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ON APPROXIMATELY FINITE-DIMENSIONAL VON NEUMAN ALGEBRAS, II

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ABSTRACT. An intrinsic characterization is given of those von Neumann algebras which are injective objects in the category of C^*-algebras with completely positive maps. For countably generated von Neumann algebras several such characterizations have been given, so it is in fact enough to observe that an injective von Neumann algebra is generated by an upward directed collection of injective countably generated sub von Neumann algebras. The present work also shows that three of the intrinsic characterizations known in the countably generated case hold in general.

1. **Introduction.** In [5], a von Neumann algebra was defined to be approximately finite-dimensional if any finite set of elements can be approximated arbitrarily closely in the *-ultrastrong topology by elements of a finite-dimensional sub von Neumann algebra.

It was shown in [5] and [4] that an approximately finite-dimensional countably generated von Neumann algebra is generated by an increasing sequence of finite-dimensional sub von Neumann algebras, and therefore by Proposition 7.3 of [7] is injective. Conversely, from work of Connes ([1], [2]) it follows that a countably generated injective von Neumann algebra is generated by an increasing net of finite-dimensional sub von Neumann algebras, and therefore is approximately finite-dimensional. This note removes the condition that the von Neumann algebra be countably generated.

2. THEOREM. *A (*-ultrastrongly) approximately finite-dimensional von Neumann algebra is injective.*

Proof. The proof is very similar to the proof of Tomiyama that a von Neumann algebra generated by an upward directed collection of injective sub von Neumann algebras is injective (Proposition 7.3 of [7]).

Let A be a von Neumann algebra which is *-ultrastrongly approximately finite-dimensional. Consider the set of pairs (F, V) where F is a finite subset of A and V is a *-ultrastrong neighbourhood of 0 in A; this set is upward directed when ordered by the relation: $(F, V) \leq (F', V')$ if $F \subset F'$ and $V \supset V'$.

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Denote this directed set by I, and for each $i=(F, V)\in I$ choose a finite-dimensional sub von Neumann algebra A_i of A such that $F\subset A_i+V$.

By Theorem 7.2 of [7], to show that A is injective it is enough to show that A' is injective. For each $i\in I$, choose a projection of norm one E_i from $B(H)$ onto A_i'. Choose a limit point E of the net (E_i) in the compact set of linear contractions of $B(H)$ into itself (with the topology of simple ultraweak convergence). Then for any $i\in I$, E_i is equal to the identity on A_i' and therefore on $A'\subset A_i'$. It follows that E is the identity on A'. Moreover, for any $i=(F, V)\in I$, and any $j\geq i$ such that $EF\subset E_jF+V$, since $F\subset A_j+V$ and $E_jF\subset A_j'$,

$$[EF, F]\subset[E_jF, F]+[V, F]\subset[E_jF, V]+[V, F].$$

If F is fixed and the *-ultrastrong neighbourhood V of 0 is chosen small, then $(E_jF)V$, FV and $(V(E_jF))^*=(E_jF)^*V^*$, $(VF)^*=F^*V^*$ are ultrastrongly small (independently of j), so $[E_jF, V]$ and $[V, F]$ are ultraweakly small; this shows that $[EF, F]=0$. We have shown that $E\mid A'=1$ and $[EA, A]=0$; this says that E is a projection from $B(H)$ onto A'. By construction $\|E\|=1$, so A' is injective.

3. Remark. Although one consequence of the work in [4] and [5] is that a countably generated approximately finite-dimensional von Neumann algebra A is injective—a special case of 2—in fact stronger results were obtained in [4] and [5]. First, an increasing generating net (indeed, sequence) of finite-dimensional sub von Neumann algebras of A was constructed without the predual A_* being assumed separable, and, secondly, in the properly infinite case, it was shown that the finite-dimensional subalgebras can be chosen to be factors.

4. Theorem. *An injective von Neumann algebra is generated by an upward directed collection of injective countably generated sub von Neumann algebras.*

Proof. Let A be an injective von Neumann algebra. Since the set of countably decomposable projections of A is upward directed with supremum 1, clearly we may suppose that A is countably decomposable.

By 6.8 of [2] the crossed product of an injective von Neumann algebra by a one-parameter automorphism group is injective, so by Takesaki's decomposition theorem for purely infinite von Neumann algebras (see 4.5, 8.2, and 8.11 of [6]) we may suppose that A is semifinite, provided that we ensure that the injective countably generated sub von Neumann algebras of A which we construct are left invariant by a given one-parameter automorphism group θ of A satisfying $\tau\circ\theta_t=e^{-t}\tau$ $(t\in\mathbb{R})$ for some faithful semifinite normal trace τ of A, in the case that such a group θ is indeed given (that is, that A is originally purely infinite).

By 11.1 of [6] and the remarks which follow it, we may suppose that there exists a finite projection p in A such that $\theta_t p \leq p$ for $t \geq 0$, and such that the supremum of $\theta_t p$ for all $t \in \mathbb{R}$ is 1.

Since pAp is finite, each sub von Neumann algebra of pAp is injective— it is the range of a projection of norm one. Choose an upward directed collection S of countably generated sub von Neumann algebras of pAp which generates pAp. For each $B \in S$, denote by $A(B)$ the von Neumann algebra generated by all $\theta_t(B)$, $t \in \mathbb{R}$. Then all $\theta_t(p)$, $t \in \mathbb{R}$, belong to $A(B)$, and all $\theta_t(p)A(B)\theta_t(p)$ are injective, so by Proposition 7.3 of [7] $A(B)$ is injective. Clearly $B_1 \subset B_2$ implies $A(B_1) \subset A(B_2)$, so $\{A(B) \mid B \in S\}$ is an upward directed collection of injective countably generated sub von Neumann algebras of A. The sub von Neumann algebra of A generated by the $A(B)$, $B \in S$, contains all $\theta_t(p)A\theta_t(p)$, $t \in \mathbb{R}$, and so is equal to A.

If a group θ as above is not given (that is, if A is semifinite to begin with), then since the set of all finite projections of A is upward directed, we may suppose that A is finite, in which case as remarked above every sub von Neumann algebra of A is injective.

5. Corollary. *An injective von Neumann algebra is (*-ultrastrongly) approximately finite-dimensional.*

Proof. A countably generated injective von Neumann algebra is a direct sum of injective von Neumann algebras with separable predual. By 7.5, 7.7, 7.8 of [2] and 1 of [1], together with 6.5 of [2] and Theorem 2 of [8], an injective von Neumann algebra with separable predual is generated by an increasing sequence of finite-dimensional sub von Neumann algebras. The Corollary then follows from 4.

6. Remark. In [3], Connes showed that an amenable von Neumann algebra is injective. From 4 it follows that, conversely, an injective von Neumann algebra is amenable—even strongly amenable—as this is known for countably generated von Neumann algebras.

7. Problem. In [4] and [5] it was shown that every countably generated injective von Neumann algebra is a direct summand of an injective bidual. Can 4 be used to show this for an arbitrary injective von Neumann algebra?

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