A NEW VIEWPOINT IN DIFFERENTIAL GEOMETRY

S. BOCHNER

1. Introduction. The law of transformation for a general mixed tensor

$$\zeta^{i_1 \dots i_p}{}_{j_1 \dots j_q} \tag{1}$$

is somewhat complicated at first sight, but algebraically it can be stated succinctly in the following manner. If, without regard to the original position of various indices, we denote the components of the tensor in some fixed simple ordering by

$$u^1, \dots, u^N, \tag{2}$$

where

$$N = n^{p+q}, \tag{3}$$

then the law of transformation is the set of linear homogeneous relations

$$u'^A(y) = a^A_B u^B(x) \tag{4}$$

[summation over B from 1 to N] in which

$$\{a^A_B\} \equiv \{a^A_B(y; x)\} \tag{5}$$

is a square matrix of dimension N, and this matrix is the Kronecker product of p n-dimensional matrices

$$\left\{\frac{\partial y_i}{\partial x_j}\right\} \tag{6}$$

and q n-dimensional matrices

$$\left\{\frac{\partial x_j}{\partial y_i}\right\}. \tag{7}$$

Also, if (1) is not a "scalar" tensor but a tensor of weight W we must add to this as a further Kronecker factor the one-dimensional "matrix"

$$\left|\frac{\partial(x)}{\partial(y)}\right|^W, \tag{8}$$

or, if W is an integer, which it need not be, we may add the W one-dimensional factors

$$\frac{\partial(x)}{\partial(y)}, \tag{9}$$

if this be preferrred.

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If we set down relations (4) with unspecified matrices (5) and examine the properties needed for the erection of a calculus of tensors we find that we first of all require certain properties of "consistency" by which to validate relations (4) intrinsically, and to that purpose the self-explanatory properties

$$a_B^A(z;y)\, a_C^B(y;x) = a_C^A(z;x) \tag{10}$$

$$a_B^A(y;x) = \delta_B^A \tag{11}$$

are obviously sufficient; and the latter ones are verifiable from the mere chain property

$$\frac{\partial z_i}{\partial y_j}\frac{\partial y_j}{\partial x_k} = \delta_k^i \tag{12}$$

without using any other feature of (6), say. If, however, we proceed to introduce the interlocking concepts of "affine connection" and "covariant differentiation"—which, after all, are the heart of differential geometry— then the feature of tangentiality as manifest in the partial derivatives (6) is of the very essence, and the matrix (6) itself [or something resembling it] must of necessity be embodied in the calculus. We have found, however, in examining the nature of the ordinary affine connection L_{ij}^k that only one of its lower indices need be interacting with our special matrix (6), or (7) to be more precise, and we have correspondingly generalized it to an object

$$L_{Bi}^A \tag{13}$$

in which the other two indices A, B will be interacting with some other matrix (5), with only properties (10) and (11) presupposed known, which other matrix will be given in addition to, and quite independently from, the classical matrix (6) proper. This separation of algebraic from analytical pre-requisites, as it were, will not only clarify things but also imply a substantial generalization of the theory, and perhaps the most tangible advantage will be as follows.

If an ordinary tensor is symmetric, $\zeta_{ij} = \zeta_{ji}$, then we first of all may view it as an object (2) with $N = n^2$, but it should be noted that the "independent" components ζ_{ij} $(i \leqslant j)$, whose number is

$$N = \tfrac{1}{2}n(n+1),$$

are likewise an object of our kind, and it will have an affine connection entirely of its own to fit its own matrix structure. Similarly, a skew symmetric tensor $\zeta_{i_1 \ldots i_p}$ with $\binom{n}{p}$ independent components corresponding to $i_1 < i_2 < \ldots < i_p$ will be such an object. In particular, for $p = n$, we have *de facto* only one component $\zeta_{1 \ldots n}$, its matrix $a_1^1(y;x)$ being the Jacobian (9), so that from our approach it is entirely indistinguishable from an ordinary scalar density, and our affine connection (13), which in the present case has only components $A = B = 1$ is nothing else but the classical contraction L_{ji}^j, or something differing from it by a vector a_i as we shall see.

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2. Vectoroids. We will introduce three positive integers n, N, ν, and their sizes will not be tied in explicitly in any manner whatsoever, although the contexts in which they will be given may imply dependencies per force. Indices $i, j, k, \ldots$ whether upper or lower will range from 1 to n, indices A, B, C from 1 to N, and indices α, β, $\gamma, \ldots$ from 1 to ν.

We take an n-dimensional manifold M_n, compact or not (it need not even be separable), and on it a fixed covering by a family or arbitrary local coordinate systems—hereafter termed "allowable"—and we are defining a mathematical object, which we will call a *vectoroid*, in the following manner. In each allowable system it has N coordinate functions (2)—the number N will be called its dimension—and on the intersection of two allowable systems $U_1(x)$, $U_2(y)$ we have relations (4) for a matrix (5) for which (11) is satisfied on the intersection of any $U(x)$ with itself and (10) is satisfied on the intersection of any three systems $U_1(x)$, $U_2(y)$, $U_3(z)$.

An assemblage of matrices with properties (10), (11) will be called a (*matrix*) *structure*. In algebraic parlance we may say that a structure constitutes a matrix "representation" of the "groupoid" of coordinate transformations

$$y_i = f_i(x_i, \ldots, x_n) \qquad i = 1, \ldots, n \tag{14}$$

prevailing on intersections of pairs of allowable systems, and in topological parlance we may add that this generates a space of affinely interlocking N-dimensional linear fibers over M_n as base space and that a vectoroid is a continuous map of the base space into the full fiber bundle, the map of each base point being a point in the fiber over it.

As already stated in the introduction, any ordinary tensor of any weight may be interpreted as a vectoroid, and this includes, for $N = 1$, also scalars of any weight. For a scalar of weight 0, absolute scalar, that is, we have

$$a_1^1(y; x) = 1. \tag{15}$$

In the introduction we have also mentioned the possibility of reducing the dimension N whenever "symmetries" occur, and a general principle for so doing may be stated as follows.

Take a non-singular constant matrix λ_B^A, denoting its inverse by μ_B^A, and transform the structure (5) into

$$b_B^A(y; x) = \lambda_C^A \, a_D^C(y; x) \mu_B^D \tag{16}$$

and, correspondingly, the vectoroid u^A into

$$v^A = \lambda_C^A \, u^C. \tag{17}$$

If now it so happens that the new structure (16) fully decomposes into structures of dimension N_1, N_2, $N_1 + N_2 = N$, then in a suitably normalization of indices we may introduce the system of relations

$$\lambda_C^A \, u^C(x) = 0, \qquad A = N_1 + 1, \ldots, N \tag{18}$$

as "symmetry conditions" to be imposed on the original vectoroid (2); and if they happen to be fulfilled, then there exists a vectoroid of dimension N_1 only, having the components

$$v^1, \ldots, v^{N_1}, \tag{19}$$

and this new vectoroid may be said to be a fair substitute for the original one.

If a structure (5) has been given and is held fast, then the vectoroid (2) previously introduced is a "contravariant" one, and its "covariant" counterpart is an object t_A for which the law of transformation is

$$t'_A(y) = a_A^B(x; y) t_B(x), \tag{20}$$

or, what is the same

$$a_A^B(y; x)\, t'_B(y) = t_A(x). \tag{21}$$

If, however, we introduce the transposed inverse matrix

$$\bar{a}_B^A(y; x) = a_A^B(x; y) \tag{22}$$

then t_A can be easily identified with a vectoroid which is contravariant with regard to the new structure, and thus the contrast between contravariant and covariant is only a relative one.

Next, if we are given several of our matrix structures, equal or not, then any Kronecker product of them is again a structure, and a vectoroid pertaining to such a structure will sometimes be denoted by the symbol

$$t_{A_1 \ldots A_p}{}^{B_1 \ldots B_q}, \tag{23}$$

and then termed a *tensoroid*, whenever its structure is a product of q straight factors and p transposed inverse factors given. Also, if, for instance, in (23) the lower index A_1 and the upper index B_1 both pertain to the same structure, then we form the contraction

$$t_{A_2 \ldots A_p}{}^{B_2 \ldots B_q} = t_{CA_2 \ldots A_p}{}^{CB_2 \ldots B_q} \tag{24}$$

as usual. In particular, if t_A, u^A belong to the same structure then

$$t_A u^A \tag{25}$$

is an absolute scalar.

If in addition to the structure (5) we are given a second structure

$$b_\beta^\alpha \qquad \alpha, \beta = 1, \ldots, \nu, \tag{26}$$

if $\nu = N$, and if we are given a tensoroid $h_{A\alpha}$ for which

$$\det |h_{A\alpha}| \neq 0, \tag{27}$$

then there exists a tensoroid $h^{A\alpha}$ for which we have

$$h_{C\alpha}\, h^{C\beta} = \delta_\alpha^\beta, \quad h_{A\gamma}\, h^{B\gamma} = \delta_A^B; \tag{28}$$

and if furthermore the two structures are identical then the pair of tensoroids $h_{A\alpha}$, $h^{A\alpha}$, which we will now denote by

$$h_{AB}, \quad h^{AB}, \tag{29}$$

may be used for pulling indices pertaining to the structure (5) up and down as usual. We will always assume that the tensoroids (29) are symmetric, although for the shifting of indices alone this requirement would not be strictly needed.

Finally we note that there always exists a matrix structure defined by the Kronecker symbol, thus

$$b^{\alpha}_{\beta}\,(y; x) = \delta^{\alpha}_{\beta} \qquad \alpha, \beta, = 1, \dots, \nu \tag{30}$$

and that a contravariant vectoroid pertaining to it has the law of transformation

$$u'^{\alpha}(y) = u^{\alpha}(x), \tag{31}$$

and thus consists of ν individual absolute scalars, and similarly for covariant vectoroids. If now we form the Kronecker product of an arbitrary structure (5) with the special structure (30), then a tensoroid $u^{A\alpha}$ pertaining to the product is a set of ν vectoroids u^A pertaining to (5) each, and any set of ν such vectoroids may be so viewed. In other words, from our general approach there is only a relative distinction between indices which are vectorial and those which are enumerative, and this merger of the two types of indices has some technical advantages.

3. Covariant differentiation. In the right side of (5) we have introduced both symbols y and x simultaneously and in a given order and in this way the consistency rules (10) and (11) have become self-explanatory. However, analytically, the individual components of the tensor (5) will be viewed as functions either of the variables $x = (x_1, \dots, x_n)$ or of the variables $y = (y_1, \dots, y_n)$ by themselves, depending on whether we view the intersection of the neighbourhoods $U_1(x)$, $U_2(y)$ as being part of the one or of the other; and in forming partial derivatives with respect to the variables x_i or y_j the variables exhibited will be those underlying.

If we are given a structure (5), and if the functions (5) and (14) belong to differentiability class C^1, then we define an affine connection

$$L^{B}_{Ai} \qquad A, B = 1, \dots, N; i = 1, \dots, n \tag{32}$$

by the law of transformation

$$a^{B}_{A}(y; x)\, \frac{\partial y_j}{\partial x_i} L'^{C}_{Bj}(y) = -\frac{\partial a^{C}_{A}(y; x)}{\partial x_i} + a^{C}_{B}(y; x)\, L^{B}_{Ai}(x), \tag{33}$$

and the following theorem can be easily verified.

THEOREM 1. *If t_A is a covariant vectoroid then the components*

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$$t_{A,i} = \frac{\partial t_A}{\partial x_i} - t_B L^B_{Ai} \tag{34}$$

constitute a tensoroid for the Kronecker product of (5) *and* (6).

Also, if (34) *is to constitute a tensoroid for arbitrary vectoroids* t_A *locally, then* (33) *must be satisfied.*

Furthermore, if u^A *is contravariant vectoroid then*

$$u^A{}_{,i} = \frac{\partial u^A}{\partial x_i} + u^B L^A_{Bi}$$

is a mixed tensoroid, that is to say, if L^B_{Ai} *is an affine connection for a structure* (5) *then* $-\ L^A_{Bi}$ *is one for the structure* (22).

It should be noted, however, that given a structure, the affine connection (32), if one exists, is determined up to an arbitrary additive tensor T^B_{Ai} only, so that associating $-\ L^A_{Bi}$ with the structure (22) is a deliberate act of normalization, pursuant to the fixed normalization of (32) for (5) itself.

If there is given a second structure (26) and if it has an affine connection $\Lambda^\beta_{\alpha i}$, then the Kronecker product of (5) and (26) admits the affine connection

$$L^{(B\beta)}_{(A\alpha)i} = \delta^\beta_\alpha L^B_{Ai} + \delta^B_A \Lambda^\beta_{\alpha i}, \tag{35}$$

and the resulting formula

$$t_{A\alpha,i} = \frac{\partial t_{A\alpha}}{\partial x_i} - t_{B\alpha}\, L^B_{Ai} - t_{A\beta}\Lambda^\beta_{\alpha i} \tag{36}$$

is the generalization of a classical one; and similarly for mixed tensoroids.

For the special structure (30), relations (33) reduce to

$$\frac{\partial y_j}{\partial x_i}\, \Lambda'^\gamma_{\alpha j} = \Lambda^\gamma_{\alpha i}, \tag{37}$$

and that means that $\Lambda^\gamma_{\alpha i}$ is a vector with regard to the index i, with α and γ being enumerative indices. In particular (37) will be satisfied by

$$\Lambda^\gamma_{\alpha i} = 0, \tag{38}$$

in which case (36) will reduce to the expression

$$t_{A\alpha,i} = \frac{\partial t_{A\alpha}}{\partial x_i} - t_{B\alpha}\, L^B_{Ai}, \tag{39}$$

but it must be stated that (38) is a deliberate "canonical" normalization for the solution of (37) and not the only one possible.

We are turning to the general expression (36). If we have

$$t_{A\alpha,i} = 0 \tag{40}$$

that is

$$\frac{\partial t_{A\alpha}}{\partial x_i} - t_{A\beta}\Lambda^{\beta}_{\alpha i} = t_{B\alpha} L^B_{A i} \tag{41}$$

and if, for $\nu = N$, $t_{A\alpha}$ has a transposed inverse $t^{A\alpha}$, then (34) implies

$$L^C_{A i} = t^{C\alpha} \frac{\partial t_{A\alpha}}{\partial x_i} - t^{C\alpha} t_{A\beta}\Lambda^{\beta}_{\alpha i}, \tag{42}$$

and thus L^C_{Ai} is uniquely determined by $\Lambda^{\beta}_{\alpha i}$. In particular for (38) we obtain

$$L^C_{A i} = t^{C\alpha} \frac{\partial t_{A\alpha}}{\partial x_i}, \tag{43}$$

and this is a generalization from vectors to vectoroids of the known theorem that if there are given n independent parallel vector fields, the affine connection is uniquely determined hereby.

Finally, if we are also given a (non-symmetric) affine connection Γ^k_{ij} to fit the structure (6), then if everything given is twice differentiable, we can form the second covariant derivative

$$t_{A,i,j} = \frac{\partial t_{A,i}}{\partial x_j} - t_{B,i} L^B_{Aj} - t_{A,k}\Gamma^k_{ij}, \tag{44}$$

and we obtain

$$t_{A,i,j} - t_{A,j,i} = t_B L^B_{Aij}, \tag{45}$$

where

$$L^B_{Aij} = \frac{\partial L^B_{Aj}}{\partial x_i} - \frac{\partial L^B_{Ai}}{\partial x_j} + L^B_{Ci} L^C_{Aj} - L^B_{Cj} L^C_{Ai} + t_{A,k}(\Gamma^k_{ji} - \Gamma^k_{ij}) \tag{46}$$

is a tensoroid. We always have

$$L^B_{Aij} = - L^B_{Aji}$$

and for

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

we also have

$$L^B_{Aij,k} + L^B_{Ajk,i} + L^B_{Aki,j} = 0.$$

4. Remark on affine connections. Affine connections are a particular case of a general type of coefficients in invariant systems of linear partial differential operators, as we are going to describe briefly.

Consider for an ordinary scalar t the expression

$$c^{ij} \frac{\partial^2 t}{\partial x_i \partial x_j} + c^k \frac{\partial t}{\partial x_k} + ct \tag{47}$$

with $c^{ij} = c^{ji}$. If this is to be formally invariant under the transformation (14) then we must have

$$c^{ij}(x) = c'^{pq}(y)\,\frac{\partial x_i}{\partial y_p}\,\frac{\partial x_j}{\partial y_q}, \tag{48}$$

$$c^k(x) = c'^{pq}(y)\,\frac{\partial^2 x_k}{\partial y_p \partial y_q} + c'^r(y)\,\frac{\partial x_k}{\partial y_r}, \tag{49}$$

$$c(x) = c(y), \tag{50}$$

and it can be easily verified that the entire system of coefficients

$$\{c^{ij}, c^k, c\}$$

constitutes a vectoroid for an appropriate structure. However the sub-system $\{c^{ij}\}$ forms a vectoroid by itself, and if we insert some special values for this subset, and if we ignore the quantity c itself which is a scalar "by accident," then the remaining coefficients c^k transform themselves by relation (49) after the manner of an affine connection.

After this preliminary instance we will now set up a very comprehensive situation. If t_A is an unspecified vectoroid for a given structure (5) and if we demand that for some given integer l the system of partial differential operators

$$O_\alpha(t) = \sum_{0 \leqslant l_1 + \ldots + l_{\lambda_1} \leqslant l} c_\alpha{}^{A l_1 \ldots l_\lambda}\,\frac{\partial^{l_1 + \ldots l_\lambda} t_A}{\partial x_1{}^{l_1} \ldots \partial x_\lambda{}^{l_\lambda}}, \qquad \alpha = 1, \ldots, \nu, \tag{51}$$

shall form a vectoroid pertaining to some other structure (26), identically in all t_A, then the set of all coefficents occuring in all individual operators (51) form a vectoroid for a suitable structure. However the subset of those coefficients for which

$$l_1 + \ldots + l_\lambda = l \tag{52}$$

form a vectoroid by themselves, and if we insert some special values for this subset then the remaining "additional" coefficients transform in the manner of an affine connection.

In the particular case (34) we have $l = 1$ and the structure (26) is the Kronecker product of (5) and (6). Thus every index $\alpha = \alpha_0$ is a pair of indices $A = A_0$, $i = i_0$ and the highest coefficients corresponding to (52) have been specialized in the following "invariant" manner: if $A = A_0, i = i_0$ then $c_\alpha{}^{A l_1 \ldots l_\lambda} = 1$, otherwise $= 0$.

5. Compact manifolds. On a compact M_n we stipulate the following data, all of differentiability class C^2: a matrix structure (5), with an affine connection (32), a positive definite symmetric tensor (29) with

$$h^{AB}{}_{,i} = 0, \tag{53}$$

that is

$$\frac{\partial h^{AB}}{\partial x_i} + h^{CB} L^A_{Ci} + h^{AC} L^B_{Ci} = 0, \tag{54}$$

an ordinary (non-symmetric) affine connection

$$\Gamma^k_{ij}, \tag{55}$$

and an ordinary positive definite symmetric tensor g^{ij} which need *not* be related to (55).

We note that the Kronecker symbol δ^B_A is a mixed tensoroid for (5), and if for any mixed tensoroid t^B_A we adopt the definition

$$t^B_{A,i} = \frac{\partial t^B_A}{\partial x_i} - t^B_C L^C_{Ai} + t^C_A L^B_{Ci}, \tag{56}$$

as we will, then we obtain

$$\delta^B_{A,i} = 0, \tag{57}$$

and on recalling the relation

$$h_{AC}h^{BC} = \delta^B_A,$$

we now obtain from (53) and (57) the further inference

$$h_{AB,i} = 0. \tag{58}$$

Therefore, if we will employ the tensor (29) for shifting indices up and down, this operation of shifting will be commutative with covariant differentiation. Thus, if for a vectoroid t_A we form the scalar "square length"

$$\phi = h^{AB}t_At_B, \tag{59}$$

then we obtain

$$\frac{\partial \phi}{\partial x_i} \equiv \phi_{,i} = 2h^{AB}t_{A,i}t_B = t_{A,i}t^A = t_At^A{}_{,i}. \tag{60}$$

If we differentiate once more we obtain

$$\tfrac{1}{2}\phi_{,i,j} = h^{AB}t_{A,i}t_{B,j} + h^{AB}t_{A,i,j}t_B, \tag{61}$$

where of course

$$\phi_{,i,j} \equiv \frac{\partial^2\phi}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial\phi}{\partial x_k}, \tag{62}$$

and $t_{A,i,j}$ is given by (41). And finally, if we introduce the invariant operator

$$\Delta\phi = g^{ij}\,\phi_{i,j} \equiv g^{ij}\,\frac{\partial^2\phi}{\partial x_i\partial x_j} - g^{ij}\,\Gamma^k_{ij}\,\frac{\partial\phi}{\partial x_k}, \tag{63}$$

we obtain the equation

$$\tfrac{1}{2}\Delta\phi = g^{ij}h^{AB}t_{A,i}t_{B,j} + h^{AB}(g^{ij}t_{A,i,j})t_B = T_1 + T_2 \tag{64}$$

which we will now analyze.

As on previous occasions[1] we will use the lemma that, since M_n is compact,

[1]See our papers: *Vector fields and Ricci-curvature*, Bull. Amer. Math. Soc., vol. 52 (1946), 776-797, and *Curvature and Betti numbers*, Ann. of Math., vol. 49 (1948), 379-390 and II in vol. 50 (1949), 77-93.

we cannot have

(65) $$\Delta\phi \geqslant 0$$

on M_n unless we have

(66) $$\Delta\phi = 0.$$

Furthermore we have identically $T_1 \geqslant 0$, and $T_1 = 0$ if and only if $t_{A,i} = 0$. Thus, if for special reasons we have

(67) $$T_2 \geqslant 0,$$

then we must of necessity have

(68) $$T_2 = 0$$

and also

(69) $$t_{A,i} = 0.$$

In order to bring about (67) we introduce some fixed symmetric matrix S_{AB} and set up the system of partial differential equations

(70) $$g^{ij}t_{A,i,j} = S_A^B t_B (\equiv S_{AB}t^B),$$

and since this implies

(71) $$T_2 = S_{AB}t^A t^B,$$

we obtain the following conclusion.

THEOREM 2. *If on our compact M_n we are given a symmetric matrix S_{AB} which is positive definite then there is no vectoroid t_A other than* 0 *which satisfies the equations* (70).

If the matrix is only semi-definite then the only solutions of (70) *are those for which we have*

$$t_{A,i} = 0, \qquad S_A^B t_B = 0,$$

simultaneously.

If ζ_a is an ordinary vector field, and if it is harmonic (curl ζ = div ζ = 0) then it satisfies the equations

$$g^{ij}\zeta_{a,i,j} = - R_{ab}\zeta^b$$

where R_{ab} is the Ricci tensor based on g^{ij}, provided the affine connection is the ordinary one; and if the vector is a Killing vector ($\zeta_{i,j} + \zeta_{j,i} = 0$) then it satisfies the alternate equations

$$g^{ij}\zeta_{a,i,j} = R_{ab}\zeta^b.$$

Thus Theorem 2 includes the following statements made previously among others.[1] If $- R_{ab}$ is positive definite there exists no harmonic vector, and if it is negative definite there exists no Killing vector, and if it is semi-definite then in either case we must have

$$\zeta_{a,i} = 0, \qquad R_{ab}\zeta^b = 0,$$

simultaneously.

In the papers cited we also had results bearing on tensors of several indices. Some of them, in particular those referring to complex manifolds could also be subsumed under Theorem 2, but the analysis would have to be rather detailed and would not be profitable in the end. However, we would like to terminate with giving a sample of another type of Theorem[2] which is especially adapted to complex spaces and which actually gave us the impetus for devising the generalization from vector to vectoroid. The theorem is largely preanalytic in the sense that no affine connection will be involved or even assumed existing, and although the theorem originated in algebraic geometry[2] no algebraic structure will be involved either.

Let M_n be a compact space of n complex (that is $2n$ real) variables for which the functions (14) are holomorphic functions, that is local power series, from the complex variables x_i to the complex variables y_j, and let the matrix structure (5) be likewise holomorphic in either set of variables, and let t_A, u^A be holomorphic fields of vectoroids on all of M_n. The contraction (25) is an absolute scalar and also holomorphic, and being so on a compact manifold it must be a numerical constant

$$t_A u^A = c. \tag{72}$$

If now there are given $N + 1$ covariant vectoroids $t^\alpha_A (\alpha = 1, \ldots, N + 1)$ then a contravariant vectoroid u^A, if existing, has to satisfy a set of relations

$$t^\alpha_A u^A = c^\alpha, \qquad \alpha = 1, \ldots, N + 1 \tag{73}$$

and the following theorem ensues.[2]

THEOREM 3. *If on a compact complex M_n there exist $N + 1$ holomorphic vectoroids t_N, if the rank of the N by $N + 1$ matrix*

$$t^\alpha_N$$

is somewhere N and if for no set of constants c^α, not all zero, is the determinant

$$\det |t^\alpha_1, \ldots, t^\alpha_N, c^\alpha_N| \qquad \alpha = 1, \ldots, N + 1$$

identically 0, *then there exists on M_n no holomorphic contravariant vectoroid u^A whatsoever.*

If for each $\beta = 1, \ldots, N + 1$ we introduce the determinant

$$D^\beta = \det |t^1_A, \ldots, t^{\beta-1}_A, t^{\beta+1}_A, \ldots, t^{N+1}_A| \qquad A = 1, \ldots, N$$

then each D^β is a holomorphic density with the density factor

$$\det a^A_B(x; y) \qquad A, B = 1, \ldots, N$$

and the conditions of the theorem can also be stated in this way that these $N + 1$ densities shall not be linearly dependent for constant complex coefficients.

Princeton University

[2] *Vector fields on complex and real manifolds*, Ann. of Math. vol. 52 (1950), 642-649.