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ON RADICALS OF SUBMODULES OF FINITELY GENERATED MODULES

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ABSTRACT. The concept of the M-radical of a submodule B of an R-module A is discussed (R is a commutative ring with identity and A is a unitary R-module). The M-radical of B is defined as the intersection of all prime submodules of A containing B. The main result of the paper is that if $\sqrt{(B:A)}$ denotes the ideal radical of $(B:A)$, then M-rad $B = \sqrt{(B:A)}\,A$, provided that A is a finitely generated multiplication module. Additionally, it is shown that if A is an arbitrary module, $\sqrt{(B:A)}\,A \subseteq \langle C\rangle \subseteq M$-rad B, where $C = \{ra \mid a \in A$ and $r^n a \in B$, for some $n \in \mathbb{Z}^+\}$.

Since the radical of an ideal plays an important role in the study of rings, one would naturally seek a counterpart in the module setting. Indeed, such a concept has been discussed [4] e.g., where the radical of a submodule B of an R-module A is defined as the radical of the annihilator ideal of A/B, that is, the radical of a submodule is still an ideal. However, some information seems to be lost here. For example, if one merely takes the $\mathbb{Z}$-module A to be $\mathbb{Z} \oplus \mathbb{Z}(\mathbb{Z} =$ integers), then for every non-zero cyclic submodule B of A, ann $A/B = 0$. Hence the radical (as defined in [4]) of every non-zero cyclic submodule of A is also zero.

In what follows all rings are commutative with identity and all modules are unitary. $I \lhd R$ means that I is an ideal of R.

We define the M-radical of a submodule B of an R-module A to be the intersection of all prime submodules of A containing B. A submodule T of A is a prime submodule provided that $T \neq A$ and for $r \in R$, $a \in A\backslash T$ such that $ra \in T$, it follows that $rA \subseteq T$. Equivalently, T is a prime submodule of A whenever $ID \subseteq T$, (with $I \lhd R$, and D a submodule of A) implies that $I \subseteq (T:A)$ or $D \subseteq T$ [3].

The problem now becomes that of characterizing (internally) the M-radical of B (denoted rad B). We solve the problem completely for submodules of finitely generated multiplication modules. A is a multiplication module provided for each submodule B of A, $B = IA$ for some $I \lhd R$. In fact, if $(B:A)$ denotes the annihilator ideal of A/B and the (ring) radical of an ideal I is denoted by $\sqrt{I}$, then the main result of the paper can be stated as follows:

Let B be a submodule of a finitely generated multiplication module A (over a ring R). Then rad $B = \sqrt{(B:A)}\,A$.

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We observe that this result fails for the example above, for if B is any non-zero cyclic submodule of $A = \mathbb{Z} \oplus \mathbb{Z}$, then $\sqrt{(B:A)}\, A = 0$. Clearly this is not rad B since $B \subseteq$ rad B. However, it is always the case that $\sqrt{(B:A)}\, A \subseteq \text{rad}\, B$, and we record this fact in the following lemma.

LEMMA 1. *Let B be a submodule of an R-module A. Then* $\sqrt{(B:A)}\, A \subseteq \text{rad}\, B$.

PROOF. If rad $B = A$ the result is immediate. Otherwise, if P is any prime submodule of A which contains B, we have $(B:A) \subseteq (P:A)$. To show that $(P:A)$ is a prime ideal, suppose that $rs \in (P:A)$, so that $rsA \subseteq P$. Either $sA \subseteq P$ or $sa \in A \backslash P$ for some $a \in A$. In the latter case since P is a prime submodule and $r(sa) \in P$, we must have $rA \subseteq P$. Thus $r \in (P:A)$ or $s \in (P:A)$ and $(P:A)$ is prime. Hence $\sqrt{(B:A)} \subseteq (P:A)$ and thus $\sqrt{(B:A)}\, A \subseteq (P:A)A \subseteq P$. Since P is an arbitrary prime submodule containing B, we have $\sqrt{(B:A)}\, A \subseteq \text{rad}\, B$.

Bass proved that if A is a finitely generated module over a commutative ring R, and if $I \lhd R$ such that $IA = A$, then $(1 - i)A = 0$ for some $i \in I$ [1, Lemma 4.6]. By a parallel argument one can actually prove the following result.

RESULT 2. *If A is a finitely generated R-module, P is a prime ideal of R containing* ann A, *and $I \lhd R$ such that $IA \subseteq PA$, then $I \subseteq P$.*

We remark that if A is a finitely generated R-module and P is a prime ideal of R containing ann A, it now follows that $(PA:A) = P$.

LEMMA 3. *If A is a finitely generated multiplication R-module and P is a prime ideal of R containing* ann A, *then PA is a prime submodule of A.*

PROOF. Note that $PA \neq A$ and suppose that $I \lhd R$ and B is a submodule of A such that $IB \subseteq PA$. If $B = KA$, $K \lhd R$, then $IB = I(KA) \subseteq PA$. Result 2 implies that $IK \subseteq P$, hence $I \subseteq P = (PA:A)$ or $K \subseteq P$, then $B = KA \subseteq PA$ and the proof is complete.

THEOREM 4. *Let A be a finitely generated multiplication R-module and let B be a submodule of A. Then rad $B = \sqrt{(B:A)}\, A$.*

PROOF. By Lemma 1, $\sqrt{(B:A)}\, A \subseteq \text{rad}\, B$. Since A is a multiplication module, rad $B = (\text{rad}\, B:A)A$. It suffices then to show that $(\text{rad}\, B:A) \subseteq \sqrt{(B:A)}$. Let P be any prime ideal such that $(B:A) \subset P$. Since P is a prime ideal containing ann $A = (0:A)$, then PA is a prime submodule of A containing $B = (B:A)A$. Hence $(\text{rad}\, B:A)A = \text{rad}\, B \subseteq PA$, so that $(\text{rad}\, B:A) \subseteq P$. Consequently, $(\text{rad}\, B:A) \subseteq \sqrt{(B:A)}$.

COROLLARY 5. *If Q is a primary submodule of the finitely generated multiplication R-module A, then rad Q is a prime submodule of A.*

(Here we have used the concept of primary submodule as defined in [2]).

PROOF. By theorems 8.2.9 and 8.3.2 of [2], $\sqrt{(Q:A)}$ is a prime ideal containing ann A. Therefore rad $Q = \sqrt{Q:A}\, A$ is a prime submodule of A by Lemma 3.

Finally, we remark that in case that A fails to satisfy the hypothesis of Theorem 4, we can produce a somewhat sharper bound for rad B, which in general is distinct from $\sqrt{(B:A)}A$. This bound is obtained by first noting that $C = \{ra | a \in A$ and $r^n a \in B$, for some $n \in \mathbb{Z}^+\} \subseteq B$, [3]. It is then not difficult to show that $\sqrt{(B:A)}A \subseteq \langle C \rangle$ (= the submodule generated by C).

Consequently, we must have in the arbitrary setting, $\sqrt{(B:A)}A \subseteq \langle C \rangle \subseteq \text{rad}\, B$. Of course, in case that A is a finitely generated multiplication R-module, these three submodules coincide (Theorem 4).

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