Ergod. Th. & Dynam. Sys. (1988), **8***, 283–299
Printed in Great Britain

Entropy and volume

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Abstract. An inequality is given relating the topological entropy of a smooth map to growth rates of the volumes of iterates of smooth submanifolds. Applications to the entropy of algebraic maps are given.

1. *Introduction*

The topological entropy of a map f is a number which roughly measures the orbit structure complexity of f. Maps with positive entropy exhibit complicated dynamics, and the larger the entropy is, the more complicated the dynamics is. It is of interest to calculate the entropy for particular maps under study, but any of the standard definitions of entropy makes this a difficult task. Our goal in this paper is to present an inequality which gives an upper bound for the topological entropy of a smooth map in terms of the growth rates of volumes of smooth submanifolds. We shall use this inequality to give simple proofs of estimates of Gromov of the entropy of real and complex polynomial maps. In many cases our inequality is likely to be an equality. Recently, Yomdin [19] has proved that equality holds for all C^∞-maps.

We proceed to state our results. First, we recall the definition of the topological entropy. If f is a continuous self-map of the compact metric space (Ω, d), ε is a positive real and n is a positive integer, then a subset $E \subset \Omega$ is called an (n, ε)-separated set if whenever $x \neq y$ in E there is an integer j in $[0, n)$ such that $d(f^j x, f^j y) > \varepsilon$. Let $r(n, \varepsilon, f)$ denote the maximal cardinality of an (n, ε)-separated set. The topological entropy $h(f)$ is defined to be

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon, f).$$

It is well known that

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f) = \sup_{\mu \in \mathcal{M}_e(f)} h_\mu(f),$$

where $\mathcal{M}(f)$ is the set of f-invariant probability measures, $\mathcal{M}_e(f)$ denotes the ergodic elements in $\mathcal{M}(f)$, and $h_\mu(f)$ denotes the metric entropy of f relative to the measure μ.

Let M be a $C^{1+\alpha}$ Riemannian manifold, and let $f: M \to M$ be a $C^{1+\alpha}$-map. Let Ω be a compact f-invariant set. For $1 \leq k \leq \dim M$, let D^k be the unit disc in R^k. A smooth k-disc in M is a $C^{1+\alpha}$-map $\gamma: D^k \to M$. The Riemannian metric on M induces a norm $|\cdot|_k$ on each exterior power $\Lambda^k T_x M$ of the tangent space $T_x M$ of x. We define the k-volume of γ to be

$$|\gamma| \underset{\text{def}}{=} \int_{D^k} |\Lambda^k T_x \gamma|_k \, d\lambda(x),$$

https://doi.org/10.1017/S0143385700009469 Published online by Cambridge University Press

where $T_x\gamma$ is the derivative of γ, $\Lambda^k T_x\gamma: \Lambda^k T_x D^k \to \Lambda^k T_{\gamma x} M$ is the induced linear map on the kth exterior power, and $d\lambda$ is the usual volume form on D^k (i.e., if $(t_1, \ldots, t_k)$ are coordinates on D^k, then $d\lambda = dt_1 \wedge \cdots \wedge dt_k$). Similarly, if $A \subset D_k$, we set

$$|\gamma|A| = \int_A |\Lambda^k T_x \gamma|_k \, d\lambda(x).$$

Let $\mathscr{A}$ be a collection of $C^{1+\alpha}$-discs in M whose dimensions vary from 1 through dim M. Let G^k be the Grassmann bundle of k-planes over Ω and let $\bigcup_k G^k(\Omega)$ be the disjoint union. We assume $M \subset \mathbb{R}^N$ for some large N, so that for $x, y \in M$ it is meaningful to write $|y - x|$, etc.

For $\gamma \in \mathscr{A}$, $\gamma: D^k \to M$ let

$$\mathrm{Lip}_\alpha(\gamma) = \sup_{\substack{x \neq y \\ x \in D^k \\ y \in D^k}} \frac{|T_y\gamma - T_x\gamma|}{|y - x|^\alpha}.$$

We say $\mathscr{A}$ is *ample* for Ω if there exists $K > 0$ such that

(1) $\inf_{|v|=1} |T_0\gamma(v)| \geq K^{-1}$ for $\gamma \in \mathscr{A}$,

(2) $\mathrm{Lip}_\alpha(\gamma) \leq K$ for $\gamma \in \mathscr{A}$,

(3) $\bigcup_{\gamma \in \mathscr{A}} \mathrm{Image}(T_0\gamma)$ is dense in $\bigcup_k G^k(\Omega)$.

In the above, 0 refers to the origin in D^k where γ is defined on D^k, and $T_0\gamma$ is the derivative of γ at 0. Note that condition (3) does not require that the $\gamma(0)$ be in Ω.

We observe that (3) forces $\mathscr{A}$ to contain many k-discs for each $1 \leq k \leq \dim M$.

It is easy to see that ample families always exist. Choose ε so that $\exp_x(v)$ is defined for each $x \in \Omega$ and $|v| \leq \varepsilon$, and $\exp_x |\{v \in T_xM: |v| \leq \varepsilon\}$ is an embedding. Consider the family $\mathscr{E} = \{\exp_x |H: x \in \Omega$ and H is a closed ε-ball about 0 in some non-zero linear subspace of $T_xM\}$. Clearly $\mathscr{E}$ is ample over Ω. Note also that countable ample families always exist. Now let V be a compact neighbourhood of Ω. For $n \in Z^+$ let $W^s(n, V) = \bigcap_{0 \leq j < n} f^{-j}(V)$. For a $C^{1+\alpha}$-disc γ: $D \to V$ we define

$$G(\gamma, f, V) = \limsup_{n \to \infty} \frac{1}{n} \log^+ (|f^{n-1} \circ \gamma|\gamma^{-1}(W^s(n, V))|),$$

where $\log^+(x) = \max(\log x, 0)$. This has a simple interpretation. Observe that $\gamma^{-1}(W^s(n, V))$ is the set of points x in D^k such that $f^j(\gamma(x)) \in V$ for $0 \leq j < n$, and $|f^{n-1} \circ \gamma|\gamma^{-1}(W^s(n, V))|$ is the k-dimensional volume of the $(f^{n-1} \circ \gamma)$-image of this set.

THEOREM 1. *Let f be a $C^{1+\alpha}$-self-map of the C^2 Riemannian manifold M. Let Ω be a compact f-invariant set, let U be a compact neighbourhood of Ω in M and let $\mathscr{A}$ be an ample family of smooth discs for Ω. Then*

$$h(f|\Omega) \leq \sup_{\gamma \in \mathscr{A}} G(\gamma, f, U).$$

Now suppose M is a complex Hermitian manifold with complex dimension $\dim_{\mathbb{C}} M$, and $f: M \to M$ is holomorphic. Let Ω be a compact f-invariant set and let U be a compact neighbourhood of Ω. For $1 \leq k \leq \dim_{\mathbb{C}} M$ let $G_c^k(M)$ be the

k-dimensional complex linear subspaces of the complex tangent spaces $\{(T_xM)_{\mathbb{C}}\}$ and set $G_c(M)=\bigcup_{1\le k\le \dim_{\mathbb{C}} M} G_c^k(M)$.

A holomorphic k-disc in M is a holomorphic map $\gamma: D_c^k \to M$, where D_c^k is the open unit ball in $\mathbb{C}^k$. Define the volume $|\gamma|$ of such a γ as follows. Let $(z_1, z_2, \ldots, z_k)$ be complex coordinates on D_c^k with $\bar{z}_i$ the complex conjugate of z_i. Set

$$|\gamma|=(-1)^{[k/2]}(i/2)^k \int_{D_c^k} \left| \bigwedge_{j=1}^{k} T_x\gamma(\partial/\partial z_j)\right| dz_1 \wedge \cdots \wedge dz_k \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_k.$$

We say $\mathscr{A}$ is *holomorphically ample* over Ω if conditions (1) and (2) above hold and

(3) $\bigcup_{\gamma\in\mathscr{A}} (T_0\gamma(T_0D_c^k))$ is dense in $\bigcup_k G_c^k(M)$,

(4) each $\gamma \in A$ is holomorphic.

THEOREM 2. *With the above notations, if $\mathscr{A}$ is holomorphically ample over Ω, then*

$$h(f|\Omega)\le \sup_{\gamma\in\mathscr{A}} G(\gamma, f, U).$$

Theorems 1 and 2 enable us to give simple proofs of the following theorem due to Gromov.

THEOREM 3 (Gromov [**2**]). (i) *Suppose $f:\mathbb{R}^N\to\mathbb{R}^N$ is given by*

$$f(x_1,\ldots,x_N)=(f_1(x_1,\ldots,x_N), f_2(x_1,\ldots,x_N),\ldots,f_N(x_1,\ldots,x_N)),$$

where each f_i is a polynomial of degree $\le d$. If Ω is any compact f-invariant set, then $h(f|\Omega)\le N\log d$.

(ii) *Suppose $f: P^N(\mathbb{C})\to P^N(\mathbb{C})$ is a holomorphic globally defined self-map of complex projective N-space, and the topological degree of f is d. Then $h(f)=\log d$.*

Remark. In the case of endomorphisms $f: P^1(\mathbb{C})\to P^1(\mathbb{C})$ (i.e. the Riemann sphere), Ljubich [**5**] has obtained a different proof of the estimate in theorem 3(ii).

THEOREM 4. *Suppose $f:\mathbb{R}^N\to\mathbb{R}^N$ and Ω are as in the hypotheses of theorem* 3(i). *If, in addition, f is injective on some neighbourhood of Ω, then $h(f|\Omega)\le (N-1)\log d$.*

Assuming theorems 1 and 2, let us prove theorem 3.

Proof of 3(i). Let $\mathscr{E}^k(\mathbb{R}^N)$ be the set of linear embeddings of $\mathbb{R}^k$ into $\mathbb{R}^N$, and let $\mathscr{E}=\bigcup_{1\le k\le N}\mathscr{E}^k(\mathbb{R}^N)$.

For $x\in\mathbb{R}^N$ let $\mathscr{A}_x^k=\{\gamma: D^k\to\mathbb{R}^N: \gamma(t)=x+L(t)$ where $L\in\mathscr{E}^k(\mathbb{R}^N)$ and $t\in D^k\}$. Clearly

$$\mathscr{A}=\bigcup_{\substack{x\in\Omega\\ 1\le k\le N}} \mathscr{A}_x^k$$

is ample for Ω. We call $\mathscr{A}$ the family of affine unit discs over Ω. Let $T>0$ be such that the ball of radius 1 about Ω is contained in $I^N=[-T,T]\times\cdots\times[-T,T]$.

For $\gamma\in\mathscr{A}_x^k$, $x\in\Omega$ let us prove that $G(\gamma, f, I^N)\le k\log d$. This and theorem 1 give theorem 3(i).

We write $f^n(x_1,\ldots,x_N)=(f_1^n(x_1,\ldots,x_N),\ldots,f_N^n(x_1,\ldots,x_N))$, where each f_i^n is a polynomial in $(c_1,\ldots,x_n)$ of degree $\le d^n$.

Let $\gamma\in\mathscr{A}_x^k$. Then

$$|f^{n-1}\circ\gamma|\gamma^{-1}(W^s(I^N,n))|\le|f^{n-1}\circ\gamma|\gamma^{-1}f^{-n+1}(I^N)|.$$

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Let $A_{n-1}=\gamma^{-1}f^{-n+1}(I^N)$ and let $t=(t_1,\ldots,t_k)$ be coordinates on D^k. Now $t\in A_{n-1}$ implies that

$$f^{n-1}(\gamma(t)) \underset{\text{def}}{=} g^{n-1}(t)=(g_1^{n-1}(t_1,\ldots,t_k),\ldots,g_N^{n-1}(t_1,\ldots,t_k))\in I^N,$$

where $g_i^{n-1}(t_1,\ldots,t_k)$ is a polynomial in $(t_1,\ldots,t_k)$ of degree $\le d^{n-1}$.

Also

$$\left[\begin{aligned}|f^{n-1}\circ\gamma|A_{n-1}| &= \int_{A_{n-1}}\left|\bigwedge_{j=1}^{k} T_t(f^{n-1}\circ\gamma)(\partial/\partial t_j)\right| dt_1\wedge\cdots\wedge dt_k\\ &\le C\max_{i_1<\cdots<i_k}\int_{A_{n-1}}\left|\frac{\partial(g_{i_1}^{n-1},\ldots,g_{i_k}^{n-1})}{\partial(t_1,\ldots,t_k)}\right| dt_1\wedge\cdots\wedge dt_k\end{aligned}\right]$$

with $C>0$ independent of n.

The last integral is the volume with multiplicities of $\pi g^{n-1}|A_{n-1}$, where π is the projection of $\mathbb{R}^N$ onto $(x_{i_1},\ldots,x_{i_k})$-space. This integral is less than or equal to $(2T)^kB_{n-1}$, where B_{n-1} is the maximum cardinality of $[\pi g^{n-1}]^{-1}(y)$, where y is a regular value of πg^{n-1}. Thus B_{n-1} may be estimated by Bezout's theorem [**8**]. Indeed, πg^{n-1} is a polynomial map from $\mathbb{R}^k$ to $\mathbb{R}^k$ whose components have degree $\le d^{n-1}$.

Let $(x_1,\ldots,x_k)$ denote coordinates in $\mathbb{R}^k$ so that

$$\pi g^{n-1}(x_1,\ldots,x_k)=(\textstyle\sum a^1_{i_1\ldots i_k}x_1^{i_1}\ldots x_k^{i_k},\ldots).$$

Let $z_i=x_i+\sqrt{-1}y_i$, $i=1,\ldots,k$ denote coordinates on $\mathbb{C}^k$ so that

$$\mathbb{R}^k=\{(z_1,\ldots,z_k): z_i=x_i+\sqrt{-1}y_i \text{ and } y_i=0 \text{ for } i=1,\ldots,k\}.$$

Thus we regard $\mathbb{R}^k$ as a subset of $\mathbb{C}^k$. The map $\pi g^{n-1}:\mathbb{R}^k\to\mathbb{R}^k$ naturally extends to the polynomial $p^{n-1}:\mathbb{C}^k\to\mathbb{C}^k$ with

$$p^{n-1}(z_1,\ldots,z_k)=(\textstyle\sum a^1_{i_1\ldots i_k}z_1^{i_1}\ldots z_k^{i_k},\sum a^2_{i_1\ldots i_k}z_1^{i_1}\ldots z_k^{i_k},\ldots).$$

Let $w\in\mathbb{R}^k$ be such that $\pi g^{n-1}(w)=y$. Let A_w denote the derivative of πg^{n-1} at w. Then A_w is a linear isomorphism from $\mathbb{R}^k$ to itself since y is a regular value of πg^{n-1}. Regarding w as a point in $\mathbb{C}^k$, let F_w be the derivative of p^{n-1} at w which we think of as a complex linear isomorphism $\mathbb{C}^k\to\mathbb{C}^k$. Looking at the induced real vector space structure $\mathbb{R}^{2k}$ of $\mathbb{C}^k$, the map F_w induces a real linear map $\tilde F_w:\mathbb{R}^{2k}\to\mathbb{R}^{2k}$, and it is easily checked that $\tilde F_w$ is an isomorphism. Hence for any y_1 in $\mathbb{C}^k$ near y there will be a unique point w_1 in $\mathbb{C}^k$ near w such that $p^{n-1}(w_1)=y_1$. By Bezout's theorem, if $y_1\in\mathbb{C}^k$ is a regular value of p^{n-1}, then the cardinality of

$$\{w\in\mathbb{C}^k: p^{n-1}(w)=y_1\}\le(\deg g_{i_1}^{n-1})\cdots(\deg g_{i_k}^{n-1})\le d^{k(n-1)}.$$

This implies that $B_{n-1}\le d^{k(n-1)}$, so $|f^{n-1}\circ\gamma|A_{n-1}|\le C_1d^{k(n-1)}$ with C_1 independent of n. Thus $G(\gamma,f,I^n)\le k\log d$. □

Proof of 3(ii). We consider $P^N(\mathbb{C})$ as the quotient space of $\mathbb{C}^{N+1}\backslash\{0\}$ under the equivalence relation $(x_0,\ldots,x_N)\sim(y_0,\ldots,y_n)$ if and only if there is a non-zero $\alpha\in\mathbb{C}\backslash\{0\}$ such that $x_i=\alpha y_i$ for all i. Let $\pi:\mathbb{C}^{N+1}\backslash\{0\}\to P^N(\mathbb{C})$ be the natural projection. We use the usual Kahler metrics on $\mathbb{C}^{N+1}\backslash\{0\}$ and $P^N(\mathbb{C})$. Our holomorphic f lifts to $\bar f(x_0,\ldots,x_N)=(f_i(x_0,\ldots,x_N))$ with f_i a homogeneous polynomial of degree, say δ. The topological degree of f is δ^N, so for theorem 3(ii) we must prove $h(f) = N \log \delta$. By Misiurewicz and Przytycki [7] we have $h(f) \geq N \log \delta$, so we need to prove $h(f) \leq N \log \delta$.

For $1 \leq k \leq N$ let $D^k_{\mathbb{C}}$ be the unit disc in $\mathbb{C}^k$, and for $x \in \mathbb{C}^{N+1} \backslash \{0\}$ let $\mathscr{A}^k_{x,\mathbb{C}} = \{\gamma : D^k_{\mathbb{C}} \to \mathbb{C}^{N+1} : \gamma(t) = x + L(t),\ t \in D^k_{\mathbb{C}}$, where $L : \mathbb{C}^k \to \mathbb{C}^{N+1}$ is a complex linear embedding and x is not in Image$(L)\}$. Let $\tilde{\mathscr{A}}_{x,\mathbb{C}} = \{\pi \circ \gamma : \gamma \in \mathscr{A}^k_{x,\mathbb{C}}\}$. Then

$$\mathscr{A} = \bigcup_{\substack{x \in \mathbb{C}^{N+1} \backslash \{0\} \\ 1 \leq k \leq N}} \tilde{\mathscr{A}}^k_{x,\mathbb{C}}$$

is holomorphically ample.

For $\tilde{\gamma} : D^k_{\mathbb{C}} \to \mathbb{C}^{N+1}$ with $\tilde{\gamma}(t) = x + L(t)$ and $x \notin \text{Image}(\tilde{\gamma})$ let H be the linear subspace of $\mathbb{C}^{N+1}$ spanned by the elements of Image $(\tilde{\gamma})$ and 0. Then $\dim_{\mathbb{C}} H = k+1$. Since $H \supset \text{Image}(\tilde{\gamma})$, it is clear that $G(\pi \circ \tilde{\gamma}, f) \leq G(\pi H, f)$. The computation of $G(\pi H, f)$ reduces to simple and well known facts about volumes of projective varieties in $P^N(\mathbb{C})$ with the Kahler metric on $P^N(\mathbb{C})$. Indeed $f^{n-1}(\pi H)$ is a (possibly singular) variety of degree $\delta^{(n-1)k}$. Thus the proof of theorem (5.22) of [**8**] together with the fact that varieties have locally finite volume [**18**] gives that $|f^{n-1}(\pi H)| = \delta^{(n-1)k}|\pi H|$.

Hence $G(\pi \circ \tilde{\gamma}, f) \leq k \log \delta$ and $\sup_{\gamma \in \mathscr{A}} G(\gamma, f) \leq N \log \delta$, thus proving theorem 3(ii). □

2. *Proofs of theorems* 1 *and* 2

Before proceeding to the proofs, let us make some comments. If our mapping f in the statement of theorem 1 were a diffeomorphism and the set Ω were uniformly hyperbolic, then it would be fairly easy to prove the theorem. It would only be slightly harder to do it if Ω were even uniformly partially hyperbolic. The proof in the general (perhaps non-invertible) case requires applying 'partially hyperbolic intuition' in neighbourhoods of orbits which begin in uniformly partially hyperbolic (possibly non-invariant) sets given by a suitable version of the recently developed Pesin–Oseledec theory. It is most natural in this context to make use of the so-called 'Lyapunov metric' which introduces uniform partial hyperbolicity in a measurable Finsler structure along 'good' orbits. The construction of such Lyapunov metrics for diffeomorphisms was first done by Pesin, and a clean treatment is given in [**1**]. In the previously known constructions of Lyapunov metrics, essential use is made of the fact that the derivative on each unit ball is uniformly bounded above and below. For non-invertible maps this derivative condition of course usually fails, and a large part of our work (theorem 2.3 below) involves constructing Lyapunov metrics without the lower bound condition on the derivative. In this construction we combine techniques introduced by Mañé in [**6**] and Fathi–Herman–Yoccoz in [**1**].

The multiplicative ergodic theorem of Oseledec [**9**] gives the asymptotic behaviour of the iterates Tf^n of the derivative Tf of a C^1-diffeomorphism as $|n| \to \infty$. For forward iterates, an elementary proof of this theorem was given by Raghunathan [**12**]. Later, Ruelle extended the theorem to C^1-maps of a Hilbert space under certain compactness assumptions [**16**], and Mañé extended the theorem to C^1-maps on a Banach space with compact derivative at each point [**6**]. Pesin [**10**] obtained stable and unstable manifold theorems for $C^{1+\alpha}$-diffeormorphisms preserving a smooth measure. Ruelle extended Pesin's results to arbitrary diffeomorphisms in [14], and then to differentiable maps in Hilbert space in [16]. Various stable and unstable manifold theorems for maps of compact manifolds are stated in [17], and Mañé considers stable and unstable manifolds for certain maps of Banach spaces [6]. A nice treatment of the Pesin stable manifold theory for diffeomorphisms is given in [1].

For us to make use of the Oseledec–Pesin theory, we will first pass to the inverse limit of f on Ω.

If $f:\Omega\to\Omega$ is a continuous self-map of the compact metric space (Ω, d), let $\hat{\Omega}=\{\mathbf{x}=(x_0, x_1, \ldots): x_i\in\Omega$ and $f(x_{i+1})=x_i$ for $i\geq 0\}$. Setting $\hat{f}((x_0, x_1, \ldots))=(fx_0, x_0, x_1, \ldots)$ and giving $\hat{\Omega}$ the relative topology as a subset of the product Ω^{Z^+}, we get that $\hat{f}:\hat{\Omega}\to\hat{\Omega}$ is a homeomorphism. If $\pi((x_0, x_1, \ldots))=x_0$, then $f\pi=\pi\hat{f}$. The pair $(\hat{\Omega}, \hat{f})$ is called the inverse limit of (Ω, f). We give $\hat{\Omega}$ the metric $d(\mathbf{x}, \mathbf{y})=\sum_{i\geq 0} 2^{-i} d(x_i, y_i)$.

PROPOSITION 2.1 (Rohlin [13]). *If f and $\hat{f}$ are as above and $\mu\in\mathcal{M}(f)$, $\hat{\mu}\in\mathcal{M}(\hat{f})$ satisfy $\pi_*\hat{\mu}=\mu$, then $h_\mu(f)=h_{\hat{\mu}}(\hat{f})$.*

COROLLARY 2.2. $h(f)=h(\hat{f})$.

Proof. Since $\pi\hat{f}=f\pi$, it follows that $h(\hat{f})\geq h(f)$. For the reverse inequality use

$$h(f)=\sup_{\mu\in\mathcal{M}(f)} h_\mu(f), \qquad h(\hat{f})=\sup_{\mu\in\mathcal{M}(\hat{f})} h_\mu(\hat{f})$$

and the fact that for $\mu\in\mathcal{M}(f)$ there is a $\hat{\mu}\in\mathcal{M}(f)$ with $\pi_*\hat{\mu}=\mu$, and $h_{\hat{\mu}}(\hat{f})=h_\mu(f)$. □

The proof of the next result makes use of techniques of Mañé [6] and Fathi–Herman–Yoccoz [1]. We thank Mañé for several conversations concerning his results in [6] and their adaptation to our setting.

THEOREM 2.3. *Let $f:X\to X$ be a homeomorphism of the compact metric space X. Let $\pi:E\to X$ be a continuous Banach bundle over X and let $L:E\to E$ be a continuous vector bundle map covering f. Let $E_x=\pi^{-1}x$. Suppose that $L_x:E_x\to E_{fx}$ is a compact map for each $x\in X$. Then there is an f-invariant Borel set $\Gamma\subset X$ with the following properties*:

(1) $\mu(\Gamma)=1$ *for every f-invariant probability measure.*

(2) *There is a splitting $E_x=E_1(x)\oplus E_2(x)\oplus E_3(x)$ depending measurably on $x\in\Gamma$ such that*

(a) $v\in E_x\Rightarrow\mathcal{X}(x, v)=\lim_{n\to\infty}(1/n)\log|L_x^n v|$ *exists in* $[-\infty, \infty)$;

(b) $v\in E_1(x)\Rightarrow\mathcal{X}(x, v)<0$, $v\in E_2(x)\Rightarrow\mathcal{X}(x, v)=0$ *and* $v\in E_3(x)\Rightarrow\mathcal{X}(x, v)>0$;

(c) $L_x(E_1(x))\subset E_1(f(x))$, $L_x(E_2(x)\oplus E_3(x))=E_2(f(x))\oplus E_3(f(x))$ *and* $L_x\,|\,E_2(x)\oplus E_3(x)$ *is an isomorphism*;

(d) $\dim E_2(x)\oplus E_3(x)<\infty$.

(3) *Suppose $\lambda_1(x)$ is a negative f-invariant measurable function and $\lambda_3(x)$, $\varepsilon(x)$ are positive f-invariant measurable functions such that* $v\in E_1(x)\Rightarrow\mathcal{X}(x, v)<$

$\lambda_1(x)+\varepsilon(x)<0$ and $v\in E_3(x)$ implies $\mathscr{X}(x, v)>\lambda_3(x)-\varepsilon(x)>0$. Then there are a measurable norm $|\cdot|'_x$ on E_x for $x\in\Gamma$ and a measurable function $A(x)$ such that

(a) $$v\in E_1(x)\Rightarrow|L_xv|'_{fx}\le e^{\lambda_1(x)+\varepsilon(x)}|v|'_x,$$

$$v\in E_2(x)\Rightarrow e^{-\varepsilon(x)}|v|'_x\le|L_xv|'_{fx}\le e^{\varepsilon(x)}|v|'_x,$$

$$v\in E_3(x)\Rightarrow|L_xv|'_{fx}\ge e^{\lambda_3(x)-\varepsilon(x)}|v|'_x;$$

(b) $|v|_x\le|v|'_x\le A(x)|v|_x$ *for* $x\in\Gamma$, $v\in E_x$;

(c) $\lim_{n\to\pm\infty}(1/n)\log A(f^nx)=0$ *for* $x\in\Gamma$.

(4) *If $\eta(x)$ is the infimum of the angles between $E_i(x)$ and $E_j(x)$ for $i\neq j$, then* $\lim_{n\to\pm\infty}(1/n)\log\eta(f^nx)=0$ *for* $x\in\Gamma$.

Proof. Using the ergodic decomposition theorem, we may assume μ is a fixed ergodic invariant measure and $\lambda_1(\cdot)$, $\lambda_3(\cdot)$ and $\varepsilon(\cdot)$ are constant functions. Following Mañé [**6**], we first cover L with a Banach bundle map $\tilde{L}$ which is injective on fibres. Let $Z^+=\{0,1,2,\ldots\}$ be the non-negative integers. Removing a μ-null set, we may assume that $E=\Omega\times F$, where F is a Banach space and $\pi:E\to\Omega$ is the projection. Let $H=\{\theta:Z^+\to F|\sup_{i\ge0}|\theta(i)|<\infty\}$ and let $|\theta|=\sup_i|\theta(i)|$. Let $a>\sup_{x\in\Omega}|L_x|$. Let $\tilde{L}:\Omega\times H\to\Omega\times H$ be defined by $\tilde{L}(x,\theta)=(fx,\tilde{L}_x\theta)$, where $\tilde{L}_x\theta=(L_x\theta(0),\theta(0)/a,\theta(1)/a^2,\ldots)$.

Then $\tilde{L}_x$ is compact for each x, and $\tilde{L}$ is a compact bundle map over f. Moreover,

$$\lim_{n\to\infty}\frac{1}{n}\log|\tilde{L}^n_x\theta|=\lim_{n\to\infty}\frac{1}{n}\log|L^n_x\theta(0)|,$$

by which we mean that each term exists if and only if the other exists and then they are equal.

According to Mañé [**6**], there is a set $\Gamma_1\subset X$ with $\mu(\Gamma_1)=1$ such that there is a splitting $H=\tilde{E}_1(x)\oplus\tilde{E}_2(x)\oplus\tilde{E}_3(x)$ for $x\in\Gamma_1$ such that

(a) $$\theta\in H\Rightarrow\mathscr{X}(x,\theta)=\lim_{n\to\infty}\frac{1}{n}\log|\tilde{L}^n_x(\theta)|=\lim_{n\to\infty}\frac{1}{n}\log|L^n_x(\theta(0))|$$

exists in $[-\infty, a)$;

(b) $$\theta\in\tilde{E}_1(x)\Rightarrow\lim_{n\to\infty}\frac{1}{n}\log|\tilde{L}^n_x(\theta)|<0,$$

$$\theta\in\tilde{E}_2(x)\Rightarrow\lim_{n\to\pm\infty}\frac{1}{n}\log|\tilde{L}^n_x(\theta)|=0,$$

$$\theta\in\tilde{E}_3(x)\Rightarrow\lim_{n\to\pm\infty}\frac{1}{n}\log|\tilde{L}^n_x(\theta)|>0;$$

(c) $\tilde{L}_x(\tilde{E}_1(x))\subset\tilde{E}_1(fx)$, $L_x(\tilde{E}_2(x)\oplus\tilde{E}_3(x))=\tilde{E}_2(fx)\oplus\tilde{E}_3(fx)$ and $\tilde{L}_x|\tilde{E}_2(x)\oplus\tilde{E}_3(x)$ is an isomorphism;

(d) dim $\tilde{E}_2(x)\oplus\tilde{E}_3(x)$ is finite;

(e) if $\eta(x)$ is the infimum of the angles betweeen the pairs of spaces $(\tilde{E}_i(x),\tilde{E}_j(x))$ for $i\neq j$, then $\lim_{n\to\pm\infty}(1/n)\log\eta(f^nx)=0$.

Now let $\tilde{\pi}:H\to F$ be the projection $\tilde{\pi}(\theta)=\theta(0)$, and let $E_i(x)=\tilde{\pi}(\tilde{E}_1(x))$ for $i=1,2,3$. Since $L_x\tilde{\pi}=\tilde{\pi}\tilde{L}_x$ for all $x\in\Gamma_1$, it is easily shown that (1) and (2) in the theorem hold. For $v \in E_1(x)$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \log |L_x^n v| < \lambda_1 + \varepsilon,$$

so there is a measurable function $C(x)$ defined on Γ_1 such that

$$|L_x^n v| \le C(x)\, e^{\lambda_1 n + \varepsilon n} |v|_x \qquad \text{for } n \ge 0,\ v \in E_1(x).$$

Thus

$$A_1(x) \equiv \sup_{n\ge 0} |L_x^n|\, e^{-n(\lambda_1+\varepsilon)} < \infty \qquad \text{for } x \in \Gamma_1.$$

Let

$$|v|_{x,1} = \sup_{n\ge 0} |L_x^n v|\, e^{-n(\lambda_1+\varepsilon)}.$$

Then

$$\begin{aligned} |L_x v|_{fx,1} &= \sup_{n\ge 0} |L_{fx}^n L_x v|\, e^{-n\lambda_1 - n\varepsilon} \\ &= e^{\lambda_1+\varepsilon}\left(\sup_{n\ge 0} |L_x^{n+1} v|\, e^{-(n+1)(\lambda_1+\varepsilon)}\right) \\ &\le e^{\lambda_1+\varepsilon} |v|_{x,1} \qquad \text{for } v \in E_1(x). \end{aligned}$$

Also

$$1 \le \frac{|v|_{1,x}}{|v|_x} \le A_1(x) \qquad \text{for } x \in \Gamma_1,\ v \in E_1(x).$$

Following Mañé [6], we will show that $\log A_1(f^{-1}x) - \log A_1(x)$ is bounded above. Consider

$$A_1(f^{-1}x) = \sup_{n\ge 0} |L_{f^{-1}x}^n|\, e^{-n(\lambda_1+\varepsilon)}.$$

For $n \ge 1$ we have

$$|L_{f^{-1}x}^n| = |L_x^{n-1} L_{f^{-1}x}| \le |L_x^{n-1}|\,|L_{f^{-1}x}|,$$

so

$$\begin{aligned} A_1(f^{-1}x) &\le \sup\,(1, \sup_{n\ge 1} |L_x^{n-1}|\,|L_{f^{-1}x}|\, e^{-n(\lambda_1+\varepsilon)}) \\ &= \sup\,(1, e^{-(\lambda_1+\varepsilon)} |L_{f^{-1}x}| \sup_{n\ge 1} |L_x^{n-1}|\, e^{-(n-1)(\lambda_1+\varepsilon)}) \\ &= \sup\,(1, |L_{f^{-1}x}|\, e^{-(\lambda_1+\varepsilon)} A_1(x)) \\ &\le A_1(x) \sup\,(1, |L_{f^{-1}x}|\, e^{-(\lambda_1+\varepsilon)}) \end{aligned}$$

since $A_1(x) \ge 1$ for all x.

https://doi.org/10.1017/S0143385700009469 Published online by Cambridge University Press

Thus

$$\log A_1(f^{-1}x) - \log A_1(x) \le \log (\sup (1, |L_{f^{-1}x}|\, e^{-(\lambda_1+\varepsilon)})),$$

which is bounded above.

Now lemma (III.8) in [6] gives $\lim_{n\to\pm\infty} (1/n) \log A_1(f^n x) = 0$. On the other hand, the family of maps $L_x : E_2(x) \oplus E_3(x) \to E_2(fx) \oplus E_3(fx)$ gives a vector bundle isomorphism of $\bigcup_{x\in\Gamma_1} E_2(x) \oplus E_3(x)$ to itself covering $f: \Gamma_1 \to \Gamma_1$. Moreover, there is a direct sum decomposition $E_2(x) \oplus E_3(x) = K_1(x) \oplus \cdots \oplus K_r(x)$ and there is a finite set of numbers $\lambda_1, \ldots, \lambda_r$ in $(-\infty, 0]$ such that $\lim_{n\to\pm\infty} (1/n) \log |L_x^{-n} v| = -\lambda_i$ for $v \in K_i(x)$. Also, if $\tilde\eta(x)$ is the minimum of the angles between the spaces $K_i(x)$ and $K_j(x)$, then $\lim_{n\to\pm\infty} (1/n) \log \tilde\eta(f^n x) = 0$. This condition implies that

$$\lim_{n\to\pm\infty} (1/n) \log |\det L_x^{-n} | E_2(x) \oplus E_3(x)| = \sum_{i=1}^{r} -\lambda_i \dim K_i(x).$$

Hence the methods of Fathi–Herman–Yoccoz [1] give a norm $|v|_{x,2}$ on $E_2(x) \oplus E_3(x)$ such that

(f) $|L_x^{-1} v|_{f^{-1}x,2} \le e^{\varepsilon + \max_i(-\lambda_i)} |v|_{x,2}$;

(g) $1 \le |v|_{x,2}/|v|_x \le A_2(x)$, where $\lim_{n\to\pm\infty} (1/n) \log A_2(f^n x) = 0$.

To see this, let $\lambda = \lambda_i$ so that $\lim_{n\to\pm\infty} (1/n) \log |L_x^n v| = \lambda$ uniformly for all $v \in K_i(x)$ with $|v| = 1$, and $L_x : K_i(x) \to K_i(fx)$ is an isomorphism for all x. Then, given $\varepsilon > 0$, there is an $N > 0$ such that $|n| \ge N \Rightarrow \lambda - \varepsilon \le (1/n) \log |L_x^n v| \le \lambda + \varepsilon \Rightarrow e^{n(\lambda-\varepsilon)}|v| \le |L_x^n v| \le e^{n(\lambda+\varepsilon)}|v|$ for all $v \in K_i(x)\backslash\{0\}$ and $n \ge N \Rightarrow$ there exist $C_1(x), C_2(x) > 0$ such that for all $n \ge 0$

$$C_1(x)\, e^{n(\lambda-\varepsilon)}|v| \le |L_x^n v| \le C_2(x)\, e^{n(\lambda+\varepsilon)}|v|.$$

For $n \le 0$ and $-n \ge N$ we have

$$e^{n(\lambda-\varepsilon)}|v| \ge |L_x^n v| \ge e^{n(\lambda+\varepsilon)}|v|.$$

This implies, for possibly different $C_1(x)$, $C_2(x)$,

$$C_2(x)\, e^{n\lambda+|n|\varepsilon}|v| \ge |L_x^n v| \ge C_1(x)\, e^{n\lambda-|n|\varepsilon}|v|.$$

This formula works for all n. Thus for $n \ge 0$ and $k \in Z$

$$|L_x^{n+k} v| \le C_2(x)\, e^{(n+k)\lambda+(n+|k|)\varepsilon}|v|$$

and

$$|L_x^k v| \ge C_1(x)\, e^{k\lambda-|k|\varepsilon}|v|,$$

so

$$e^{k\lambda+|k|\varepsilon}|v| \le C_1(x)^{-1}\, e^{2|k|\varepsilon}|L_x^k v|$$

$$\Rightarrow |L_x^{n+k} v| \le C_2(x) C_1(x)^{-1}\, e^{n(\lambda+\varepsilon)+2|k|\varepsilon}|L_x^k v|$$

$$\Rightarrow A_2(x) \equiv \sup_{\substack{n\ge 0\\ k\in Z\\ |v|\ne 0}} \frac{|L_x^{n+k} v|}{|L_x^k v|}\, e^{-n(\lambda+\varepsilon)-2|k|\varepsilon} < \infty.$$

Let $|v|_{x,2}=\sup_{n\geq 0}|L_x^n v|\,e^{-n(\lambda+\varepsilon)}$. Then $1\leq |v|_{x,2}/|v|\leq A_2(x)$. Now

$$\begin{aligned}|L_x v|_{fx,2}&=\sup_{n\geq 0}|L_{fx}L_x v|\,e^{-n(\lambda+\varepsilon)}\\&=e^{\lambda+\varepsilon}\sup_{n\geq 1}|L_x^n v|\,e^{-n(\lambda+\varepsilon)}\\&\leq e^{\lambda+\varepsilon}|v|_{x,2}.\end{aligned}$$

Also

$$A_2(fx)=\sup_{\substack{n\geq 0\\ k\in Z\\ |v|\neq 0}}\frac{|L_{fx}^{n+k}v|}{|L_{fx}^k v|}\,e^{-n(\lambda+\varepsilon)-2|k|\varepsilon}.$$

So, putting $v=L_x w$, we get

$$A_2(fx)=\sup_{\substack{n\geq 0\\ k\in Z\\ |w|\neq 0}}\frac{|L_x^{n+k+1}w|}{|L_x^{k+1}w|}\,e^{-n(\lambda+\varepsilon)-2|k|\varepsilon}.$$

Since $|-2|k|\varepsilon-(-2|k+1|\varepsilon)|=2\varepsilon$ for all k, we see that $A_2(fx)\leq A_2(x)\,e^{2\varepsilon}$ for all x. Similarly, $A_2(f^{-1}x)\leq A_2(x)\,e^{2\varepsilon}$ for all x. Assuming we have defined $|v|_{x,2}$ as above for $v\in K_i(x)$, denote this norm by $|v|_{x,2,i}$. For $v=(v_1,\ldots,v_r)\in E_2(x)\oplus E_3(x)$ let $|v|_{x,2}=\sup_i|v|_{x,2,i}$. Applying this to $L^{-1}|E_2(x)\oplus E_3(x)$ gives

$$|L_x^{-1}v|_{f^{-1}x,2}\leq e^{-(\min\lambda_i)+\varepsilon}|v|_{x,2},$$

which is equivalent to (f). Statement (g) follows by setting $A_2(x)$ to be the maximum of the $A_2(x)$ obtained for the $K_i(x)$.

Now for $v=(v_1,v_2)$ with $v_1\in E_1(x)$, $v_2\in E_2(x)\oplus E_3(x)$, set $|v|'=\max(|v_1|,|v_2|)$ and $A(x)=\max(A_1(x),A_2(x))$. This proves theorem 2.3. □

Proof of theorem 1. We know that $h(f)=h(\hat f)$ and $h(\hat f)=\sup_{\mu\in\mathcal{M}_e(\hat f)}h_\mu(\hat f)$. Fix an ergodic $\mu\in\mathcal{M}(\hat f)$. Given $\varepsilon_0>0$, we will find $\gamma\in\mathcal{A}$ such that $h_\mu(\hat f)<G(\gamma,f,U)+\varepsilon_0$. This will imply that $h(f)=h(\hat f)\leq\sup_{\gamma\in\mathcal{A}}G(\gamma,f,U)$ as required.

If $\pi:\hat\Omega\to\Omega$ is the projection and $\mathbf{x}\in\hat\Omega$, it will be convenient to denote $\pi\mathbf{x}$ by x. Excluding a μ-null set in $\hat\Omega$ and its π-image in Ω, we may assume that $T_\Omega M=\Omega\times\mathbb{R}^m$. We first apply theorem 2.3 to the vector bundle map $T\hat f:\hat\Omega\times\mathbb{R}^m\to\hat\Omega\times\mathbb{R}^m$ over $\hat f:\hat\Omega\to\hat\Omega$, where $T\hat f(\mathbf{x},v)=(\hat f\mathbf{x},T_x f(v))$. We obtain a set $\Gamma\subset\hat\Omega$, splitting $E_1\oplus E_2\oplus E_3$, and functions $\lambda_1(\cdot)$, $\lambda_3(\cdot)$, $\varepsilon(\cdot)$, etc. Ergodicity of μ implies that $\lambda_1(\mathbf{x})$, $\lambda_3(\mathbf{x})$ and $\varepsilon(\mathbf{x})$ may be assumed to be constant functions, which we denote by λ_1, λ_3 and ε. Note that we may choose ε arbitrarily small; we will see how small later. We may as well suppose that $h_\mu(\hat f)>0$. Then proposition 2.1 gives $h_{\pi_*\mu}(f)>0$. By the Ruelle entropy inequality [**15**], we then have that $x\in\pi\Gamma$ has at least one positive characteristic exponent. This implies that $E_3(\mathbf{x})\neq\{0\}$ for $\mathbf{x}\in\Gamma$.

Let $M_1>0$ be a constant such that

(1) for y near x in local coordinates $|T_y f-T_x f|\leq M_1|y-x|^\alpha$;

(2) $|f(y)-f(x)|\leq M_1|y-x|$.

We now work in local coordinates about points $z \in M$. We shall denote by d the metric in both Ω and $\hat{\Omega}$. For $d(\mathbf{x}, \mathbf{y}) \le \delta$ we consider $E_i(\mathbf{x})$ as a subspace of T_yM by translation, and we define $|v|'_{\mathbf{y}} = |v|'_{\mathbf{x}}$ for $v \in T_yM$. Then $v \in E_1(\mathbf{x})$

$$\begin{aligned}\Rightarrow |T_yf(v)|_{\hat{f}(\mathbf{y})} = |T_yf(v)|'_{\hat{f}(\mathbf{x})} &= |T_xf(v) + T_yf(v) - T_xf(v)|'_{\hat{f}(\mathbf{x})} \\ &\le e^{\lambda_1+\varepsilon}|v|'_{\mathbf{x}} + A(\hat{f}(\mathbf{x}))|T_yf(v) - T_xf(v)|_{\hat{f}(\mathbf{x})} \\ &\le e^{\lambda_1+\varepsilon}|v|'_{\mathbf{x}} + A(\hat{f}(\mathbf{x}))M_1|y-x|^{\alpha}|v|_{\mathbf{x}} \\ &\le e^{\lambda_1+\varepsilon}|v|'_{\mathbf{x}} + A(\hat{f}(\mathbf{x}))M_1|y-x|^{\alpha}|v|'_{\mathbf{x}}.\end{aligned}$$

If we let

$$\tilde{\varepsilon}(\mathbf{x}) = \min\left(1, \left[\frac{e^{\lambda_1+2\varepsilon} - e^{\lambda_1+\varepsilon}}{A(\hat{f}(x))M_1}\right]^{1/\alpha}\right),$$

then $|y-x| \le \tilde{\varepsilon}(\mathbf{x})$ and $v \in E_1(\mathbf{y}) \subset T_yM$ imply

$$(3) \qquad |T_yf(v)|'_{\hat{f}(\mathbf{y})} \le e^{\lambda_1+2\varepsilon}|v|'_{\mathbf{y}}.$$

Also we have $\lim_{n\to\pm\infty}(1/n)\log\tilde{\varepsilon}(\hat{f}^n\mathbf{x}) = 0$ since $\lim_{n\to\pm\infty}(1/n)\log A(\hat{f}^n\mathbf{x}) = 0$. Let us show that there is a subset $\tilde{\Gamma} \subset \Gamma$ with $\mu(\tilde{\Gamma}) = 1$ and a real-valued measurable function $\varepsilon_1(\mathbf{x})$ defined on $\tilde{\Gamma}$ such that $0 < \varepsilon_1(\mathbf{x}) \le \tilde{\varepsilon}(\mathbf{x})$, and for $\mathbf{x} \in \tilde{\Gamma}$

$$(4) \qquad \varepsilon_1(\hat{f}^k\mathbf{x}) \ge \varepsilon_1(\mathbf{x})\, e^{-|k|\varepsilon} \qquad \text{for } k \in Z.$$

For each $\mathbf{x} \in \Gamma$ there are constants $C_1, C_2 > 0$ such that $C_1\, e^{-n\varepsilon} \le \tilde{\varepsilon}(\hat{f}^n\mathbf{x}) \le C_2\, e^{n\varepsilon}$ for $n \ge 0$. Let $b(\mathbf{x}) = \inf_{n\ge0}\tilde{\varepsilon}(\hat{f}^n\mathbf{x})\, e^{n\varepsilon}$. It is easy to show that $0 < b(\mathbf{x}) \le \tilde{\varepsilon}(\mathbf{x})$ and $b(\hat{f}\mathbf{x}) \ge e^{-\varepsilon}b(\mathbf{x})$ for $x \in \Gamma$. Since $\log b(\hat{f}\mathbf{x}) - \log b(\mathbf{x})$ is bounded below, lemma (III.8) in [**6**] gives $\lim_{n\to\pm\infty}(1/n)\log b(\hat{f}^n\mathbf{x}) = 0$ on a subset $\tilde{\Gamma}$ with $\mu(\tilde{\Gamma}) = 1$. For $\mathbf{x} \in \tilde{\Gamma}$ there is a constant $C_1 > 0$ such that $b(\hat{f}^n\mathbf{x}) \ge C_1\, e^{n\varepsilon}$ for $n \le 0$. Letting $\varepsilon_1(\mathbf{x}) = \inf_{n\le0} b(\hat{f}^n\mathbf{x})\, e^{-\varepsilon n}$, it is easy to show that $\varepsilon_1(\hat{f}\mathbf{x}) \ge e^{-\varepsilon}\varepsilon_1(\mathbf{x})$ and $\varepsilon_1(\hat{f}^{-1}\mathbf{x}) \ge e^{-\varepsilon}\varepsilon_1(\mathbf{x})$ for $\mathbf{x} \in \tilde{\Gamma}$. Then (4) follows. From now on let us rename $\tilde{\Gamma}$ as Γ and assume (4) holds for $\mathbf{x} \in \Gamma$.

Reducing $\varepsilon_1(\mathbf{x})$ if necessary, we may assume that for $|y-x| \le \varepsilon_1(\mathbf{x})$

$$(5) \qquad v \in E_2(\mathbf{x}) \Rightarrow e^{-2\varepsilon}|v|'_{\mathbf{y}} \le |T_yf(v)|'_{\hat{f}(\mathbf{y})} \le e^{2\varepsilon}|v|'_{\mathbf{y}}$$

and

$$(6) \qquad v \in E_3(\mathbf{x}) \Rightarrow |T_yf(v)|'_{\hat{f}(y)} \ge e^{\lambda_3-2\varepsilon}|v|'_{\mathbf{y}}.$$

The norms $|\ |'_{\mathbf{y}}$ enable us to define diameters of sets and volumes of submanifolds in $\pi B_{\varepsilon_1(\mathbf{x})}(\mathbf{x})$ as well as norms of linear maps defined between subspaces $H_z \subset T_zM$ and $H_w \subset T_wM$ for $z, w \in \pi B_{\varepsilon_1(\mathbf{x})}(\mathbf{x})$. We will denote such objects by 'primes'. Thus diam$'(A)$ denotes the diameter of a set A induced by the metric on some $\pi B_{\varepsilon_1(\mathbf{x})}(\mathbf{x})$ induced by $|\ |'_{\mathbf{x}}$, etc. Since the angle functions $\eta(\cdot)$ have subexponential growth along orbits in Γ, we may assume there is a measurable function $B: \Gamma \to \mathbb{R}^+$ such that for any smooth k-disc γ in $\pi B_{\varepsilon_1(\mathbf{x})}(\mathbf{x})$

$$(7) \qquad |\gamma| \ge B(\mathbf{x})|\gamma|' \ge B(\mathbf{x})|\gamma|$$

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and

$$\text{(8)} \qquad \lim_{n\to\infty} \frac{1}{n} \log B(\hat{f}^n \mathbf{x}) = 0.$$

Let $\exp_x$ denote the exponential map associated to the Riemannian metric. Estimates (3), (5) and (6) give that Tf contracts along the $E_1(\mathbf{x})$-direction, expands along the $E_3(\mathbf{x})$-direction and is nearly an isometry along the $E_2(\mathbf{x})$-direction. Let $D_i(\mathbf{x}) = \exp_x (B_{\varepsilon_1(\mathbf{x})}(0) \cap E_i(\mathbf{x}))$. We assume $\varepsilon_1(\mathbf{x})$ is small enough so that $\pi B_{\varepsilon_1(\mathbf{x})} \subset U$ for all $\mathbf{x} \in \Gamma$, $\exp_x (B_{\varepsilon_1(\mathbf{x})}(0))$ is diffeomorphic to $D(\mathbf{x}) \equiv D_1(\mathbf{x}) \times D_2(\mathbf{x}) \times D_3(\mathbf{x})$, and $\operatorname{diam}' D_i(\mathbf{x}) \le 1$. We identify the two sets $\exp_x (B_{\varepsilon_1(x)}(0))$ and $D(\mathbf{x})$, and use the norms $|\;|_{\mathbf{y}}$ and $|\;|'_{\mathbf{y}}$. Standard graph transform estimates (e.g. as in [3]) yield the following fact.

For $\sigma_i \in (0, 1)$, $i = 1, 2, 3$, let $\sigma_i E_i(\mathbf{x}) = \{v \in E_i(\mathbf{x}) \colon |v|' \le \sigma_i\}$ and let

$$\sigma_1 D_1(\mathbf{x}) \times \sigma_2 D_2(\mathbf{x}) \times \sigma_3 D_3(\mathbf{x}) = \exp_x (\sigma_1 E_1(\mathbf{x}) + \sigma_2 E_2(\mathbf{x}) + \sigma_3 E_3(\mathbf{x})).$$

Let $\pi_i \colon D(\mathbf{x}) \to D_i(\mathbf{x})$ be the projection for $i = 2, 3$. Suppose

$$\gamma = \{(g(u, v), u, v) \colon u \in \sigma_2 D_2(\mathbf{x}),\ v \in \sigma_3 D_3(\mathbf{x})\},$$

where $g \colon \sigma_2 D_2(\mathbf{x}) \times \sigma_3 D_3(\mathbf{x}) \to \sigma_1 D_1(\mathbf{x})$ is C^1 and $|Tg|' \le 1$.

Then

(9) there is a C^1-function

$$g_1 \colon e^{-2\varepsilon} \sigma_2 D_2(\hat{f}(\mathbf{x})) \times \sigma_3 D_3(\hat{f}(\mathbf{x})) \to \sigma_1 D_1(\mathbf{x})$$

with $|Tg_1|' \le 1$ such that

$$\text{(a)} \qquad f(\gamma) \supset \gamma_1 \equiv \{(g_1(u, v), u, v) \colon u \in \varepsilon^{-2\varepsilon} \sigma_2 D_2(\hat{f}(\mathbf{x})),\ v \in \sigma_3 D_3(\hat{f}(\mathbf{x}))\}$$

and

$$\text{(b)} \qquad \operatorname{diam}' (\pi_2 f(\gamma)) \le e^{2\varepsilon}(2\sigma_2).$$

Let us recall the following fact from [4]. Let

$$d_{\hat{f},n}(\mathbf{x}, \mathbf{y}) = \max_{0 \le j < n} d(\hat{f}^i \mathbf{x}, \hat{f}^i \mathbf{y}).$$

For $\varepsilon > 0$, $n \in Z^+$ let $N(\varepsilon, n)$ denote the minimal cardinality of a collection of $d_{\hat{f},n}\varepsilon$-balls whose union has μ-measure greater than $\frac{1}{2}$. Then

$$h_\mu(\hat{f}) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(\varepsilon, n).$$

Now let Λ be a compact set with $\mu(\Lambda) > \frac{1}{2}$. For each $\varepsilon > 0$, $n \in Z^+$ let $F(\varepsilon, n)$ be a maximal (n, ε)-separated set *in* Λ. Then $\{B_{\varepsilon, d_{\hat{f},n}}(\mathbf{x}) \colon \mathbf{x} \in F(\varepsilon, n)\}$ is a covering of Λ, so $N(\varepsilon, n) \le \operatorname{card} F(\varepsilon, n)$. Here and in the sequel we let card E denote the cardinality of the set E. Thus

$$h_\mu(\hat{f}) \leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card} F(\varepsilon, n). \tag{10}$$

Next choose a compact set $\Lambda \subset \Gamma$ such that $\mu(\Lambda) > \frac{1}{2}$, the bundles $E_1(\cdot)$, $E_2(\cdot)$, $E_3(\cdot)$ and the functions $A(\cdot), \eta(\cdot), \varepsilon_1(\cdot), B(\cdot)$ are continuous on Λ. Let $0 < M_2 < \frac{1}{2}\min(\inf\{\varepsilon_1(\mathbf{x}): \mathbf{x} \in \Lambda\}, \inf\{B(\mathbf{x}): \mathbf{x}\in\Lambda\})$ be such that

$$M_2^{-1}\, e^{\varepsilon n} \geq M_2\, e^{-\varepsilon n} \tag{11a}$$

and

$$B(\hat{f}^n\mathbf{x}) \geq M_2\, e^{-n\varepsilon} \tag{11b}$$

for $n \geq 0$ and $\mathbf{x} \in \Lambda$.

It follows from the methods in [6] that if $\mathbf{x}\in\Gamma$, and

$$W^s_{\mathrm{loc}}(\mathbf{x}) = \pi\{\mathbf{y} \in B_{\varepsilon_1(\mathbf{x})}(\mathbf{x}) : d(\hat{f}^n\mathbf{y}, \hat{f}^n\mathbf{x}) \leq \varepsilon_1(\mathbf{x})\, \mathrm{e}^{(\lambda_1+\varepsilon)n} \text{ for } n \geq 0\},$$

then $W^s_{\mathrm{loc}}(\mathbf{x})$ is a $C^{1+\alpha}$-disc in M tangent at x to $E_1(\mathbf{x})$.

For each $\mathbf{x}\in\Lambda$ choose a C^1-disc $W(\mathbf{x})$ through x tangent at x to $E_2(\mathbf{x}) \oplus E_3(\mathbf{x})$. Assume M_2 small enough that if $\mathbf{x}\in\Lambda$ and $\mathbf{y}\in B_{M_2}(\mathbf{x})\cap\Lambda$, then $W^s_{\mathrm{loc}}(\mathbf{y}) \cap W(\mathbf{x}) \neq \varnothing$. By (9) there is an $\varepsilon_2 \in (0, M_2)$ such that

$$h_\mu(\hat{f}) < \limsup_{n\to\infty} \frac{1}{n} \log(\max\{\operatorname{card}(E): E \text{ is an } (n, \varepsilon_2)\text{-separated set in } \Lambda\}) + \varepsilon_0.$$

For large n let E_n be a maximal (n, ε_2)-separated set in Λ.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$ be finitely many points in Λ such that $\Lambda \subset \bigcup_{j=1}^s B_{M_2}(\mathbf{x}_i)$. Then there clearly is some $\mathbf{x}_j$ such that

$$h_\mu(\hat{f}) < \varepsilon_0 + \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card}(E_n \cap B_{M_2}(\mathbf{x}_j)).$$

Now choose $\gamma\in\mathscr{A}$ such that $W = \operatorname{Image}\gamma$ is C^1 near enough to $W(\mathbf{x}_j)$ so that $\mathbf{y}\in B_{M_2}(\mathbf{x}_j)\cap\Lambda \Rightarrow W^s_{\mathrm{loc}}(\mathbf{y}) \cap W$ is a unique point, say $z(\mathbf{y})$.

We estimate $\operatorname{card}(E_n \cap B_{M_2}(\mathbf{x}_j))$ in terms of $G(\gamma, f, U)$. We may choose $\sigma_2, \sigma_3 > 0$ independent of $\mathbf{y}$ in $\Lambda\cap B_{M_2}(\mathbf{x}_j)$ such that W contains a set W_1 which contains $z(\mathbf{y})$ and is the graph of a C^1-function $g: \sigma_2 D_2(\mathbf{y}) \times \sigma_3 D_3(\mathbf{y}) \to D_1(\mathbf{y})$ with $|Tg|' \leq 1$, $g(0,0) = z(\mathbf{y})$. Let $\varepsilon_3 \in (0, \varepsilon)$. We may assume by ergodicity that there is an $N(\varepsilon_3) > 0$ such that $\mathbf{x}\in\Lambda$, $n \geq N(\varepsilon_3) \Rightarrow$ there is a $k \in [(1-\varepsilon_3)n, n)$ such that $\hat{f}^k\mathbf{x}\in\Lambda$. For $\mathbf{y}\in E_n \cap B_{M_2}(\mathbf{x})$ let

$$\tilde{B}_n(\mathbf{y}) = \pi \bigcap\nolimits_{0\leq k\leq n-1} \hat{f}^{-k} B_{\varepsilon_1(\hat{f}^k\mathbf{y})}(\hat{f}^k\mathbf{y}).$$

Let

$$W_n(\mathbf{y}) = \operatorname{Image} g|\, e^{-4\varepsilon(n-1)}(\varepsilon_2/4)\varepsilon_1(\hat{f}^{n-1}\mathbf{y})\sigma_2 D_2(\mathbf{y}) \times \sigma_3 D_3(\mathbf{y}).$$

Then $W_n(\mathbf{y})$ is a piece of W containing $z(\mathbf{y})$ and

$$\operatorname{diam}'(\pi_2 W_n(\mathbf{y})) \leq \varepsilon_1(\hat{f}^{n-1}\mathbf{y})\, e^{-4\varepsilon(n-1)}(\varepsilon_2/2)\sigma_2.$$

Let $\tilde{W}_n(\mathbf{y}) = \tilde{B}_n(\mathbf{y}) \cap W_n(\mathbf{y})$. By (9b)

$$\operatorname{diam}' (\pi_2 f^k \tilde{W}_n(\mathbf{y})) \le e^{2k\varepsilon} \varepsilon_1(\hat{f}^{n-1}\mathbf{y})\, e^{-4\varepsilon(n-1)}(\varepsilon_2/2)\sigma_2.$$

For $0 \le k \le n-1$ and $0 \le j \le n-1-k$ we have

$$f^{k+j}(\tilde{W}_n(\mathbf{y})) \subset \pi B_{\varepsilon_1(\hat{f}^{k+j}\mathbf{y})}(\hat{f}^{k+j}\mathbf{y}),$$

so $\operatorname{diam}' (\pi_3 f^k \tilde{W}_n(\mathbf{y})) \le 2e^{-(\lambda_3 - 2\varepsilon)(n-1-k)}\varepsilon_1(\hat{f}^{n-1}\mathbf{y})$.

Hence

$$\operatorname{diam} (f^k \tilde{W}_n(\mathbf{y})) \le \operatorname{diam}' (f^k \tilde{W}_n(\mathbf{y}))$$

$$\le \varepsilon_1(\hat{f}^{n-1}\mathbf{y}) \max (e^{-4\varepsilon(n-1)+2k\varepsilon}(\varepsilon_2/2)\sigma_2, 2e^{-(\lambda_3-2\varepsilon)(n-1-k)}).$$

Since $\hat{f}^k\mathbf{y} \in \Lambda$ for some $k \in [(1-\varepsilon_3)n, n)$ and $\sup \varepsilon_1(\mathbf{x}) \le 1$, we have

$$\varepsilon_1(\hat{f}^{n-1}\mathbf{y}) \le e^{\varepsilon_3 n} \le e^{\varepsilon n}.$$

Letting T be the maximum diameter of any fibre of $\pi: \hat{\Omega} \to \Omega$, note that if $K \in Z^+$ is greater than the largest integer in $1 + (\log T - \log \varepsilon_2/4)/(\log 2))$, then for $\mathbf{y}, \mathbf{z} \in \hat{\Omega}$, $d(\mathbf{y}, \mathbf{z}) \le 2 \max_{0 \le i \le K} d(x_i, y_i) + \varepsilon_2/4$. Also we can pick $k_1 = k_1(\varepsilon, \varepsilon_2, \sigma_2) > K$ such that $k \in [0, n-1-k_1) \Rightarrow \operatorname{diam} (f^k \tilde{W}_n(\mathbf{y})) < \varepsilon_2/8$. This means that if $\mathbf{y} \ne \mathbf{z}$ in $E_n \cap B_{M_2}(\mathbf{x}_j)$ and $k \in (k_1, n-1-k_1)$ is such that $d(\hat{f}^k\mathbf{y}, \hat{f}^k\mathbf{z}) > \varepsilon_2$, then $d(f^k y, f^k z) > \varepsilon_2/4$. Hence, if there is a $k \in (k_1, n-1-k_1)$ such that $d(\hat{f}^k\mathbf{y}, \hat{f}^k\mathbf{z}) > \varepsilon_2$, then $\tilde{W}_n(\mathbf{y}) \cap \tilde{W}_n(\mathbf{z}) = \varnothing$.

Pick a subset $E'_n \subset E_n \cap B_{M_2}(\mathbf{x}_j)$ such that card $E'_n \ge C$ card E_n with C independent of n and $\mathbf{y} \ne \mathbf{z}$ in $E'_n \Rightarrow d(\hat{f}^k\mathbf{y}, \hat{f}^k\mathbf{z}) > \varepsilon_2$ for some $k \in [0, n-1-k_1)$.

Next observe that, by (9a), $f^{n-1}\tilde{W}_n(\mathbf{y})$ contains the graph of a C^1-function g_1: $e^{-6\varepsilon(n-1)}(\varepsilon_2/4)\varepsilon_1(\hat{f}^{n-1}\mathbf{y})\sigma_2 D_2(\hat{f}^{n-1}\mathbf{y}) \times \varepsilon_1(\hat{f}^{n-1}\mathbf{y})\sigma_3 D_3(\hat{f}^{n-1}\mathbf{y}) \to D_1(\hat{f}^{n-1}\mathbf{y})$ with $g_1(0,0) = \hat{f}^{n-1}(z(\mathbf{y}))$ and $|Tg_1|' \le 1$. This gives

$$|f^{n-1}(\tilde{W}_n(\mathbf{y}))|' \ge (e^{-6\varepsilon(n-1)}(\varepsilon_2/4)\varepsilon_1(\hat{f}^{n-1}\mathbf{y})\sigma_2)^{\dim D_2(\hat{f}^{n-1}\mathbf{y})}(\varepsilon_1(\hat{f}^{n-1}\mathbf{y})\sigma_3)^{\dim D_3(\hat{f}^{n-1}\mathbf{y})}$$
$$= C_1\, e^{-6\varepsilon(n-1)\dim D_2(\hat{f}^{n-1}(\mathbf{y}))}\varepsilon_1(\hat{f}^{n-1}\mathbf{y})^{\dim D_2 + \dim D_3}$$

with C_1 independent of n. By (7), (11b) and the fact that $f^j\mathbf{y} \in \Lambda$ for some $j \in [(1-\varepsilon_3)n, n)$, we have

$$|f^{n-1}(\tilde{W}_n(\mathbf{y}))| \ge B(\hat{f}^{n-1}(\mathbf{y}))|f^{n-1}(\tilde{W}_n(\mathbf{y}))|'$$
$$\ge M_2\, e^{-\varepsilon_3(n-1)} C_1\, e^{-6\varepsilon(n-1)\dim M}(M_2^{\dim M})\, e^{-\varepsilon_3(n-1)\dim M}$$
$$\ge C_2\, e^{-8\varepsilon(n-1)\dim M}.$$

This gives

$$|f^{n-1}(W \cap W^s(n, U))| \ge (\text{card } E'_n) C_2\, e^{-8\varepsilon(n-1)\dim M}.$$

For ε small enough, this gives $h_\mu(\hat{f}) < G(\gamma, f, U) + \varepsilon_0$ to complete the proof of theorem 1. □

Remark. The methods of the proof of theorem 1 and some more or less well known volume estimates can be used to extend Przytycki's inequality for the topological

https://doi.org/10.1017/S0143385700009469 Published online by Cambridge University Press

entropy of a diffeomorphism [11] to the case of $C^{1+\alpha}$-mappings. The inequality in question is the following. Let $\Lambda(T_xf)$ denote the induced map of the linear map T_xf on the full exterior algebra of the tangent space T_xM. Let λ denote Lebesgue measure in M. Przytycki's inequality states that

$$h(f)\le \limsup_{n\to\infty}\frac{1}{n}\log\int_M |\Lambda(T_xf^n)|\,d\lambda(x)$$

for a $C^{1+\alpha}$-diffeomorphism $f: M\to M$. To extend this for a $C^{1+\alpha}$-map in general, one proceeds as follows. From the proof of theorem 1 it is sufficient to show that

$$\operatorname{card}(E_n\cap B_{M_2}(\mathbf{x}_j))\le C\,e^{\varepsilon n}\int_M |\Lambda(T_xf^n)|\,d\lambda(x),$$

where ε is small. Let $\eta_n(\mathbf{y})$ be the Lebesgue measure of the set $\tilde{B}_n(\mathbf{y})$ defined in the proof of theorem 1. Then $C_1\,e^{-\varepsilon n}\le \eta_n(\mathbf{y})|\Lambda^k(T_xf^n)|\le C_2\,e^{\varepsilon n}$ for $n\ge 0$, where $k=\dim E_3(\mathbf{x}_j)$, ε is small and $x\in\tilde{B}_n(\mathbf{y})$. Thus

$$\begin{aligned}
h_\mu(f)&\le \limsup_{n\to+\infty}\frac{1}{n}\log\sum_{\mathbf{y}\in E_n\cap B_{M_2}} 1+\varepsilon_0\\
&\le \limsup_{n\to\infty}\frac{1}{n}\log\sum_{\mathbf{y}\in E_n\cap B_{M_2}} C_1^{-1}\,e^{\varepsilon n}\eta_n(\mathbf{y})\inf_{x\in\tilde{B}_n(\mathbf{y})}|\Lambda^k(T_xf^n)|+\varepsilon_0\\
&\le \limsup_{n\to\infty}\frac{1}{n}\log C_1^{-1}\,e^{\varepsilon n}\int_M |\Lambda^k(T_xf^n)|\,d\lambda(x)+\varepsilon_0,
\end{aligned}$$

where C_1, C_2 are constants and ε, ε_0 are arbitrarily small.

Proof of theorem 2. This is the same as the proof of theorem 1 except that we must only use discs γ in a holomorphically ample family $\mathcal{A}$. For this it suffices to show that $E_2(\mathbf{x})\oplus E_3(\mathbf{x})$ is a complex linear subspace of the complex tangent space $(T_xM)_{\mathbb{C}}$. As above, let $\pi:\hat{\Omega}\to\Omega$ be the projection. Let $(T\Omega)_{\mathbb{C}}$ be the restriction of the complex tangent bundle $(TM)_{\mathbb{C}}$ to Ω, and let $\tau: (T\Omega)_{\mathbb{C}}\to\Omega$ be the projection. We may assume (excluding a μ-null set) that the pull-back bundle $\pi^*(\tau,(T\Omega)_{\mathbb{C}},\Omega)$ is $\hat{\Omega}\times\mathbb{C}^m$ with $m=\dim_{\mathbb{C}} M$. Let $T\hat{f}:\hat{\Omega}\times\mathbb{C}^m\to\hat{\Omega}\times\mathbb{C}^m$ be defined by

$$T\hat{f}(\mathbf{x},v)=(\hat{f}\mathbf{x},T_xf(v))$$

as before. Then $E_2(\mathbf{x})\oplus E_3(\mathbf{x})$ is the set of vectors v in $\mathbb{C}^m$ such that there is a sequence $v_0,v_1,v_2,\ldots$ such that $v_0=v$, $T_{\pi\hat{f}^{-n}\mathbf{x}}f(v_n)=v_{n-1}$ for $n\ge 1$, and $\limsup_{n\to\infty}(1/n)\log|v_n|\le 0$. Since each T_zf is complex linear, it follows that $v\in E_2(\mathbf{x})\oplus E_3(\mathbf{x})$ implies $\sqrt{-1}v\in E_2(\mathbf{x})\oplus E_3(\mathbf{x})$. Thus the latter space is complex linear. □

3. *Examples*

We present some examples to show that the inequalities in theorems 3 and 4 are sharp. Let $I=[0,1]$ be the unit interval, and let $\alpha>0$. Let g_d be a real polynomial of degree d with the following properties:

(1) There are d disjoint closed intervals $I_1,\ldots,I_d$ contained in I such that $g_d(I_i)=I$ for $i=1,\ldots,d$.

(2) For $x \in I_i$, $|(d/dx)g_d(x)| > 1 + \alpha$.

(3) For $x \in I \backslash \bigcup_{j=1}^{d} I_j$ and $n \geq 1$ we have $g_d^n(x) \notin I$.

To construct g_d, just take a polynomial with critical points at $\{k/d : 0 < k < d\}$ and $|g_d(k/d)|$ large for each $0 < k < d$, $g_d(0) = 0$, $g_d(1/d) > 0$, $(d/dx)g_d(0) > 0$, $g(1) = 0$ for d even and $g(1) = 1$ for d odd. It is well known and easily verified that $\Omega_d = \bigcap_{k \geq 0} g_d^{-k}(I)$ is a compact g_d-invariant set with $h(g_d | \Omega_d) = \log d$. If

$$f = \underbrace{g_d \times g_d \times \cdots \times g_d}_{N \text{ times}} : \mathbb{R}^N \to \mathbb{R}^N,$$

then f is a polynomial map whose component functions have degree d. Also

$$h(f | \Omega_d \times \Omega_d \times \cdots \times \Omega_d) = N \log d,$$

so the first inequality in theorem 2 is sharp.

Now consider the map

$$f_b(y_1, \ldots, y_{N+1}) = (y_2, g_d(y_2) - by_3, g_d(y_3) - by_4, \ldots, g_d(y_{N+1}) - by_1).$$

For $b = 0$

$$\begin{aligned} f_b(y_1, \ldots, y_{N+1}) &= f_0(y_1, \ldots, y_{N+1}) \\ &= (y_2, g_d(y_2), g_d(y_3), \ldots, g_d(y_{N+1})) \\ &= \tilde{g} \times \underbrace{g_d \times \cdots \times g_d}_{(N-1) \text{ times}}, \end{aligned}$$

where $\tilde{g}(y_1, y_2) = (y_2, g_d(y_2))$. It is easily seen that $\tilde{g}$ collapses the plane onto the graph Γ of g_d. Let $\hat{\Omega} = \bigcap_{n \in Z} \tilde{g}^n([0, 1] \times [0, 1])$. Then $\hat{\Omega}$ is a compact zero-dimensional subset of Γ, and for $x \in \Gamma$, $T\tilde{g}$ uniformly expands non-zero tangent vectors to Γ at x. Also $h(\tilde{g} | \hat{\Omega}) = \log d$. Furthermore, Tf_0 uniformly expands non-zero vectors in

$$T\Gamma \times \underbrace{T\mathbb{R}^1 \times \cdots \times T\mathbb{R}^1}_{(N-1) \text{ times}}$$

at points $(y_2, g_d(y_2), y_3, \ldots, y_{N+1})$ in

$$\hat{\Omega} \times \underbrace{\Omega_d \times \cdots \times \Omega_d}_{(N-1) \text{ times}}$$

and, if Ω' is defined to be

$$\hat{\Omega} \times \underbrace{\Omega_d \times \cdots \times \Omega_d}_{(N-1) \text{ times}},$$

then $h(f_0 | \Omega') = N \log d$. For $b \neq 0$ and small, f_b is injective and it is a small perturbation of the map f_0 which is hyperbolic over Ω'. With a slight modification of known techniques, one can show that f_b has a compact invariant set Ω'_b near Ω_0 such that $f_b | \Omega'_b$ is topologically conjugate to the inverse limit of the map f_0: $\Omega' \to \Omega'$. Thus $h(f_b | \Omega'_b) = h(f_0 | \Omega') = N \log d$ and the second inequality in theorem 2 is sharp.

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