x, D_x)u_0, \quad v(0, \cdot) = r_0(x, D_x)u_0,$$

where $r(t, x, D_x)u_0 \in \mathcal{G}^\infty_{2,2}((-T, T) \times \mathbb{R}^n)$ and $r_0(x, D_x)u_0 \in \mathcal{G}^\infty_{2,2}(\mathbb{R}^n)$. Theorem 4.1 yields the $\mathcal{G}^\infty$-regularity of v, i.e. $v \in \mathcal{G}^\infty_{2,2}((-T, T) \times \mathbb{R}^n)$. □

5. Microlocal investigation of the solution of a generalized hyperbolic Cauchy problem

This section presents a microlocal investigation of the solution $u \in \mathcal{G}(\mathbb{R}^{n+1})$ of the hyperbolic Cauchy problem studied in §§ 3 and 4. First, we will concentrate on the microlocal properties of u, viewed as a generalized function in both the variables t and x ($\mathrm{WF}_{\mathcal{G}^\infty} u$), and, second, we will fix t and investigate the generalized function $u(t, \cdot) \in \mathcal{G}(\mathbb{R}^n)$ microlocally ($\mathrm{WF}_{\mathcal{G}^\infty} u(t, \cdot)$). Since u can be written as the action of a generalized Fourier integral operator $F_\phi(b)$ on the initial data $u_0 \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n)$, we focus our attention on the microlocal properties of generalized Fourier integral operators. We begin with some abstract theoretical results that we will finally apply to the special case of $u = F_\phi(b)u_0$ under suitable assumptions on the phase function ϕ.

5.1. Microlocal properties of generalized Fourier integral operators: the wavefront set

We begin by recalling some results obtained in [**26**]. Let Ω be an open subset of $\mathbb{R}^n$. As a preliminary step we observe that when $\phi \in \tilde{\Phi}(\Omega \times \mathbb{R}^p)$, $U \subseteq \bar{U} \Subset \Omega$, $\Gamma \subseteq \mathbb{R}^n \setminus 0$, $V \subseteq \Omega \times \mathbb{R}^p \setminus 0$, then

$$\operatorname*{Inf}_{\substack{y\in U,\,\xi\in\Gamma\\(y,\theta)\in V}} \frac{|\xi - \nabla_y\phi(y,\theta)|}{|\xi|+|\theta|} := \left[\left(\inf_{\substack{y\in U,\,\xi\in\Gamma\\(y,\theta)\in V}} \frac{|\xi - \nabla_y\phi_\varepsilon(y,\theta)|}{|\xi|+|\theta|}\right)_\varepsilon\right]$$

is a well-defined element of $\tilde{\mathbb{C}}$.

In the following, $W^{\mathrm{sc}}_{\phi,a}$ denotes the set of all points $(x_0,\xi_0) \in T^*(\Omega)\setminus 0$ with the property that, for all relatively compact open neighbourhoods $U(x_0)$ of x_0, for all open conic neighbourhoods $\Gamma(\xi_0) \subseteq \mathbb{R}^n \setminus 0$ of ξ_0 and for all open conic neighbourhoods V of cone $\operatorname{supp} a \cap C^{\mathrm{sc}}_\phi$ such that $V \cap (U(x_0)\times\mathbb{R}^p\setminus 0) \neq \emptyset$, the generalized number

$$\operatorname*{Inf}_{\substack{y\in U(x_0),\,\xi\in\Gamma(\xi_0)\\(y,\theta)\in V\cap(U(x_0)\times\mathbb{R}^p\setminus 0)}} \frac{|\xi - \nabla_y\phi(y,\theta)|}{|\xi|+|\theta|} \tag{5.1}$$

is not slow-scale invertible.

Theorem 5.1. *Let $I_\phi(a)$ be the functional in $\mathcal{L}(\mathcal{G}_{\mathrm{c}}(\Omega),\tilde{\mathbb{C}})$ given by*

$$v \to \int_{\Omega\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(y,\xi)} a(y,\xi)v(y)\,\mathrm{d}y\,\text{đ}\xi,$$

where ϕ is a slow-scale generalized phase function and a is a regular symbol of order m. The $\mathcal{G}^\infty$-wavefront set of $I_\phi(a)$ is contained in the set $W^{\mathrm{sc}}_{\phi,a}$.

The previous theorem can clearly be stated for a slow-scale generalized phase function $\phi(x,y,\xi)$ (in the variable (y,ξ)) and a regular amplitude $a(x,y,\xi)$ on $\Omega'\times\Omega\times\mathbb{R}^p$, with an open subset Ω' of $\mathbb{R}^{n'}$. In this case,

$$I_\phi(a)v = \int_{\Omega'\times\Omega\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(x,y,\xi)} a(x,y,\xi)v(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\text{đ}\xi$$

is the kernel of the generalized Fourier integral operator

$$A\colon \mathcal{G}_{\mathrm{c}}(\Omega) \to \mathcal{G}(\Omega')\colon u \to \int_{\Omega\times\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(x,y,\xi)} a(x,y,\xi)u(y)\,\mathrm{d}y\,\text{đ}\xi.$$

We will therefore use the notation K_A for the functional $I_\phi(a)$. The next theorem relates $\mathrm{WF}_{\mathcal{G}^\infty} Au$ to $\mathrm{WF}_{\mathcal{G}^\infty} K_A$ and $\mathrm{WF}_{\mathcal{G}^\infty} u$.

We recall that, given $E \subseteq \Omega'\times\Omega\times\mathbb{R}^{n'}\times\mathbb{R}^n$, $E' \subseteq (\Omega'\times\mathbb{R}^{n'})\times(\Omega\times\mathbb{R}^n)$ is the set of all (x,ξ,y,η) such that $(x,y,\xi,-\eta)\in E$. A subset E' defines a relation $E'\circ G$ of $\Omega\times\mathbb{R}^n$ with $\Omega'\times\mathbb{R}^{n'}$ when $G \subseteq \Omega\times\mathbb{R}^n$. More precisely,

$$E'\circ G := \{(x,\xi)\in\Omega'\times\mathbb{R}^{n'} : \exists (y,\eta)\in G,\ (x,\xi,y,\eta)\in E'\}.$$

Theorem 5.2. *Let $\phi(x, y, \xi)$ be a slow-scale generalized phase function in the variable (y, ξ) and let $a(x, y, \xi)$ be a regular amplitude on $\Omega' \times \Omega \times \mathbb{R}^p$. Let A be the corresponding Fourier integral operator with kernel $K_A \in \mathcal{L}(\mathcal{G}_c(\Omega' \times \Omega), \tilde{\mathbb{C}})$ and let u be a generalized function in $\mathcal{G}_c(\Omega)$. If*

$$\mathrm{WF}'_{\mathcal{G}^\infty} K_A \circ \mathrm{WF}_{\mathcal{G}^\infty} u \subseteq T^*(\Omega') \setminus 0, \tag{5.2}$$

then

$$\mathrm{WF}_{\mathcal{G}^\infty} Au \subseteq (\mathrm{WF}'_{\mathcal{G}^\infty} K_A \circ \mathrm{WF}_{\mathcal{G}^\infty} u) \cup (\mathrm{WF}'_{\mathcal{G}^\infty} K_A \circ (\operatorname{sing\,supp}_{\mathcal{G}^\infty} u \times \{0\})), \tag{5.3}$$

where the right-hand side is a conic subset of $T^*(\Omega') \setminus 0$.

Proof. Let L be a neighbourhood of $\operatorname{sing\,supp}_{\mathcal{G}^\infty} u$. Let Λ be the right-hand side of (5.3) with $\operatorname{sing\,supp}_{\mathcal{G}^\infty} u$ replaced by L. It is clear that Λ is conic, and the inclusion $\Lambda \subseteq T^*(\Omega') \setminus 0$ follows from (5.2).

We prove that if $(x_0, \xi_0) \notin \Lambda$, then $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} Au$. From this it follows that $\mathrm{WF}_{\mathcal{G}^\infty} Au \subseteq \Lambda$. At this point, since L is chosen arbitrarily and u has compact support, we get the desired inclusion (5.3).

Let K_1 be a neighbourhood of x_0, and let $K_2 = \pi_\Omega(\operatorname{sing\,supp}_{\mathcal{G}^\infty} K_A \cap (K_1 \times L))$. From the assumptions on phase function and amplitude we have that A maps $\mathcal{G}^\infty$ into $\mathcal{G}^\infty$. It follows that, for a cut-off function ψ that is identically 1 on a neighbourhood of $\operatorname{sing\,supp}_{\mathcal{G}^\infty} u$, we can write $\operatorname{sing\,supp}_{\mathcal{G}^\infty} Au = \operatorname{sing\,supp}_{\mathcal{G}^\infty} A(\psi u)$. If $K_2 = \emptyset$, then taking ψ such that $\operatorname{supp}(\psi u) \subseteq L$ we have that $Au|_{K_1}$ can be written as $\int K_A(x, y)\psi u(y)\,\mathrm{d}y$, where $K_A(x, y)$ is $\mathcal{G}^\infty$-regular. Hence, $Au|_{K_1} \in \mathcal{G}^\infty$ and $(x_0, \xi) \notin \mathrm{WF}_{\mathcal{G}^\infty} Au$ for each $\xi \neq 0$.

We assume that K_2 is not empty and for $y_0 \in K_2$ we set

$$\Sigma_0 = \{(\xi, \eta) \colon (x_0, y_0, \xi, \eta) \in \mathrm{WF}_{\mathcal{G}^\infty} K_A\},$$
$$\Gamma_0 = \{\eta \colon (y_0, \eta) \in \mathrm{WF}_{\mathcal{G}^\infty} u\}.$$

We claim that there exist

- a conic open neighbourhood $\tilde{\Sigma}_0 \subseteq \mathbb{R}^{n'+n} \setminus 0$ of Σ_0,
- a conic open neighbourhood $\tilde{\Gamma}_0 \subseteq \mathbb{R}^n \setminus 0$ of Γ_0,
- a conic neighbourhood $W \subseteq \mathbb{R}^{n'} \setminus 0$ of ξ_0,
- a number $\delta > 0$ and neighbourhoods V_1 and V_2 of x_0 and y_0, respectively,

such that

$$W \cap (\tilde{\Sigma}'_0 \circ \tilde{\Gamma}_0) = \emptyset, \tag{5.4}$$

$$V_1 \times V_2 \times \{(\xi, \eta) \colon \xi \in W,\ |\eta| \leqslant \delta|\xi|\} \cap \mathrm{WF}_{\mathcal{G}^\infty} K_A = \emptyset, \tag{5.5}$$

$$\tilde{\Sigma}_0 \cap \{(\xi, \eta) \colon -\eta \in \tilde{\Gamma}_0,\ |\xi| \leqslant \delta|\eta|\} = \emptyset. \tag{5.6}$$

But $(x_0, \xi_0) \notin \Lambda$ implies that $(x_0, \xi_0) \notin \mathrm{WF}'_{\mathcal{G}^\infty} K_A \circ \mathrm{WF}_{\mathcal{G}^\infty} u$. Therefore, since Σ_0, Γ_0 and $\Sigma_0 \circ \Gamma_0$ are closed and conic, there exist neighbourhoods W, $\tilde{\Sigma}_0$ and $\tilde{\Gamma}_0$, as above, satisfying (5.4). If $(x_0,\xi_0)\notin\Lambda$, then $(x_0,y_0,\xi_0,0)\notin \mathrm{WF}_{\mathcal{G}^\infty}K_A$. Indeed, $(x_0,y_0,\xi_0,0)\in \mathrm{WF}_{\mathcal{G}^\infty}K_A$ implies that $(x_0,\xi_0)\in\mathrm{WF}_{\mathcal{G}^\infty}K'_A\circ(L\times\{0\})\subseteq\Lambda$. It follows that there exist neighbourhoods V_1 and V_2, as above, and $\delta>0$ such that (5.5) holds. Finally, we have that $\{(0,\eta)\colon -\eta\in\Gamma_0\}\cap\Sigma_0=\emptyset$. Indeed, if $(x_0,0,y_0,\eta)\in\mathrm{WF}_{\mathcal{G}^\infty}K_A$ and $(y_0,-\eta)\in\mathrm{WF}_{\mathcal{G}^\infty}u$, then $(x_0,0)\in\mathrm{WF}'_{\mathcal{G}^\infty}K_A\circ\mathrm{WF}_{\mathcal{G}^\infty}u$. This is absurd since $\mathrm{WF}'_{\mathcal{G}^\infty}K_A\circ\mathrm{WF}_{\mathcal{G}^\infty}u\subseteq T^*(\Omega')\setminus 0$. This assertion yields (5.6) for a suitable choice of $\tilde\Sigma_0$ and $\tilde\Gamma_0$. By shrinking V_1 and V_2 we can also impose that

$$V_1\subseteq K_1,\quad (V_2\times(\tilde\Gamma_0)^{\mathrm{c}})\cap\mathrm{WF}_{\mathcal{G}^\infty}u=\emptyset,\quad (V_1\times V_2\times(\tilde\Sigma_0)^{\mathrm{c}})\cap\mathrm{WF}_{\mathcal{G}^\infty}K_A=\emptyset,\qquad(5.7)$$

where the complements are intended in $\mathbb{R}^n\setminus 0$ and $\mathbb{R}^{n'+n}\setminus 0$, respectively.

Letting $\alpha\in\mathcal{C}^\infty_{\mathrm{c}}(V_1)$ be identically 1 near x_0, we have to prove that $\widehat{\alpha Au}$ is $\mathcal{G}^\infty_{\mathscr{S},0}$-regular on a certain conic neighbourhood of ξ_0. Let $\psi\in\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ be identically 1 on a neighbourhood of $\operatorname{sing\,supp}_{\mathcal{G}^\infty}u$ and with $\operatorname{supp}\psi\subseteq L$. Since $\alpha A((1-\psi)u)\in\mathcal{G}^\infty_{\mathrm{c}}(\Omega')$ we have that

$$\widehat{\alpha Au}=\widehat{\alpha A(\psi u)}\ \text{modulo}\ \mathcal{G}^\infty_{\mathscr{S}}.$$

We write that

$$\widehat{\alpha A\psi u}(\xi)=\langle K_A,\alpha(x)\mathrm{e}^{-\mathrm{i}x\xi}\psi(y)u(y)\rangle.$$

From $K_2=\pi_\Omega(\operatorname{sing\,supp}_{\mathcal{G}^\infty}K_A\cap(K_1\times L))$ we have that we can restrict our attention to $y\in K_2$. Taking a partition of unity in a neighbourhood of K_2 (based on the neighbourhoods V_2 with the property above) we observe that $\widehat{\alpha A\psi u}(\xi)$ is a finite sum of terms of the form

$$\langle fK_A,\alpha(x)\mathrm{e}^{-\mathrm{i}x\xi}\beta(y)u(y)\rangle,$$

where $f\in\mathcal{C}^\infty_{\mathrm{c}}(V_1\times V_2)$ and $\beta\in\mathcal{C}^\infty_{\mathrm{c}}(V_2)$. Applying the Fourier and inverse Fourier transforms we can write that

$$\langle fK_A,\alpha(x)\mathrm{e}^{-\mathrm{i}x\xi}\beta(y)u(y)\rangle=\int_{\mathbb{R}^{n'+n}}\widehat{fK_A}(\theta,\eta)\hat\alpha(\xi-\theta)\widehat{\beta u}(-\eta)\,\text{đ}\theta\,\text{đ}\eta.$$

From (5.7) we now have that the tempered generalized function $\widehat{fK_A}$ is $\mathcal{G}^\infty_{\mathscr{S},0}$-regular outside $\tilde\Sigma_0$ and that $\hat\alpha(\xi-\theta)\widehat{\beta u}(-\eta)$ is $\mathcal{G}^\infty_{\mathscr{S},0}$-regular with respect to (θ,η) outside $\{(\theta,\eta)\colon -\eta\in\tilde\Gamma_0,\ |\theta|\leqslant\delta|\eta|\}$. This fact combined with (5.6) guarantees that the integral above is absolutely convergent. Note that

$$|\hat\alpha(\xi-\theta)|\leqslant c_N(1+|\xi-\theta|)^{-N}$$

for arbitrary $N\in\mathbb{N}$ and that if W' is a sufficiently small conic open neighbourhood of ξ_0 with $\bar W'\subseteq W$, then there exists $c>0$ such that $|\xi-\theta|\geqslant c(|\xi|+|\theta|)$ for all $\xi\in W'$ and all $\theta\in W^{\mathrm{c}}$. Hence, the estimate

$$|\hat\alpha(\xi-\theta)|\leqslant c'_N(1+|\xi|+|\theta|)^{-N}$$

holds for all $\xi \in W'$ and all $\theta \in W^{\mathrm{c}}$. It follows that

$$\int_{\theta \notin W} \widehat{fK_A}(\theta,\eta)\hat{\alpha}(\xi-\theta)\widehat{\beta u}(-\eta)\,đ\theta\,đ\eta \in \mathcal{G}^{\infty}_{\mathscr{S},0}(W'). \tag{5.8}$$

In order to understand the integration over $W \times \mathbb{R}^n$ we begin by proving that $\widehat{fK_A}(\theta,\eta)\widehat{\beta u}(-\eta)$ is $\mathcal{G}^{\infty}_{\mathscr{S},0}$-regular in (θ,η) for $\theta \in W$ and $\eta \in \mathbb{R}^n$. From (5.5) we have that $\widehat{fK_A}$ is $\mathcal{G}^{\infty}_{\mathscr{S},0}$-regular in (θ,η) for $\theta \in W$ and $|\eta| \leqslant \delta|\theta|$. If $|\eta| \geqslant \delta|\theta|$ and $-\eta \notin \tilde{\Gamma}_0$, (5.7) implies that $\widehat{\beta u}$ is $\mathcal{G}^{\infty}_{\mathscr{S},0}$-regular in η and, therefore, also in (θ,η). Finally, when $\theta \in W$ and $-\eta \in \tilde{\Gamma}_0$, then $(\theta,\eta) \notin \tilde{\Sigma}_0$ from (5.4). Since $\widehat{fK_A}$ is $\mathcal{G}^{\infty}_{\mathscr{S},0}$-regular outside $\tilde{\Sigma}_0$, we conclude that $\widehat{fK_A}(\theta,\eta)\widehat{\beta u}(-\eta)$ has the same regularity property. At this point, it is clear that the generalized function given by

$$\int_{\theta \in W} \widehat{fK_A}(\theta,\eta)\hat{\alpha}(\xi-\theta)\widehat{\beta u}(-\eta)\,đ\theta\,đ\eta \tag{5.9}$$

is $\mathcal{G}^{\infty}_{\mathscr{S},0}$-regular on W. A combination of (5.8) and (5.9) yields that $\widehat{\alpha A u} \in \mathcal{G}^{\infty}_{\mathscr{S},0}(W')$. This means that $(x_0,\xi_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} Au$ and completes the proof. □

We recall that the $\mathcal{G}^\infty$-microsupport of a generalized symbol a on $\Omega \times \mathbb{R}^n$ is the complement of the set of points (x_0,ξ_0) with the following property: there exist a representative $(a_\varepsilon)_\varepsilon$ of a, a relatively compact open neighbourhood U of x_0, a conic neighbourhood $\Gamma \subseteq \mathbb{R}^n \setminus 0$ of ξ_0 and a natural number $N \in \mathbb{N}$ such that,

$$\forall m \in \mathbb{R},\ \forall \alpha,\beta \in \mathbb{N}^n,\ \exists c>0,\ \exists \eta \in (0,1],\ \forall (x,\xi) \in U \times \Gamma,\ \forall \varepsilon \in (0,\eta],$$
$$|\partial_\xi^\alpha \partial_x^\beta a_\varepsilon(x,\xi)| \leqslant c\langle\xi\rangle^m \varepsilon^{-N}. \tag{5.10}$$

The $\mathcal{G}^\infty$-microsupport is denoted by $\mu\,\mathrm{supp}_{\mathcal{G}^\infty} a$.

Remark 5.3. Condition (5.2) is satisfied when $A = a(x,D)$ is a generalized pseudodifferential operator with regular symbol. In this case we have from [**26**, Remark 5.15] that if $(x,y,\xi,\eta) \in \mathrm{WF}_{\mathcal{G}^\infty} K_A = \mathrm{WF}_{\mathcal{G}^\infty} K_{a(x,D)}$, then $x=y$ and $\eta = -\xi$, with $\xi \neq 0$ and $(x,\xi) \in \mu\,\mathrm{supp}_{\mathcal{G}^\infty} a$. Hence,

$$\mathrm{WF}'_{\mathcal{G}^\infty} K_{a(x,D)} \circ \mathrm{WF}_{\mathcal{G}^\infty} u \subseteq \{(x,\xi)\colon (x,x,\xi,-\xi) \in \mathrm{WF}_{\mathcal{G}^\infty} K_{a(x,D)}\} \cap \mathrm{WF}_{\mathcal{G}^\infty} u,$$

which implies that $\mathrm{WF}'_{\mathcal{G}^\infty} K_{a(x,D)} \circ \mathrm{WF}_{\mathcal{G}^\infty} u \subseteq T^*(\Omega) \setminus 0$. Since

$$\mathrm{WF}'_{\mathcal{G}^\infty} K_{a(x,D)} \circ (\mathrm{sing\,supp}_{\mathcal{G}^\infty} u \times \{0\}) = \emptyset,$$

Theorem 5.2 yields, for $u \in \mathcal{G}_{\mathrm{c}}(\Omega)$, that

$$\begin{aligned}\mathrm{WF}_{\mathcal{G}^\infty} a(x,D)u &\subseteq \mathrm{WF}'_{\mathcal{G}^\infty} K_{a(x,D)} \circ \mathrm{WF}_{\mathcal{G}^\infty} u\\ &\subseteq \{(x,\xi)\colon (x,x,\xi,-\xi) \in \mathrm{WF}_{\mathcal{G}^\infty} K_{a(x,D)}\} \cap \mathrm{WF}_{\mathcal{G}^\infty} u\\ &\subseteq \mu\,\mathrm{supp}_{\mathcal{G}^\infty} a \cap \mathrm{WF}_{\mathcal{G}^\infty} u.\end{aligned}$$

Note that the same inclusion was obtained in [**23**, Theorem 3.6].

We now consider a special class of generalized Fourier integral operators that fulfil (5.2) and we rephrase Theorem 5.2 for them. This requires some preliminary results concerning the set C_ϕ of singular points and the set $W^{\mathrm{sc}}_{\phi,a}$ defined before Theorem 5.1.

Proposition 5.4. *Let* $[(\phi)_\varepsilon] \in \tilde{\Phi}(\Omega \times \mathbb{R}^p)$ *and let* $\phi \in \mathcal{C}^1(\Omega \times \mathbb{R}^p \setminus 0)$ *such that* $\lim_{\varepsilon\to 0} \phi_\varepsilon = \phi$ *in* $\mathcal{C}^1(\Omega \times \mathbb{R}^p \setminus 0)$*. It then follows that*

(i) $C^{\mathrm{sc}}_{[(\phi_\varepsilon)_\varepsilon]} \subseteq \{(x,\xi) \in T^*(\Omega) \setminus 0\colon \nabla_\xi \phi(x,\xi) = 0\}$,

(ii) *for any generalized symbol* a,

$$W^{\mathrm{sc}}_{[(\phi)_\varepsilon],a} \subseteq \{(x, \nabla_x \phi(x,\theta))\colon \theta \neq 0,\ (x,\theta) \in \operatorname{cone\,supp} a,\ \nabla_\theta \phi(x,\theta) = 0\}.$$

Proof. (i) From the limit property we obtain that, for any compact subset K of Ω, $\sup_{x\in K,\,|\xi|=1} |\nabla_\xi \phi_\varepsilon(x,\xi) - \nabla_\xi \phi(x,\xi)|$ tends to 0. If, for some (x_0,ξ_0), $\nabla_\xi\phi(x_0,\xi_0) \neq 0$, then $|\nabla_\xi \phi(x,\xi)| \geqslant c$ on a relatively compact neighbourhood $U(x_0)$ and on a conic neighbourhood $\Gamma(\xi_0)$. Since for all $n \in \mathbb{N}$ there exists $\varepsilon_n \in (0,1]$ such that

$$\sup_{x\in U(x_0),\,\xi\in\Gamma(\xi_0)} |\nabla_\xi \phi_\varepsilon(x,\xi) - \nabla_\xi \phi(x,\xi)| \leqslant \frac{1}{n}$$

for all $\varepsilon \in (0,\varepsilon_n)$, we get that

$$|\nabla_\xi \phi_\varepsilon(x,\xi)| \geqslant |\nabla_\xi \phi(x,\xi)| - |\nabla_\xi \phi_\varepsilon(x,\xi) - \nabla_\xi \phi(x,\xi)| \geqslant c - \frac{1}{n}$$

for all $x \in U(x_0)$, $\xi \in \Gamma(\xi_0)$ and $\varepsilon \in (0,\varepsilon_n)$. Choosing n large enough we have that the net

$$\inf_{x\in U(x_0),\,\xi\in\Gamma(\xi_0)} |\nabla_\xi \phi_\varepsilon(x,\xi)|$$

is slow-scale invertible, that is, $(x_0,\xi_0) \notin C^{\mathrm{sc}}_{[(\phi_\varepsilon)_\varepsilon]}$.

(ii) If $(x_0,\xi_0) \in W^{\mathrm{sc}}_{[(\phi)_\varepsilon],a}$, then we find sequences of neighbourhoods $U_n(x_0)$, $\Gamma_n(\xi_0)$ and $V_n(\operatorname{cone\,supp} a \cap C^{\mathrm{sc}}_{[(\phi_\varepsilon)_\varepsilon]})$ and sequences of points $x_n \in U_n(x_0)$, $\xi_n \in \Gamma_n(\xi_0)$, $(x_n,\theta_n) \in V_n$ such that

$$|\xi_n - \nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)| \leqslant |\log \varepsilon_n|^{-1}(|\xi_n| + |\theta_n|),$$

where ε_n tends to 0. It is not restrictive to assume that θ_n has norm 1. Hence,

$$|\xi_n - \nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)| \leqslant |\log \varepsilon_n|^{-1}(|\xi_n| + 1). \tag{5.11}$$

Passing to subsequences, we have that x_n converges to x_0 and θ_n converges to a certain θ_0, with $|\theta_0| = 1$ and $(x_0,\theta_0) \in \operatorname{cone\,supp} a \cap C^{\mathrm{sc}}_{[(\phi_\varepsilon)_\varepsilon]}$. The first assertion of this proposition implies that $\nabla_\theta \phi(x_0,\theta_0) = 0$. The sequence $|\xi_n|$ is bounded. Indeed,

$$|\xi_n| \leqslant |\xi_n - \nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)| + |\nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)| \leqslant |\log \varepsilon_n|^{-1}(|\xi_n| + 1) + |\nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)|,$$

where from $\lim_{n\to\infty} |\nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)| = |\nabla_x \phi(x_0,\theta_0)|$ it follows that $|\nabla_x \phi_{\varepsilon_n}(x_n,\theta_n)| \leqslant 1$ for n large enough. Thus,

$$|\xi_n|(1 - |\log \varepsilon_n|^{-1}) \leqslant |\log \varepsilon_n|^{-1} + 1$$

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results in an upper bound for $|\xi_n|$. As a consequence, passing again to subsequences, we find some $\xi' = \lambda\xi_0$, $\lambda > 0$, such that $\xi_n \to \xi'$. Passing to the limit in (5.11), we deduce that $\xi' = \lambda\xi_0 = \nabla_x\phi(x_0, \theta_0)$. In conclusion, we have proved that if $(x_0, \xi_0) \in W^{\mathrm{sc}}_{[(\phi)_\varepsilon],a}$, then there exists $\theta_0' \neq 0$ such that $(x_0, \theta_0') \in \operatorname{cone\,supp} a$, $\nabla_\theta\phi(x_0, \theta_0') = 0$ and $\xi_0 = \nabla_x\phi(x_0, \theta_0')$. □

We are ready to consider a Fourier integral operator with phase function

$$[(\phi_\varepsilon(x, \xi))_\varepsilon] - y\xi.$$

From the previous statements we obtain the following microlocal result.

Theorem 5.5. *Let $[(\phi_\varepsilon(x, \xi) - y\xi)_\varepsilon]$ be a slow-scale generalized phase function in the variable (y, ξ) and let $a(x, y, \xi)$ be a regular generalized amplitude on $\Omega' \times \Omega \times \mathbb{R}^n$. Let A be the corresponding Fourier integral operator with kernel $K_A \in \mathcal{L}(\mathcal{G}_c(\Omega' \times \Omega), \widetilde{\mathbb{C}})$ and let u be a generalized function in $\mathcal{G}_{\mathrm{c}}(\Omega)$. If there exists $\phi \in \mathcal{C}^1(\Omega' \times \mathbb{R}^n \setminus 0)$ with $\nabla_x\phi(x, \xi) \neq 0$ for $\xi \neq 0$ such that $\phi_\varepsilon \to \phi$ in $\mathcal{C}^1(\Omega' \times \mathbb{R}^n \setminus 0)$, then*

(i) $\mathrm{WF}'_{\mathcal{G}^\infty} K_A \circ \mathrm{WF}_{\mathcal{G}^\infty} u \subseteq T^*(\Omega') \setminus 0$,

(ii) $\mathrm{WF}_{\mathcal{G}^\infty} Au \subseteq \{(x, \nabla_x\phi(x, \theta))\colon (x, \nabla_\theta\phi(x, \theta), \theta) \in \operatorname{cone\,supp} a,\ (\nabla_\theta\phi(x, \theta), \theta) \in \mathrm{WF}_{\mathcal{G}^\infty} u\}$.

Proof. An application of Proposition 5.4 to the phase function $[(\phi_\varepsilon(x, \xi) - y\xi)_\varepsilon]$ yields that

$$WF_{\mathcal{G}^\infty} K_A \subseteq \{(x, y, \nabla_x\phi(x, \theta), -\theta)\colon \theta \neq 0,\ (x, y, \theta) \in \operatorname{cone\,supp} a,\ \nabla_\theta\phi(x, \theta) = y\}, \tag{5.12}$$

i.e.

$$WF_{\mathcal{G}^\infty} K_A \subseteq \{(x, \nabla_\theta\phi(x, \theta), \nabla_x\phi(x, \theta), -\theta)\colon \theta \neq 0,\ (x, y, \theta) \in \operatorname{cone\,supp} a\}.$$

From the hypothesis on $\nabla_x\phi$ we have that if $(x, \xi) \in \mathrm{WF}'_{\mathcal{G}^\infty} K_A \circ \mathrm{WF}_{\mathcal{G}^\infty} u$, then $\xi = \nabla_x\phi(x, \theta)$ for some $\theta \neq 0$. It follows that $\xi \neq 0$. From (5.12) and Theorem 5.2 we easily obtain inclusion (ii). □

We now apply Theorem 5.5 to the case of a generalized Fourier integral operator solving a hyperbolic Cauchy problem as in § 3. We obtain the following microlocal result.

Proposition 5.6. *Let*

$$u(t, x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}[(\phi_\varepsilon(t, x, \eta))_\varepsilon]} b(t, x, \eta)\hat{u}_0(\eta)\,\text{đ}\eta,$$

where b is a regular generalized symbol of order 0 and $[(\phi_\varepsilon(t, x, \eta) - y\eta)_\varepsilon]$ is a slow-scale generalized phase function in the variable (y, η). Assume that there exists $\phi \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \setminus 0)$ with $\nabla_{(t,x)}\phi(t, x, \eta) \neq 0$ for $\eta \neq 0$ such that $\phi_\varepsilon \to \phi$ in $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \setminus 0)$. Then,

$$\mathrm{WF}_{\mathcal{G}^\infty} u \subseteq \{(t, x, \nabla_t\phi(t, x, \eta), \nabla_x\phi(t, x, \eta))\colon (t, x, \eta) \in \operatorname{cone\,supp} b,\ (\nabla_\eta\phi(t, x, \eta), \eta) \in \mathrm{WF}_{\mathcal{G}^\infty} u_0\}.$$

Example 5.7. We demonstrate how Proposition 5.6 can be used in a simple example. It concerns a transport equation with discontinuous coefficients and distributional data taken from [**46**, § 5]. More precisely, let

$$D_t u(t,x) = -H(t-1) D_x u(t,x), \quad u(0,x) = \delta(x),$$

with $t \in [0,+\infty)$ and $x \in \mathbb{R}$. Here, H denotes the Heaviside function and δ denotes the Dirac measure. Take $\rho \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\rho \subseteq [-1,1]$, non-negative, symmetric and with integral equal to 1. Let $(\omega^{-1}(\varepsilon))_\varepsilon$ be a slow-scale net with $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$. The solution $u \in \mathcal{G}([0,+\infty)\times\mathbb{R})$ of the Cauchy problem can be written, at the level of representatives, as

$$u_\varepsilon(t,x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(x-\varLambda_\varepsilon(t))\eta}\hat{\rho}_\varepsilon(\eta)\,\text{đ}\eta,$$

where

$$\varLambda_\varepsilon(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1-\omega(\varepsilon), \\ \displaystyle\int_{1-\omega(\varepsilon)}^{t} \lambda_\varepsilon(z)\,\mathrm{d}z, & 1-\omega(\varepsilon) \leqslant t \leqslant 1+\omega(\varepsilon), \\ t-1, & t \geqslant \omega(\varepsilon)+1, \end{cases}$$

and

$$\lambda_\varepsilon(t) = \int_{-\infty}^{(t-1)/\omega(\varepsilon)} \rho(z)\,\mathrm{d}z.$$

In particular, $\varLambda_\varepsilon(t)$ converges to

$$\varLambda(t) := \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ t-1, & t \geqslant 1, \end{cases}$$

for $\varepsilon \to 0$. When $t \neq 1$ we are working under the hypotheses of Proposition 5.6. Indeed, $\nabla_{(t,x)}(x-\varLambda(t))\eta = (-\varLambda'(t)\eta,\eta) \neq 0$ for $\eta \neq 0$. It follows that

$$\mathrm{WF}_{\mathcal{G}^\infty}(u|_{0\leqslant t<1}) \subseteq \{(t,0,0,\eta) : \eta \neq 0,\ 0 \leqslant t < 1\}$$

and

$$\mathrm{WF}_{\mathcal{G}^\infty}(u|_{t>1}) \subseteq \{(t,t-1,-\eta,\eta) : \eta \neq 0,\ t > 1\}.$$

The fact that $\varLambda$ is not differentiable at $t = 1$ does not allow us to apply Proposition 5.6 in a neighbourhood of $t = 1$. The wavefront set at $t = 1$ has been computed by direct calculations in [**46**].

Another class of generalized Fourier integral operators satisfying (5.2) can be found when solving a hyperbolic Cauchy problem where the principal part depends only on t. More precisely, we will deal with phase functions of the type

$$[(x\eta + \varphi_\varepsilon(t,\eta) - y\eta)_\varepsilon],$$

where φ_ε is homogeneous of order 1 in η. Note that unlike Proposition 5.6 we do not require any convergence as $\varepsilon \to 0$, but simply that φ_ε does not depend on x. We consider the Cauchy problem

$$D_t u = \sum_{j=1}^{n} a_{1,j}(t) D_j u + a_0(t,x)u, \quad u(0,\cdot) = u_0 \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n).$$

A typical example of $\varphi = [(\varphi_\varepsilon)_\varepsilon]$ is given by

$$\varphi(t,\eta) = \sum_{j=1}^{n} \biggl(\int_0^t a_{1,j}(\sigma)\,\mathrm{d}\sigma \biggr) \eta_j,$$

where

$$\gamma_j(x,t,s) = x_j + \int_s^t a_{1,j}(\sigma)\,\mathrm{d}\sigma$$

are the components of the characteristic curves corresponding to the operator

$$D_t - \sum_{j=1}^{n} a_{1,j}(t) D_j.$$

Proposition 5.8. *Let*

$$u(t,x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}[(x\eta+\varphi_\varepsilon(t,\eta))_\varepsilon]} b(t,x,\eta)\hat{u}_0(\eta)\,\mathrm{đ}\eta := Bu_0,$$

where b is a regular generalized symbol of order 0 and $[(x\eta+\varphi_\varepsilon(t,\eta)-y\eta)_\varepsilon]$ is a slow-scale generalized phase function in the variable (y,η). If the net

$$\biggl(\sup_{|t|\leqslant T,\, \eta\neq 0} \partial_t \varphi_\varepsilon\Bigl(t, \frac{\eta}{|\eta|}\Bigr) \biggr)_\varepsilon$$

is bounded for all $T > 0$, then

$$\mathrm{WF}_{\mathcal{G}^\infty} u \subseteq (\mathrm{WF}'_{\mathcal{G}^\infty} K_B \circ \mathrm{WF}_{\mathcal{G}^\infty} u_0) \cup (\mathrm{WF}'_{\mathcal{G}^\infty} K_B \circ (\operatorname{sing\,supp}_{\mathcal{G}^\infty} u_0 \times \{0\})). \tag{5.13}$$

Proof. The microlocal result (5.13) is directly obtained from Theorem 5.2 by checking that the inclusion

$$\mathrm{WF}'_{\mathcal{G}^\infty} K_B \circ \mathrm{WF}_{\mathcal{G}^\infty} u_0 \subseteq T^*(\mathbb{R}\times\mathbb{R}^n) \setminus 0$$

holds. We prove something stronger. We prove that if $(t_0,x_0,y_0,\tau_0,\xi_0,\eta_0) \in \mathrm{WF}_{\mathcal{G}^\infty} K_B$, then $\xi_0 = -\eta_0$. Thus, $(t_0,x_0,y_0,0,0,\eta_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} K_B$ if $\eta_0 \neq 0$.

By Theorem 5.1, for the point $(t_0,x_0,y_0,\tau_0,\xi_0,\eta_0) \in \mathrm{WF}_{\mathcal{G}^\infty} K_B$ we can find sequences of neighbourhoods $U_n(t_0,x_0,y_0)$, $\Gamma_n(\tau_0,\xi_0,\eta_0)$ and sequences of points $(t_n,x_n,y_n) \in U_n(t_0,x_0,y_0)$, $(\tau_n,\xi_n,\eta_n) \in \Gamma_n$ and θ_n such that

$$|(\tau_n,\xi_n,\eta_n) - (\partial_t\varphi_{\varepsilon_n}(t_n,\theta_n),\theta_n,-\theta_n)| \leqslant |\log\varepsilon_n|^{-1}(|(\tau_n,\xi_n,\eta_n)| + |\theta_n|),$$

where ε_n tends to 0. Again, we may assume that θ_n has norm 1. Hence,

$$|(\tau_n, \xi_n, \eta_n) - (\partial_t \varphi_{\varepsilon_n}(t_n, \theta_n), \theta_n, -\theta_n)| \leqslant |\log \varepsilon_n|^{-1}(|(\tau_n, \xi_n, \eta_n)| + 1).$$

From the hypothesis on $\partial_t \varphi_\varepsilon$ it follows that

$$\begin{aligned} |(\tau_n, \xi_n, \eta_n)| &\leqslant |\log \varepsilon_n|^{-1}(|(\tau_n, \xi_n, \eta_n)| + 1) + |(\partial_t \varphi_{\varepsilon_n}(t_n, \theta_n), \theta_n, -\theta_n)| \\ &\leqslant |\log \varepsilon_n|^{-1}|(\tau_n, \xi_n, \eta_n)| + c, \end{aligned}$$

which means that the sequence $|(\tau_n, \xi_n, \eta_n)|$ is bounded. Passing to subsequences, we have that (τ_n, ξ_n, η_n) converges to $\lambda(\tau_0, \xi_0, \eta_0)$ and θ_n converges to some $\theta_0 \neq 0$. Finally,

$$\begin{aligned} |(\xi_n, \eta_n) - (\theta_n, -\theta_n)| &\leqslant |(\tau_n, \xi_n, \eta_n) - (\partial_t \varphi_{\varepsilon_n}(t_n, \theta_n), \theta_n, -\theta_n)| \\ &\leqslant |\log \varepsilon_n|^{-1}(|(\tau_n, \xi_n, \eta_n)| + 1) \end{aligned}$$

yields that

$$(\lambda \xi_0, \lambda \eta_0) = (\theta_0, -\theta_0),$$

i.e. $\xi_0 = -\eta_0$. □

5.2. The Hamiltonian flow and the $\mathcal{G}^\infty$-wavefront set at fixed time

We now fix t and investigate the microlocal properties of the solution $u(t, \cdot)$ of (3.1). As in the previous subsection, we begin with some theoretical results on generalized Fourier integral operators of the type

$$F_\phi(b)u(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\phi(x,\eta)} b(x, \eta) \hat{u}(\eta) \, \mathrm{d}\eta$$

and we then consider the specific case of a strictly hyperbolic Cauchy problem. The following theorem is modelled on [**41**, Theorem 4.1.6] and makes use of the composition formula for generalized operators in [**21**].

Theorem 5.9. *Let Ω be an open subset of $\mathbb{R}^n$, let ϕ be a slow-scale regular generalized symbol on $\Omega_x \times \mathbb{R}^n_\eta$ homogeneous of order 1 in η, let b be a slow-scale regular generalized symbol of order 0 on $\Omega \times \mathbb{R}^n$ compactly supported in x and let $u \in \mathcal{G}_c(\Omega)$. Let $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ and $(y_0, \eta_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} u$. If there exist*

- *a representative $(\phi_\varepsilon)_\varepsilon$ of ϕ such that $\nabla_x \phi_\varepsilon \neq 0$ and*

$$\left(\inf_{x \in K,\, \eta \in \mathbb{R}^n \setminus 0} \left| \nabla_x \phi_\varepsilon \left(x, \frac{\eta}{|\eta|} \right) \right| \right)_\varepsilon$$

  is slow-scale strictly non-zero for all $K \Subset \Omega$,

- *conic neighbourhoods W and Z of (y_0, η_0) and (x_0, ξ_0), respectively, with $W \subset (\mathrm{WF}_{\mathcal{G}^\infty} u)^{\mathrm{c}}$,*

- *$\varepsilon_0 \in (0, 1]$ and a slow-scale net $(\lambda_\varepsilon)_\varepsilon$*

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such that

$$(x,\nabla_x\phi_\varepsilon(x,\eta))\in Z \quad\Rightarrow\quad |y-\nabla_\eta\phi_\varepsilon(x,\eta)|\geqslant\lambda_\varepsilon^{-1} \quad \textit{for all } \varepsilon\in(0,\varepsilon_0] \textit{ and all } (y,\eta)\notin W \tag{5.14}$$

holds, then

$$(x_0,\xi_0)\notin \mathrm{WF}_{\mathcal{G}^\infty}(F_\phi(b)u). \tag{5.15}$$

Proof. By the definition of the $\mathcal{G}^\infty$-wavefront set we can find a conic neighbourhood W_1 of (y_0,η_0) such that $W\subset W_1\subset(\mathrm{WF}_{\mathcal{G}^\infty}u)^{\mathrm{c}}$. From [**23**, Lemma 3.4], we can find a symbol $c\in S^0(\Omega\times\mathbb{R}^n)$ with $\mu\,\mathrm{supp}\,c\subseteq W_1$ and such that $c(x,\eta)=1$ on W when $|\eta|\geqslant 1$. In addition, we can construct c such that $\mathrm{supp}_x\,c\Subset\Omega$. From [**23**, Theorem 3.6], we have that $\mathrm{WF}_{\mathcal{G}^\infty}c(x,D)u\subseteq\mathrm{WF}_{\mathcal{G}^\infty}u\cap\mu\,\mathrm{supp}\,c=\emptyset$. It follows that $c(x,D)u\in\mathcal{G}^\infty_{\mathrm{c}}(\Omega)$. We now write that

$$F_\phi(b)u=F_\phi(b)c(y,D)u+F_\phi(b)e(y,D)u,$$

where $e(y,\eta)=1-c(y,\eta)$. Since $c(y,D)u\in\mathcal{G}^\infty_{\mathrm{c}}(\Omega)$, we obtain from the mapping properties of a generalized Fourier integral operator that $F_\phi(b)c(y,D)u$ is also $\mathcal{G}^\infty$-regular. This means that

$$\mathrm{WF}_{\mathcal{G}^\infty}F_\phi(b)u=\mathrm{WF}_{\mathcal{G}^\infty}F_\phi(b)e(y,D)u.$$

We therefore prove that the implication

$$(y_0,\eta_0)\notin\mathrm{WF}_{\mathcal{G}^\infty}(u)\quad\Rightarrow\quad(x_0,\xi_0)\notin\mathrm{WF}_{\mathcal{G}^\infty}(F_\phi(b)e(y,D)u) \tag{5.16}$$

is true. Making use of the characterization of the $\mathcal{G}^\infty$-wavefront set stated in [**23**, Theorem 3.11], it is sufficient to prove that, for some pseudodifferential operator $a(x,D)$ with $a\in S^0(\Omega\times\mathbb{R}^n)$ such that $\mathrm{supp}\,a\subseteq Z$ and $a(x,\xi)=1$ on some smaller neighbourhood Z_1 when $|\xi|\geqslant 1$, we have that $a(x,D)F_\phi(b)(e(y,D)u)\in\mathcal{G}^\infty(\Omega)$. We are working under the hypotheses of the composition theorem in [**21**] (see [**21**, Theorem 5.11]). Hence, modulo a $\mathcal{G}^\infty$-regularizing operator that does not affect the wavefront set, we have that the operator $a(x,D)F_\phi(b)(e(y,D)u)$ is the generalized Fourier integral operator

$$F_\phi(b_1)(e(y,D)u)(x)=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(x,\eta)}b_1(x,\eta)\int_\Omega\mathrm{e}^{-\mathrm{i}y\eta}e(y,D)u(y)\,\mathrm{d}y\,\mathrm{đ}\eta,$$

where $b_1\in\mathcal{G}^{\mathrm{sc}}_{S^0(\Omega\times\mathbb{R}^n)}$ has the asymptotic expansion

$$\sum_\alpha\frac{\partial_\xi^\alpha a(x,\nabla_x\phi(x,\eta))}{\alpha!}D_z^\alpha(\mathrm{e}^{\mathrm{i}\bar\phi(z,x,\eta)}b(z,\eta))\Big|_{z=x}.$$

At the level of representatives, this means that

$$b_{1,\varepsilon}(x,\eta)\sim\sum_\alpha\frac{\partial_\xi^\alpha a(x,\nabla_x\phi_\varepsilon(x,\eta))}{\alpha!}D_z^\alpha(\mathrm{e}^{\mathrm{i}\bar\phi_\varepsilon(z,x,\eta)}b_\varepsilon(z,\eta))\Big|_{z=x}$$

and, since we argue modulo $\mathcal{G}^\infty$, we can consider $b_{1,\varepsilon}$ as a sum of a convergent series obtained from the asymptotic expansion above, as in [**21**, Theorem 2.2]. Hence,

$b_{1,\varepsilon}(x,\eta)=0$ if $(x,\nabla_x\phi_\varepsilon(x,\eta))\notin Z$ and $\varepsilon\in(0,\varepsilon_0]$. In the following, we complete the proof by showing that $F_\phi(b_1)e(y,D)$ is an integral operator with kernel in $\mathcal{G}^\infty(\Omega\times\Omega)$. This follows from the fact that the transposed operator ${}^{\mathrm{t}}e(y,D)^{\mathrm{t}}F_\phi(b_1)$ has a kernel in $\mathcal{G}^\infty(\Omega\times\Omega)$. Working at the level of representatives, we have that the kernel of ${}^{\mathrm{t}}e(y,D)^{\mathrm{t}}F_\phi(b_1)$ is given by

$$v\to\int_{\Omega\times\Omega\times\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\phi_\varepsilon(x,\eta)-y\eta)}({}^{\mathrm{t}}e)(y,-\eta)b_{1,\varepsilon}(x,\eta)v(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{đ}\eta,\tag{5.17}$$

where ${}^{\mathrm{t}}e(y,\eta)\sim\sum_\alpha(\alpha!)^{-1}D_\eta^\alpha D_y^\alpha e(y,-\eta)$. Again arguing modulo $\mathcal{G}^\infty$, it is not a restriction to assume that ${}^{\mathrm{t}}e(y,-\eta)=0$ when $(y,\eta)\in W$. It follows that in the integral (5.17) we may assume that $(x,\nabla_x\phi_\varepsilon(x,\eta))\in Z$ and $(y,\eta)\notin W$ for $\varepsilon\in(0,\varepsilon_0]$. Thus, by integrating by parts we have that

$$\begin{aligned}\int_{\Omega\times\Omega\times\mathbb{R}^n}&\mathrm{e}^{\mathrm{i}(\phi_\varepsilon(x,\eta)-y\eta)}({}^{\mathrm{t}}e)(y,-\eta)b_{1,\varepsilon}(x,\eta)v(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{đ}\eta\\&=\int_{\Omega\times\Omega\times\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\phi_\varepsilon(x,\eta)-y\eta)}\frac{\Delta_\eta^N({}^{\mathrm{t}}e(y,-\eta)b_{1,\varepsilon}(x,\eta))}{|\nabla_x\phi_\varepsilon(x,\eta)-y|^{2N}}\,\mathrm{đ}\eta v(x,y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where we can make use of hypothesis (5.14). Since the nets involved in the integral

$$\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}(\phi_\varepsilon(x,\eta)-y\eta)}\frac{\Delta_\eta^N({}^{\mathrm{t}}e(y,-\eta)b_{1,\varepsilon}(x,\eta))}{|\nabla_x\phi_\varepsilon(x,\eta)-y|^{2N}}\,\mathrm{đ}\eta\tag{5.18}$$

are of slow-scale type and $|\nabla_x\phi_\varepsilon(x,\eta)-y|^{2N}\geqslant\lambda_\varepsilon^{-2N}$, where $(\lambda_\varepsilon)_\varepsilon$ is also a slow-scale net, we conclude that (5.18) generates a generalized function in $\mathcal{G}^\infty(\Omega\times\Omega)$. □

Theorem 5.9 applies to the solution

$$u(t,x)=F_\phi(b)(u_0)(t,x)=\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\phi(t,x,\eta)}b(t,x)\hat{u}_0(\eta)\,\mathrm{đ}\eta$$

(see Proposition 3.6) of (3.1) under a certain convergence assumption on the Hamiltonian flow.

Theorem 5.10. *Let $u\in\mathcal{G}(\mathbb{R}^{n+1})$ be the unique solution of the Cauchy problem*

$$D_tu=\sum_{j=1}^n a_{1,j}(t,x)D_ju+a_0(t,x)u,\quad u(0,\cdot)=u_0\in\mathcal{G}_{\mathrm{c}}(\mathbb{R}^n).$$

Under hypotheses (h_2) *on the coefficients* $a_{1,j}$, $j=1,\dots,n$, *and* (i_2) *on the coefficients* a_0, *let*

$$\phi(t,x,\eta)=\sum_{h=1}^n\gamma_h(x,t,0)\eta_h$$

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be the solution of (3.2) and let $b(t,x) \in \mathcal{G}^\infty(\mathbb{R}^{n+1})$ be as in (3.6). Let ϕ have a representative $(\phi_\varepsilon)_\varepsilon$ satisfying the following conditions for any $t \in \mathbb{R}$:

(i) *$\nabla_x \phi_\varepsilon(t,x,\eta) \neq 0$ for all ε, x and $\eta \neq 0$ and the net*

$$\inf_{x \in K \Subset \mathbb{R}^n,\, \eta \neq 0} \left| \nabla_x \phi_\varepsilon\left(t, x, \frac{\eta}{|\eta|}\right) \right|$$

is slow-scale strictly non-zero;

(ii) *the Jacobian $(\partial_j \gamma_{i,\varepsilon}(x,t,0))_{i,j}$ is invertible and the solution $(y,\eta) = \chi_{t,\varepsilon}(x,\xi)$ of the system*

$$\xi = \nabla_x \phi_\varepsilon(t,x,\eta),$$
$$y = \nabla_\eta \phi_\varepsilon(t,x,\eta)$$

converges to a limit χ_t in $\mathcal{C}(T^*(\mathbb{R}^n) \setminus 0, T^*(\mathbb{R}^n) \setminus 0)$ as $\varepsilon \to 0$;

(iii) *χ_t defines a bijection on $T^*(\mathbb{R}^n) \setminus 0$.*

Then,

$$\mathrm{WF}_{\mathcal{G}^\infty} u(t,\cdot) \subseteq \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0).$$

Proof. We denote the linear map corresponding to the Jacobian $(\partial_j \gamma_{i,\varepsilon}(x,t,0))_{i,j}$ by $\Gamma_\varepsilon(t,x)$. The invertibility of $\Gamma_\varepsilon(t,x)$ allows us to solve the Hamilton–Jacobi system above with respect to the variable (y,η). More precisely, we have that

$$\chi_{t,\varepsilon}(x,\xi) = (\gamma_{1,\varepsilon}(x,t,0), \dots, \gamma_{n,\varepsilon}(x,t,0), (\Gamma_\varepsilon^*)^{-1}(t,x)\xi)$$

and

$$\chi_{t,\varepsilon}^{-1}(y,\eta) = (\gamma_{1,\varepsilon}(y,0,t), \dots, \gamma_{n,\varepsilon}(y,0,t); \Gamma_\varepsilon^*(t, \gamma_\varepsilon(y,0,t))\eta),$$

where $\gamma_\varepsilon(y,0,t) = (\gamma_{1,\varepsilon}(y,0,t), \dots, \gamma_{n,\varepsilon}(y,0,t))$.

Let $(x_0,\xi_0) \in T^*(\mathbb{R}^n) \setminus 0$ and let $(y_0,\eta_0) = \chi_t(x_0,\xi_0)$. The limit in (ii) entails the following assertion: for all conic neighbourhoods W of (y_0,η_0) there exists a conic neighbourhood Z of (x_0,ξ_0) and some $\varepsilon_0 \in (0,1]$ such that $\chi_{t,\varepsilon}(x,\xi) \in W$ for all $(x,\xi) \in Z$ and $\varepsilon \in (0,\varepsilon_0]$. In other words, if $(x, \nabla_x \phi_\varepsilon(t,x,\eta)) \in Z$ and $\varepsilon \in (0,\varepsilon_0]$, then $(\nabla_\eta \phi_\varepsilon(t,x,\eta), \eta) \in W$. Thus, $(x, \nabla_x \phi_\varepsilon(t,x,\eta)) \in Z$ and $\varepsilon \in (0,\varepsilon_0]$ imply, for all W_1 with $\bar{W} \subseteq W_1$, the assertion that

$$\exists c > 0,\ \forall (y,\eta) \notin W_1, \quad |y - \nabla_\eta \phi_\varepsilon(t,x,\eta)| \geqslant c.$$

Combining hypothesis (i) with this result, we are working under the assumptions of Theorem 5.9 for fixed t. Therefore, that $(y_0,\eta_0) = \chi_t(x_0,\xi_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} u_0$ implies that $(x_0,\xi_0) \notin \mathrm{WF}_{\mathcal{G}^\infty} u(t,\cdot)$ or, in other words, that

$$\mathrm{WF}_{\mathcal{G}^\infty}(u(t,\cdot)) \subseteq \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0). \tag{5.19}$$

□

Remark 5.11. When the coefficients $a_{1,j}$ and a_0 are $\mathcal{C}^\infty$-functions, Theorem 5.10 recovers the propagation of singularities result obtained in [**23**, Proposition 4.3]. In this case, arguing by time reversal, one proves the inclusion $\chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0) \subseteq \mathrm{WF}_{\mathcal{G}^\infty} u(t,\cdot)$ and, therefore, $\mathrm{WF}_{\mathcal{G}^\infty} u(t,\cdot) = \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0)$.

5.3. Examples

This subsection contains some examples of first-order hyperbolic Cauchy problems whose principal part depends only on t. Applying the results of the previous sections, we can express the Colombeau solution by means of a generalized FIO formula and perform a microlocal investigation.

Proposition 5.12. *Let $u \in \mathcal{G}(\mathbb{R}^{n+1})$ be the unique solution of the Cauchy problem*

$$D_t u = \sum_{j=1}^n a_{1,j}(t) D_j u + a_0(t,x)u, \quad u(0,\cdot) = u_0 \in \mathcal{G}_{\mathrm{c}}(\mathbb{R}^n).$$

Under hypotheses (h_2) *and* (i_2), *let*

$$\phi(t,x,\eta) = \sum_{h=1}^n \gamma_h(x,t,0)\eta_h$$

be the solution of (3.2) and let $b(t,x) \in \mathcal{G}^\infty(\mathbb{R}^{n+1})$ be as in (3.6). If there exists a choice of representatives $(a_{1,j,\varepsilon})_\varepsilon$ such that

$$\Lambda_{j,\varepsilon}(t) = -\int_0^t a_{1,j,\varepsilon}(z)\,\mathrm{d}z, \quad j = 1,\dots,n,$$

depends continuously on ε and

$$\lim_{\varepsilon\to 0} \Lambda_{j,\varepsilon}(t) = \Lambda_j(t) \quad \text{for all } t \in \mathbb{R},\ j = 1,\dots,n,$$

then

$$\mathrm{WF}_{\mathcal{G}^\infty} u(t,\cdot) = \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0),$$

where

$$\chi_t^{-1}(y,\eta) = (y_1 + \Lambda_1(t),\dots,y_n + \Lambda_n(t),\eta).$$

Proof. The components $\gamma_{j,\varepsilon}(x,t,s)$ of the characteristic curves are the solutions of the Cauchy problems

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma_{j,\varepsilon}(x,t,s) = -a_{1,j,\varepsilon}(s), \quad \gamma_{j,\varepsilon}(x,t,t) = x_j.$$

Hence,

$$\gamma_{j,\varepsilon}(x,t,s) = \Lambda_{j,\varepsilon}(s) + x_j - \Lambda_{j,\varepsilon}(t)$$

and

$$\phi_\varepsilon(t,x,\eta)=\sum_{j=1}^{n}(x_j-\Lambda_{j,\varepsilon}(t))\eta_j.$$

It follows that hypotheses (i) and (ii) of Theorem 5.10 are satisfied, with $\chi_{t,\varepsilon}(x,\xi)$ given by

$$\xi=\eta,\qquad y=\gamma_\varepsilon(x,t,0)=x-\Lambda_\varepsilon(t).$$

Now,

$$\chi_t(x,\xi)=\lim_{\varepsilon\to 0}\chi_{t,\varepsilon}(x,\xi)=(x-\Lambda(t),\xi)$$

is a bijection on $T^*(\mathbb{R}^n)\setminus 0$ with $\chi_t^{-1}(y,\eta)=(y+\Lambda(t),\eta)$. From Theorem 5.10 we have that

$$\mathrm{WF}_{\mathcal{G}^\infty}(u(t,\cdot))\subseteq\chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty}u_0)=\{(y+\Lambda(t),\eta)\colon (y,\eta)\in\mathrm{WF}_{\mathcal{G}^\infty}u_0\}.$$

We will now argue by time reversal. We fix $t_0\in\mathbb{R}$ and we set $v(t,x)=u(t_0-t,x)$. From the original Cauchy problem we obtain that

$$D_t v=-\sum_{j=1}^{n}a_{1,j}(t_0-t)D_j v(t,x)-a_0(t_0-t,x)v,$$

with initial condition $v(0,x)=u(t_0,x)$. The corresponding characteristic curves are

$$\gamma_{j,\varepsilon}(x,t,s)=x_j+\int_0^s a_{j,\varepsilon}(t_0-z)\,\mathrm{d}z-\int_0^t a_{,j,\varepsilon}\,(t_0-z)\,\mathrm{d}z$$

and therefore

$$\gamma_{j,\varepsilon}(x,t,0)=x_j-\Lambda_{j,\varepsilon}(t_0-t)+\Lambda_{j,\varepsilon}(t_0).$$

In particular, denoting the corresponding Hamiltonian flow by $\tilde{\chi}^{-1}_{t_0,\varepsilon}$, we have that

$$\tilde{\chi}_{t_0,\varepsilon}(x,\xi)=(x+\Lambda_\varepsilon(t_0),\xi),\qquad \tilde{\chi}^{-1}_{t_0,\varepsilon}(y,\eta)=(y-\Lambda_\varepsilon(t_0),\eta).$$

Moreover,

$$\tilde{\chi}_{t_0}(x,\xi):=\lim_{\varepsilon\to 0}\tilde{\chi}_{t_0,\varepsilon}(x,\xi)=\lim_{\varepsilon\to 0}(x+\Lambda_\varepsilon(t_0),\xi)=(x+\Lambda(t_0),\xi)$$

and

$$\tilde{\chi}^{-1}_{t_0}(y,\eta):=\lim_{\varepsilon\to 0}\tilde{\chi}^{-1}_{t_0,\varepsilon}(y,\eta)=(y-\Lambda(t_0),\eta).$$

Hence,

$$\mathrm{WF}_{\mathcal{G}^\infty}(v(t_0,\cdot))\subseteq\tilde{\chi}^{-1}_{t_0}(\mathrm{WF}_{\mathcal{G}^\infty}(v(0,\cdot)))$$

or, in other words,

$$\mathrm{WF}_{\mathcal{G}^\infty}(u_0)\subseteq\tilde{\chi}^{-1}_{t_0}(\mathrm{WF}_{\mathcal{G}^\infty}(u(t_0,\cdot))),$$

which implies that

$$\tilde{\chi}_{t_0}(\mathrm{WF}_{\mathcal{G}^\infty}(u_0))\subseteq\mathrm{WF}_{\mathcal{G}^\infty}(u(t_0,\cdot)).$$

Since $\tilde{\chi}_{t_0} = \chi_{t_0}^{-1}$, we deduce that

$$\chi_{t_0}^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0) \subseteq \mathrm{WF}_{\mathcal{G}^\infty}(u(t_0, \cdot)).$$

In conclusion,

$$\mathrm{WF}_{\mathcal{G}^\infty} u(t, \cdot) = \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} u_0).$$

□

Example 5.13 (a transport equation with discontinuous coefficient and distributional data). We continue studying the Cauchy problem from Example 5.7. By applying the previous proposition, we can undertake a microlocal investigation of the solution $u \in \mathcal{G}([0, +\infty) \times \mathbb{R})$. Using the nets and notation of Example 5.7, we easily see that the assumptions of Proposition 5.12 are fulfilled. In particular, the Hamiltonian flow is given by

$$\chi_{t,\varepsilon}(x, \xi) = (x - \Lambda_\varepsilon(t), \xi), \qquad \chi_{t,\varepsilon}^{-1}(y, \eta) = (y + \Lambda_\varepsilon(t), \eta),$$

with $\Lambda_\varepsilon(t)$ defined as in Example 5.7 and having limit

$$\Lambda(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ t - 1, & t \geqslant 1. \end{cases}$$

Hence,

$$\chi_t(x, \xi) := \lim_{\varepsilon \to 0} \chi_{t,\varepsilon}(x, \xi) = (x - \Lambda(t), \xi), \qquad \chi_t^{-1}(y, \eta) = \lim_{\varepsilon \to 0} \chi_{t,\varepsilon}^{-1}(y, \eta) = (y + \Lambda(t), \eta).$$

In conclusion, for each $t \geqslant 0$ we have that

$$\mathrm{WF}_{\mathcal{G}^\infty}(u(t, \cdot)) = \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty} \iota(\delta)) = \{(\Lambda(t), \eta) \colon \eta \neq 0\}.$$

Example 5.14 (an example in higher dimension). We can adapt the previous example for higher space dimensions. For instance, let

$$D_t u(t, x_1, x_2) = -a_1 H(t-1) D_{x_1} u(t, x_1, x_2) - a_2 H(t-1) D_{x_2} u(t, x_1, x_2),$$
$$u(0, x) = \psi(x_1, x_2) \delta_C(x_1, x_2),$$

where $a_1, a_2 \in \mathbb{R}$, δ_C is the Dirac measure along a smooth, simple curve $C \in \mathbb{R}^2$ and $\psi \in \mathcal{D}(\mathbb{R}^2)$. In this case, the components of the characteristic curves are given by

$$\gamma_{1,\varepsilon}(x_1, t, s) = a_1 \Lambda_\varepsilon(s) + x_1 - a_1 \Lambda_\varepsilon(t), \qquad \gamma_{2,\varepsilon}(x_2, t, s) = a_2 \Lambda_\varepsilon(s) + x_2 - a_2 \Lambda_\varepsilon(t),$$

with Λ_ε as above. This yields that

$$\chi_{t,\varepsilon}(x, \xi) = ((x_1 - a_1 \Lambda_\varepsilon(t), x_2 - a_2 \Lambda_\varepsilon(t)), (\xi_1, \xi_2)),$$
$$\chi_{t,\varepsilon}^{-1}(y, \eta) = ((y_1 + a_1 \Lambda_\varepsilon(t), y_2 + a_2 \Lambda_\varepsilon(t)), (\eta_1, \eta_2))$$

and gives the limit

$$\chi_t^{-1}(y, \eta) = ((y_1 + a_1 \Lambda(t), y_2 + a_2 \Lambda(t)), (\eta_1, \eta_2)).$$

In conclusion,

$$
\begin{aligned}
&\mathrm{WF}_{\mathcal{G}^\infty}(u(t,\cdot)) \\
&\quad = \chi_t^{-1}(\mathrm{WF}_{\mathcal{G}^\infty}(\psi\delta_C)) \\
&\quad = \{(x_1 + a_1\Lambda(t), x_2 + a_2\Lambda(t), \xi_1, \xi_2)\colon (x_1,x_2) \in \operatorname{supp}\psi,\ (\xi_1,\xi_2) \in N(x_1,x_2)\},
\end{aligned}
$$

where $N(x_1, x_2)$ denotes the set of non-zero conormal directions of C.

Example 5.15 (an example with the Dirac measure as a coefficient). We consider the Cauchy problem

$$D_t u(t,x) = -\delta(t-1) D_x u(t,x), \quad u(0,x) = u_0(x),$$

with $u_0 \in \mathcal{G}_\mathrm{c}(\mathbb{R})$. In the regularization of $\delta(t-1)$ via convolution with a mollifier we use $\rho \in \mathcal{D}(\mathbb{R})$ as in Example 5.7 and a slow-scale net $(\omega^{-1}(\varepsilon))_\varepsilon$, with $0 < \omega(\varepsilon) < 1$ and $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$. The solution is represented by

$$u_\varepsilon(t,x) = u_{0\varepsilon}(x - \lambda_\varepsilon(t)),$$

with $\lambda_\varepsilon(t)$ as in Example 5.7. Since $\operatorname{supp}\rho \subseteq [-1,1]$ and $\int_{-1}^0 \rho(z)\,\mathrm{d}z = \frac{1}{2}$, we have that $\lambda_\varepsilon(1) = \frac{1}{2}$ for all $\varepsilon \in (0,1]$, and thus

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon(t) =: \lambda(t) = \begin{cases} 1, & t > 1, \\ \frac{1}{2}, & t = 1, \\ 0, & t < 1. \end{cases}$$

By Proposition 5.12,

$$\mathrm{WF}_{\mathcal{G}^\infty}(u(t,\cdot)) = \{(x + \lambda(t), \xi)\colon (x,\xi) \in \mathrm{WF}_{\mathcal{G}^\infty} u_0\}$$

for all $t \in \mathbb{R}$. Note that the limits of the flows $\chi_{t,\varepsilon}(x,\xi)$ are homeomorphisms at each fixed t, while the limiting two-dimensional characteristic coordinate change $(t,x) \to (t, x - \lambda(t))$ is still bijective, but no longer continuous. Such a situation is admitted by Proposition 5.12.

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References

1. L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, *Invent. Math.* **158** (2004), 227–260.
2. L. Ambrosio, Transport equation and Cauchy problem for non-smooth vector fields, in *Calculus of variations and nonlinear partial differential equations* (ed. L. Ambrosio *et al.*), Lecture Notes in Mathematics, Volume 1927, pp. 1–41 (Springer, 2008).
3. H. A. Biagioni, Generalized solutions to nonlinear first-order systems, *Monatsh. Math.* **118** (1994), 7–20.

4. H. A. Biagioni and M. Oberguggenberger, Generalized solutions to the Korteweg–de Vries and the regularized long-wave equations, *SIAM J. Math. Analysis* **23** (1992), 923–940.
5. F. Bouchut and F. James, One-dimensional transport equations with discontinuous coefficients, *Nonlin. Analysis* **32** (1998), 891–933.
6. M. Cicognani, Esistenza, unicità e propagazione della regolarità della soluzione del problema di Cauchy per certi operatori strettamente iperbolici con coefficienti lipschitziani rispetto al tempo, *Ann. Univ. Ferrara Sci. Mat.* **33** (1987), 259–292.
7. F. Colombini and N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, *Duke Math. J.* **77** (1995), 657–698.
8. F. Colombini and N. Lerner, Uniqueness of continuous solutions for BV vector fields, *Duke Math. J.* **111** (2002), 357–384.
9. F. Colombini and G. Métivier, The Cauchy problem for wave equations with non-Lipschitz coefficients: application to continuation of solutions of some nonlinear wave equations, *Annales Scient. Éc. Norm. Sup.* **41** (2008), 177–220.
10. F. Colombini, E. De Giorgi and S. Spagnolo, Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps, *Annali Scuola Norm. Sup. Pisa IV* **6** (1979), 511–559.
11. E. D. Conway, Generalized solutions of linear differential equations with discontinuous coefficients and the uniqueness question for multidimensional quasilinear conservation laws, *J. Math. Analysis Applic.* **18** (1967), 238–251.
12. L. de Simon and G. Torelli, Linear second-order differential equations with discontinuous coefficients in Hilbert spaces, *Annali Scuola Norm. Sup. Pisa IV* **1** (1974), 131–154.
13. R. J. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98** (1989), 511–547.
14. J. J. Duistermaat, *Fourier integral operators*, Progress in Mathematics, Volume 130 (Birkhäuser, 1996).
15. J. J. Duistermaat and L. Hörmander, Fourier integral operators, II, *Acta Math.* **128** (1972), 183–269.
16. E. Erlacher and M. Grosser, Ordinary differential equations in algebras of generalized functions, in *Pseudo-differential operators, generalized functions and asymptotics*, Operator Theory: Advances and Applications, Volume 231, pp. 253–270 (Birkhäuser, 2012).
17. R. Fernandez, On the Hamilton–Jacobi equation in the framework of generalized functions, *J. Math. Analysis Applic.* **382** (2011), 487–502.
18. C. Garetto, Topological structures in Colombeau algebras: topological $\tilde{\mathbb{C}}$-modules and duality theory, *Acta Appl. Math.* **88** (2005), 81–123.
19. C. Garetto, Topological structures in Colombeau algebras: investigation of the duals of $\mathcal{G}_c(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$, *Monatsh. Math.* **146** (2005), 203–226.
20. C. Garetto, Microlocal analysis in the dual of a Colombeau algebra: generalized wave front sets and noncharacteristic regularity, *New York J. Math.* **12** (2006), 275–318.
21. C. Garetto, Generalized Fourier integral operators on spaces of Colombeau type, in *New developments in pseudo-differential operators* (ed. L. Rodino and M. W. Wong), Operator Theory: Advances and Applications, Volume 189, pp. 137–184 (Birkhäuser, 2008).
22. C. Garetto, Fundamental solutions in the Colombeau framework: applications to solvability and regularity theory, *Acta Appl. Math.* **102** (2008), 281–318.
23. C. Garetto and G. Hörmann, Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities, *Proc. Edinb. Math. Soc.* **48** (2005), 603–629.
24. C. Garetto and M. Oberguggenberger, Symmetrisers and generalised solutions for strictly hyperbolic systems with singular coefficients, eprint (arXiv:1104.2281 [math.AP], 2011).

25. C. Garetto, T. Gramchev and M. Oberguggenberger, Pseudodifferential operators with generalized symbols and regularity theory, *Electron. J. Diff. Eqns* **2005** (2005), 1–43.

26. C. Garetto, G. Hörmann and M. Oberguggenberger, Generalized oscillatory integrals and Fourier integral operators, *Proc. Edinb. Math. Soc.* **52** (2009), 351–386.

27. I. M. Gel′fand, Some questions of analysis and differential equations, *Usp. Mat. Nauk* **14** (1959), 3–19.

28. M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer, *Geometric theory of generalized functions with applications to general relativity*, Mathematics and Its Applications, Volume 537 (Kluwer Academic, Dordrecht, 2001).

29. S. Haller and G. Hörmann, Comparison of some solution concepts for linear first-order hyperbolic differential equations with non-smooth coefficients, *Publ. Inst. Math.* **84** (2008), 123–157

30. L. Hörmander, Fourier integral operators, I, *Acta Math.* **127** (1971), 79–183.

31. L. Hörmander, *Lectures on nonlinear hyperbolic differential equations* (Springer, 1997).

32. G. Hörmann, First-order hyperbolic pseudodifferential equations with generalized symbols, *J. Math. Analysis Applic.* **293** (2004), 40–56.

33. G. Hörmann and M. de Hoop, Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients, *Acta Appl. Math.* **67** (2001), 173–224.

34. G. Hörmann and M. Oberguggenberger, Elliptic regularity and solvability for partial differential equations with Colombeau coefficients, *Electron. J. Diff. Eqns* **2004** (2004), 1–30.

35. G. G. Hörmann and C. Spreitzer, Symmetric hyperbolic systems in algebras of generalized functions and distributional limits, *J. Math. Analysis Applic.* **388** (2012), 1166–1179.

36. G. Hörmann, M. Kunzinger and R. Steinbauer, Wave equations on non-smooth space-times, in *Asymptotic properties of solutions to hyperbolic equations* (ed. M. Ruzhansky and J. Wirth), Progress in Mathematics, Volume 301 (Birkhäuser, 2012).

37. G. Hörmann, M. Oberguggenberger and S. Pilipović, Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients, *Trans. Am. Math. Soc.* **358** (2006), 3363–3383.

38. A. E. Hurd and D. H. Sattinger, Questions of existence and uniqueness for hyperbolic equations with discontinuous coefficients, *Trans. Am. Math. Soc.* **132** (1968), 159–174.

39. N. N. Kuznecov, On hyperbolic systems of linear equations with discontinuous coefficients, *Computat. Math. Math. Phys.* **3** (1963), 394–412.

40. F. Lafon and M. Oberguggenberger, Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case, *J. Math. Analysis Applic.* **160** (1991), 93–106.

41. M. Mascarello and L. Rodino, *Partial differential equations with multiple characteristics*, Mathematical Topics, Volume 13 (Akademie, Berlin, 1997).

42. M. Nedeljkov, S. Pilipović and D. Scarpalézos, *The linear theory of Colombeau generalized functions* (Longman Scientific and Technical, Harlow, 1998).

43. M. Oberguggenberger, Hyperbolic systems with discontinuous coefficients: generalized solutions and a transmission problem in acoustics, *J. Math. Analysis Applic.* **142** (1989), 452–467.

44. M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Pitman Research Notes in Mathematics, Volume 259 (Longman, Harlow, 1992).

45. M. Oberguggenberger, Case study of a nonlinear, nonconservative, non-strictly hyperbolic system, *Nonlin. Analysis* **19** (1992), 53–79.

46. M. Oberguggenberger, Hyperbolic systems with discontinuous coefficients: generalized wavefront sets, in *New developments in pseudo-differential operators* (ed. L. Rodino and M. W. Wong), Operator Theory: Advances and Applications, Volume 189, pp. 117–136 (Birkhäuser, 2008).
47. F. Poupaud and M. Rascle, Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients, *Commun. PDEs* **22** (1997), 337–358.
48. M. Ruzhansky, On local and global regularity of Fourier integral operators, in *New developments in pseudo-differential operators* (ed. L. Rodino and M. W. Wong), Operator Theory: Advances and Applications, Volume 189, pp. 185–200 (Birkhäuser, 2009).
49. M. E. Taylor, *Pseudodifferential operators and nonlinear PDEs* (Birkhäuser, 1991).