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INVERSE SEMIGROUP HOMOMORPHISMS VIA PARTIAL GROUP ACTIONS

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This paper constructs all homomorphisms of inverse semigroups which factor through an E-unitary inverse semigroup; the construction is in terms of a semilattice component and a group component. It is shown that such homomorphisms have a unique factorisation $\beta\alpha$ with α preserving the maximal group image, β idempotent separating, and the domain I of β E-unitary; moreover, the P-representation of I is explicitly constructed. This theory, in particular, applies whenever the domain or codomain of a homomorphism is E-unitary. Stronger results are obtained for the case of F-inverse monoids.

Special cases of our results include the P-theorem and the factorisation theorem for homomorphisms from E-unitary inverse semigroups (via idempotent pure followed by idempotent separating). We also deduce a criterion of McAlister–Reilly for the existence of E-unitary covers over a group, as well as a generalisation to F-inverse covers, allowing a quick proof that every inverse monoid has an F-inverse cover.

1. Introduction and Main Results

The class of E-unitary inverse semigroups has received special attention in the semigroup theory literature. These are inverse semigroups with an idempotent pure homomorphism to a group. McAlister's P-theorem [**5**], under a reformulation below, states that all E-unitary inverse semigroups can be constructed as the "semidirect product" of a group and a semilattice where the group acts partially on the semilattice. A result of Munn and Reilly [**8**] shows that every homomorphism from an E-unitary inverse semigroup factors as an idempotent pure homomorphism followed by an idempotent separating homomorphism.

In this paper, we generalise both these results simultaneously. If S is an inverse semigroup, σ_S will denote its minimal group congruence, $G(S)$ its maximal group image and $E(S)$ its set of idempotents. We give an explicit construction of all homomorphisms from S which factor through an E-unitary inverse semigroup. The construction builds

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the homomorphism out of pieces defined on $E(S)$ and on $G(S)$. More precisely (undefined terms will be defined in the text):

THEOREM 1.1. *Let $\varphi : S \to T$ be a homomorphism of inverse semigroups. Then the following are equivalent:*

1. *The image under φ of each σ_S-class of S is compatible;*
2. *$\varphi = \beta_0\alpha_0$ with $\alpha_0 : S \to I_0$, $\beta_0 : I_0 \to T$ homomorphisms and I_0 E-unitary;*
3. *$\varphi = \beta\alpha$ with $\alpha : S \to I$ a surjective, maximal group image preserving homomorphism, $\beta : I \to T$ idempotent separating, and I E-unitary;*
4. *There is a dual prehomomorphism $\rho : G(S) \to C(T)$ such that $\varphi(s) \leqslant \rho(\sigma(s))$ for all $s \in S$;*
5. *There is a compatible pair $\psi : E(S) \to E(T)$, $\rho : G(S) \to C(T)$ such that $\varphi(s) = \psi(ss^{-1})\rho(\sigma(s))$.*

Moreover, α and β are unique, there is a unique minimal choice of (ψ, ρ), and I is the P-semigroup $P_{(\psi,\rho)}$ associated to (ψ, ρ).

Note that by considering the case when S is E-unitary, we obtain the aforementioned result of Munn and Reilly [**8**]. Theorem 1.1 allows the construction of a P-representation of an E-unitary inverse semigroup from any idempotent separating image.

Roughly speaking, a compatible pair is a way of coordinatising a homomorphism into two components: one component is a semilattice homomorphism; the other is a dual prehomomorphism from a group. In the case of an E-unitary inverse semigroup acting on a set X, the theorem (or rather a variant thereof proved below) states that the action is obtained from a partial group action of the maximal group image (see [**2, 4, 11**] for the importance of partial group actions) by restricting the domains of the various group elements. More specifically, we have

THEOREM 1.2. *Let S be an inverse semigroup and X a set. Suppose S acts on X with the property: $s\ \sigma_S\ t$ implies $s\cdot$ and $t\cdot$ agree on the overlap of their domains. Then there is a partial action of $G(S)$ on X and an action of $E(S)$ on S such that, for all $s \in S$, the action of $\sigma(s)(s^{-1}s)\sigma(s)$ is the action of ss^{-1}, and $sx = (ss^{-1})\sigma(s)x$; the converse holds as well. In particular, if S is E-unitary, this is always the case.*

The action of $E(S)$ in the above theorem is unique, while there is a unique minimal choice of the partial group action (where minimality is in terms of the size of the domain of the action of the elements).

It is shown in [**3**], based on earlier work of Exel [**2**], that dual prehomomorphisms $\varphi : G \to T$, where G is a group, correspond bijectively to homomorphisms $\psi : G^{\mathrm{Pr}} \to T$ where G^{Pr} is the prefix expansion of Birget–Rhodes [**1**]. This is an F-inverse monoid with maximal group image G, finite $\mathcal{D}$-classes, and a transparent structure. Thus, in some sense, homomorphisms from arbitrary E-unitary inverse semigroups can be reduced to homomorphisms from semilattices and from F-inverse monoids with finite $\mathcal{D}$-classes.

As a corollary of Theorem 1.1 we shall be able to deduce the following result of McAlister and Reilly [**6**].

Corollary 1.3. *An inverse semigroup S has an E-unitary cover over a group G if and only if there is a dual prehomomorphism $\rho : G \to C(S)$ such that, for $s \in S$, $s \leqslant \rho(g)$ for some $g \in G$.*

We note that the equivalence of 1, 2, and 3 can be deduced in a straightforward manner from the usual factorisation theorem [**4**, Theorem 7.5.3]. But we prefer the constructive approach afforded by compatible pairs since it both builds explicitly the P-representation of I, and provides a simple coordinatisation into semilattice and group components.

We also extend Theorem 1.1 to F-inverse monoids (where things simplify because $C(T)$ is not needed). In particular we shall be able to deduce the following generalisation of Corollary 1.3.

Corollary 1.4. *An inverse monoid S has an F-inverse cover over a group G if and only if there is a dual prehomomorphism $\rho : G \to S$ such that, for $s \in S$, $s \leqslant \rho(g)$ for some $g \in G$.*

We shall then easily obtain the following result of [**8**] which clearly implies the McAlister-Tilson covering theorem.

Theorem 1.5. *Let S be an inverse monoid, then S has an F-inverse cover. As a consequence, every inverse semigroup has an E-unitary cover.*

2. Preliminaries and Notation

Fix an inverse semigroup S. Recall that the *natural partial order* on S is given by $s \leqslant t$ if $s = ss^{-1}t$. We let $\sigma : S \to G(S)$ denote the canonical projection.

One says that $s, t \in S$ are *compatible*, written $s \sim t$, if $st^{-1}, s^{-1}t \in E(S)$. Intuitively, if S acts faithfully by partial bijections on the left of a set X, then s and t are compatible precisely when their union (as a relation) is a partial bijection. The relation $\sim$ is not in general transitive; in fact, S is E-unitary if and only if $\sim$ is transitive, in which case $\sim = \sigma_S$ [**4**, Theorems 2.4.4 and 2.4.6].

A subset A of S is said to be *compatible* if each pair of its elements is compatible. Following Schein [**9**], a subset which is a compatible order ideal is called *permissible*. For example, for $s \in S$, $[s] = \{t \in S \mid t \leqslant s\}$ is easily seen to be permissible. If S is E-unitary, $\sigma^{-1}(g)$ is permissible for all $g \in G(S)$. One says that S is *complete* if each compatible subset has a join. For instance, the symmetric inverse monoid on X, $I(X)$, is complete. Observe that if A is any subset of $I(X)$ having a join, A must be compatible.

An inverse semigroup is called *infinitely distributive* if multiplication distributes over arbitrary joins (whenever they exist). One can show S is infinitely distributive if and only if $E(S)$ is. Thus $I(X)$ is infinitely distributive. Let $C(S)$ be the set of permissible subsets of S. Then $C(S)$ is an inverse semigroup under setwise multiplication. There is a natural embedding of S in $C(S)$ by $s \mapsto [s]$. One can show [**4**, Theorems 1.4.24 and 1.4.25] that $C(S)$ is complete and infinitely distributive, and that the above embedding is universal. The Preston–Wagner representation affords another embedding of S into a complete, infinitely distributive inverse semigroup.

A map $\varphi : S \to T$ is called a *prehomomorphism* if $\varphi(st) \leqslant \varphi(s)\varphi(t)$. This is equivalent to asking that φ preserve order and that if $s^{-1}s = tt^{-1}$, then $\varphi(st) = \varphi(s)\varphi(t)$ [**4**]. Also φ is a homomorphism if and only if it preserves products of idempotents.

One says that a map $\varphi : S \to T$ of inverse semigroups is a *dual prehomomorphism* if $\varphi(s)\varphi(t) \leqslant \varphi(st)$ and $\varphi(s^{-1}) = \varphi(s)^{-1}$. Dual prehomomorphisms take idempotents to idempotents. Indeed

$$\varphi(e) = \varphi(e)\varphi(e)^{-1}\varphi(e) = \varphi(e)^3 \leqslant \varphi(e^3) = \varphi(e).$$

If $\varphi : S \to T$ is a homomorphism, we set $\ker\varphi = \varphi^{-1}(E(T))$ and $\mathrm{tr}(\varphi) = \varphi|_{E(S)}$. We use a similar notation for congruences. Any congruence is uniquely determined by its kernel and its trace [**4**]. A surjective, idempotent separating homomorphism is called a *cover*.

If $(X, \leqslant)$ is a partially ordered set, we use $I(X, \leqslant)$ for the inverse monoid of partial order isomorphisms of X whose domains are order ideals. If G is a group, a partial action of G on X consists of a dual prehomomorphism $\rho : G \to I(X, \leqslant)$ with $\rho(1) = 1$. One usually writes gx for $\rho(g)(x)$. If X is set ordered by equality, this is the usual notion (see [**2, 4**]). If X is a semilattice, then the partial action automatically preserves meets.

If Y is a semilattice and G is a group acting partially on Y with the property that, for each $g \in G$, there exists $y \in Y$ with $g^{-1}y$ defined, then one can define an inverse semigroup

$$P(Y, G) = \{(y, g) \in Y \times G \mid \exists g^{-1}y\}.$$

One defines a product by

$$(y, g)(x, h) = \big(g(g^{-1}y \wedge x), gh\big).$$

It is easy to check this is well defined and that $(y, g)^{-1} = (g^{-1}y, g^{-1})$. Furthermore, the projection to G is an idempotent pure, surjective homomorphism whence $P(Y, G)$ is E-unitary. The P-theorem states that all E-unitary inverse semigroups arise in this manner. Note: the usual statement of the P-theorem differs slightly from, but is equivalent to, ours; see [**3**].

3. Constructing Homomorphisms

We begin with a general method of constructing homomorphisms sending σ_S-classes to compatible subsets; of course, we aim to show that all such homomorphisms so arise.

3.1. Compatible Pairs Let S, T be inverse semigroups. Suppose $\psi : E(S) \to E(T)$ is a homomorphism and $\rho : G(S) \to T$ is a dual prehomomorphism. We say (ψ, ρ) is a *compatible pair* if

$$\rho(g)\psi(s^{-1}s)\rho(g)^{-1} = \psi(ss^{-1}) \text{ for all } s \in \sigma^{-1}(g). \tag{3.1}$$

The intuition behind this is to axiomatise what is necessary to be able to define a homomorphism from $P(Y, G)$ to T by taking the product of a map defined on Y and a map defined on G.

Proposition 3.1. *Let (ψ, ρ) be a compatible pair.*

1. *$\psi(s^{-1}s) \leqslant \rho(\sigma(s))^{-1}\rho(\sigma(s))$ whence $e \leqslant s^{-1}s$ for some $s \in \sigma^{-1}(g)$ implies $\psi(e) \leqslant \rho(g)^{-1}\rho(g)$.*
2. *$e \leqslant s^{-1}s \implies \rho(\sigma(s))\psi(e)\rho(\sigma(s))^{-1} = \psi(ses^{-1})$.*
3. *$G(S)$ acts partially on $\psi(E(S))$ by $gf = \rho(g)f\rho(g)^{-1}$ if $f = \psi(e)$ with $e \leqslant s^{-1}s$ for some $s \in \sigma^{-1}(g)$. Also, if $s \in \sigma^{-1}(g)$, then $g^{-1}\psi(ss^{-1})$ is defined.*

Proof: For 1, observe that

$$\psi(s^{-1}s)\rho(g)^{-1}\rho(g) = \rho(g)^{-1}\rho(g)\psi(s^{-1}s)\rho(g)^{-1}\rho(g) = \psi(s^{-1}s)$$

by two applications of (3.1). The second statement is clear.

For 2,

$$\rho(g)\psi(e)\rho(g)^{-1} = \rho(g)\psi\big((se)^{-1}(se)\big)\rho(g)^{-1} = \psi\big((se)(se)^{-1}\big)$$

by (3.1).

For 3, observe first that $g\psi(e) \in \psi(E(S))$ by 2. To see that $g\cdot$ has domain an order ideal and that it preserves order, let $e \leqslant s^{-1}s$ with $s \in \sigma^{-1}(g)$ and $\psi(f) \leqslant \psi(e)$. Then $\psi(ef) = \psi(f)$ and $ef \leqslant s^{-1}s$ whence $g\psi(f)$ is defined. Moreover, $sefs^{-1} \leqslant ses^{-1}$ so $g\psi(f) \leqslant g\psi(e)$ by 2. Note that $1\psi(e) = \psi(e)$ by (3.1) with $s = e$.

Observe that if $e \leqslant s^{-1}s$, then $ses^{-1} \leqslant ss^{-1}$ and $s^{-1}ses^{-1}s = e$. It immediately follows that, for $f \in \psi(E(S))$, $g^{-1}(gf) = f$ whenever gf is defined whence $g\cdot$ and $g^{-1}\cdot$ are inverses.

Finally, suppose $h(gf)$ is defined with $f \in \psi(E(S))$. Since $h(gf)$ and $g^{-1}(gf)$ are defined, there exist $e, e' \in E(S)$ with $\psi(e) = gf = \psi(e')$, $e \leqslant t^{-1}t$ some $t \in \sigma^{-1}(h)$, and $e' \leqslant ss^{-1}$ some $s \in \sigma^{-1}(g)$. Then $ee' \leqslant ss^{-1}, t^{-1}t$ and $\psi(ee') = gf$ whence $h(gf) = \psi(tee't^{-1})$. Also $f = g^{-1}\psi(ee') = \psi(s^{-1}ee's)$. But

$$s^{-1}ee's \leqslant s^{-1}(ss^{-1}t^{-1}t)s = (ts)^{-1}ts \text{ and } ts \in \sigma^{-1}(hg), \text{ so}$$
$$(hg)f = \psi\big((ts)s^{-1}ee's(ts)^{-1}\big) = \psi(tee't^{-1}) = h(gf).$$

The last statement is clear. ∎

Thus, given a compatible pair (ψ, ρ), we can define the associated E-unitary inverse semigroup $P_{(\psi,\rho)} = P(\psi(E(S)), G(S))$. Observe that Proposition 3.1 (1) implies that if $g\psi(e)$ is defined, $\psi(e) \leqslant \rho(g)^{-1}\rho(g)$. We shall use this repeatedly in the sequel.

THEOREM 3.2. *Let (ψ, ρ) be a compatible pair. Then:*

1. *$\alpha : S \to P_{(\psi,\rho)}$ defined by $\alpha(s) = (\psi(ss^{-1}), \sigma(s))$ is a surjective homomorphism, preserving maximal group images. Moreover, if S is E-unitary, α is injective if and only if ψ is;*
2. *$\beta : P_{(\psi,\rho)} \to T$ given by $\beta(f, g) = f\rho(g)$ is an idempotent separating homomorphism;*
3. *$\beta\alpha(e) = \psi(e)$ for $e \in E(S)$.*

PROOF: By Proposition 3.1 (3), α is well defined. Since ψ is a homomorphism, α induces a homomorphism on the idempotents. So it suffices to show α is a prehomomorphism. We first observe that (3.1) implies $\sigma(s)^{-1}\psi(ss^{-1}) = \psi(s^{-1}s)$. Suppose $s^{-1}s = tt^{-1}$. Then

$$(3.2) \qquad (\psi(ss^{-1}), \sigma(s))(\psi(tt^{-1}), \sigma(t)) = \Big(\sigma(s)\Big((\sigma(s)^{-1}\psi(ss^{-1}))\psi(tt^{-1})\Big), \sigma(st)\Big).$$

The first coordinate of the righthand side of (3.2) is then

$$\sigma(s)(\psi(s^{-1}s)\psi(tt^{-1})) = \sigma(s)\psi(s^{-1}s) = \psi(ss^{-1}).$$

Since $(st)(st)^{-1} = ss^{-1}$, it follows that α is a prehomomorphism. To see that α is surjective, note that if $(f, g) \in P_{(\psi,\rho)}$, then, since $g^{-1}f$ is defined, $f = \psi(e)$ with $e \leqslant ss^{-1}$ for some $s \in \sigma^{-1}(g)$. But

$$\alpha(es) = (\psi(ess^{-1}e), \sigma(es)) = (f, g)$$

so α is surjective. If α is injective, ψ must be. Suppose S is E-unitary and ψ is injective. Then α is idempotent separating, so it suffices to show α is idempotent pure. But if $\alpha(s) = (\psi(ss^{-1}), \sigma(s))$ is an idempotent, then $\sigma(s) = 1$ and hence $s \in E(S)$.

To prove 2, consider

$$(3.3) \qquad \beta(e, g)\beta(f, h) = e\rho(g)f\rho(h).$$

Since $g^{-1}e$ is defined $e \leqslant \rho(g)\rho(g)^{-1}$, whence the righthand side of (3.3) is

$$(3.4) \qquad \rho(g)\rho(g)^{-1}e\rho(g)f\rho(h) = \rho(g)(g^{-1}e)f\rho(h).$$

But since $g(g^{-1}e)$ is defined and the domain of $g\cdot$ is an order ideal, $g((g^{-1}e)f)$ is defined whence $(g^{-1}e)f \leqslant \rho(g)^{-1}\rho(g)$. This lets us transform the righthand side of (3.4) into

$$\rho(g)(g^{-1}e)f\rho(g)^{-1}\rho(g)\rho(h) = g((g^{-1}e)f)\rho(g)\rho(h).$$

Observe that $g^{-1}e \leqslant \rho(g)^{-1}\rho(g)$ and $f \leqslant \rho(h)\rho(h)^{-1}$ (recall: $h^{-1}e$ is defined) imply

$$g\big((g^{-1}e)f\big) \leqslant \rho(g)\big(\rho(g)^{-1}\rho(g)\rho(h)\rho(h)^{-1}\big)\rho(g)^{-1} = \rho(g)\rho(h)\big(\rho(g)\rho(h)\big)^{-1}. \tag{3.5}$$

Denote the left hand side of (3.5) by q. Using $\rho(g)\rho(h) \leqslant \rho(gh)$, we see

$$q\rho(g)\rho(h) = q\big(\rho(g)\rho(h)\big)\big(\rho(g)\rho(h)\big)^{-1}\rho(gh) = q\rho(gh).$$

On the other hand, $(e,g)(f,h) = (q,gh)$, so β is a homomorphism. To see that β is idempotent separating, first observe that $\rho(1)$ is an idempotent and, for $e \in E(S)$, $\psi(e) \leqslant \rho(1)$ by Proposition 3.1 (1). Hence $\beta(\psi(e),1) = \psi(e)\rho(1) = \psi(e)$. We now deduce 3 since $\beta\alpha(e) = \beta(\psi(e),1)$. □

As an immediate corollary, we have:

Corollary 3.3. *Let (ψ,ρ) be a compatible pair. Set $\varphi(s) = \psi(ss^{-1})\rho\big(\sigma(s)\big)$. Then φ is a homomorphism; in fact, $\varphi = \beta\alpha$ where α is a surjective, maximal group image preserving homomorphism and β is idempotent separating with E-unitary domain.*

Note that Theorem 3.2 shows that to prove the P-theorem, we just need to show that given an E-unitary inverse semigroup S, there is a compatible pair (ψ,ρ) with ψ an injective homomorphism. We shall soon see that such a pair can be constructed from any idempotent separating congruence on S.

Lemma 3.4. *Let $\varphi : S \to T$ be a homomorphism.*

1. $s_1\,\sigma_S\,s_2 \implies \varphi(s_1)\,\sigma_T\,\varphi(s_2)$.
2. $s \sim t \implies \varphi(s) \sim \varphi(t)$.

Proof: For 1, if $u \leqslant s_1, s_2$, then $\varphi(u) \leqslant \varphi(s_1), \varphi(s_2)$ so $\varphi(s_1)\,\sigma_T\,\varphi(s_2)$.

For 2, $st^{-1}, s^{-1}t \in E(S)$, implies $\varphi(s)\varphi(t)^{-1} = \varphi(st^{-1}) \in E(T)$ and, dually, $\varphi(s)^{-1}\varphi(t) \in E(T)$. □

Corollary 3.5. *Let $\alpha : S \to I$ and $\beta : I \to T$ be homomorphisms of inverse semigroups with I E-unitary. Then, for $g \in G(S)$, $\beta\alpha\big(\sigma^{-1}(g)\big)$ is compatible.*

Proof: By Lemma 3.4 (1), $\alpha\big(\sigma^{-1}(g)\big)$ is contained in a single σ_I-class. Since I is E-unitary, it follows $\alpha\big(\sigma^{-1}(g)\big)$ is compatible whence, by Lemma 3.4 (2), $\beta\alpha\big(\sigma^{-1}(g)\big)$ is compatible. □

In particular, $\varphi = \beta\alpha$, constructed above, sends σ_S-classes to compatible subsets.

We now state a lemma which is useful in constructing compatible pairs.

Lemma 3.6. *Suppose $\varphi : S \to T$ is a homomorphism and $\rho : G(S) \to T$ is a dual prehomomorphism such that $\varphi(s) \leqslant \rho\big(\sigma(s)\big)$ for all $s \in S$. Then (ψ,ρ) is a compatible pair, where $\psi = \varphi|_{E(S)}$, and $\varphi = \beta\alpha$; that is, $\varphi(s) = \psi(ss^{-1})\rho\big(\sigma(s)\big)$ for all $s \in S$.*

Proof: Since $\varphi(s) \leqslant \rho\big(\sigma(s)\big)$, the following equalities hold:

1. $\rho\big(\sigma(s)\big)\varphi(s)^{-1}\varphi(s) = \varphi(s)$;

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2. $\varphi(s)^{-1}\varphi(s)\rho(\sigma(s))^{-1} = \varphi(s)^{-1}$;
3. $\varphi(s) = \varphi(s)\varphi(s)^{-1}\rho(\sigma(s))$.

By taking products of the corresponding sides of 1 and 2, we obtain

$$\rho(\sigma(s))\psi(s^{-1}s)\rho(\sigma(s))^{-1} = \psi(ss^{-1}),$$

verifying (3.1). To complete the proof, observe that 3 states precisely that $\varphi(s) = \psi(ss^{-1})\rho(\sigma(s))$. ∎

We now define an ordering on compatible pairs; the definition is motivated by Lemma 3.7 below. For compatible pairs (ψ_1, ρ_1), (ψ_2, ρ_2) (where $\psi_i : E(S) \to E(T)$, $\rho_i : G(S) \to T$, $i = 1, 2$), we write $(\psi_1, \rho_1) \leqslant (\psi_2, \rho_2)$ if $\psi_1 = \psi_2$ and $\rho_1(g) \leqslant \rho_2(g)$ for all $g \in G(S)$. This is clearly a partial order on the set of compatible pairs.

Lemma 3.7. *Suppose $(\psi_1, \rho_1) \leqslant (\psi_2, \rho_2)$. Then both pairs induce the same factorisation $\beta\alpha : S \to T$. Furthermore, for any compatible pairs (ψ_1, ρ_1), (ψ_2, ρ_2) with $\psi_1(ss^{-1})\rho_1(\sigma(s)) = \psi_2(ss^{-1})\rho_2(\sigma(s))$, $\psi_1 = \psi_2$.*

Proof: Let us write the associated factorisations as $\beta_1\alpha_1$ and $\beta_2\alpha_2$. Also, let $\psi = \psi_1 = \psi_2$. First we show the actions are the same. Let $e \leqslant s^{-1}s$ with $s \in \sigma^{-1}(g)$. Then

$$\rho_1(g)\psi(e)\rho_1(g)^{-1} = \psi(ses^{-1}) = \rho_2(g)\psi(e)\rho_2(g)^{-1}$$

by Proposition 3.1 (2). It follows now that $\alpha_1 = \alpha_2$ so we may drop the subscripts. Since α is onto, to show $\beta_1 = \beta_2$ it suffices to show $\beta_1\alpha = \beta_2\alpha$. Now $\psi(ss^{-1}) \leqslant \rho_1(\sigma(s))\rho_1(\sigma(s))^{-1}$ by Proposition 3.1 (1). So

$$\begin{aligned}\beta_2\alpha(s) = \psi(ss^{-1})\rho_2(\sigma(s)) &= \psi(ss^{-1})\rho_1(\sigma(s))\rho_1(\sigma(s))^{-1}\rho_2(\sigma(s)) \\ &= \psi(ss^{-1})\rho_1(\sigma(s)) = \beta_1\alpha(s),\end{aligned}$$

the penultimate equality following because $\rho_1(\sigma(s)) \leqslant \rho_2(\sigma(s))$.

The last statement follow from Proposition 3.1 (3). ∎

3.2. The Complete, Infinitely Distributive Case We begin with the case T is complete and infinitely distributive. This theorem is inspired by Lawson and Kellendonk's rendition [**3**] of the P-theorem in terms of partial group actions (which, in turn, was inspired by Schein's and Munn's proofs of the P-theorem [**10, 7**]). Fix a homomorphism $\varphi : S \to T$ of inverse semigroups such that each σ_S-class is sent to a compatible subset and such that T is complete and infinitely distributive.

Lemma 3.8. *Define $\rho : G(S) \to T$ by*

$$\rho(g) = \bigvee \varphi(\sigma^{-1}(g)).$$

Then ρ is a dual prehomomorphism.

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PROOF: Indeed,

$$\rho(g_1)\rho(g_2) = \bigvee \varphi(\sigma^{-1}(g_1)) \bigvee \varphi(\sigma^{-1}(g_2)) = \bigvee \varphi(\sigma^{-1}(g_1))\varphi(\sigma^{-1}(g_2)) \tag{3.6}$$

by [4, Proposition 1.4.20]. Since $\sigma^{-1}(g_1)\sigma^{-1}(g_2) \subseteq \sigma^{-1}(g_1g_2)$, it follows that the right hand side of (3.6) is less than $\bigvee \sigma^{-1}(g_1g_2) = \rho(g_1g_2)$.

Also, $\rho(g^{-1}) = \bigvee \varphi(\sigma^{-1}(g^{-1})) = \bigvee \varphi(\sigma^{-1}(g))^{-1} = \rho(g)^{-1}$ since inversion preserves joins. ∎

THEOREM 3.9. *Let $\varphi : S \to T$ be a homomorphism of inverse semigroups such that each σ_S-class is sent to a compatible subset and T is complete and infinitely distributive. Then there is a unique minimal compatible pair $\psi : E(S) \to E(T)$, $\rho : G(S) \to T$ such that $\varphi = \beta\alpha$ (constructed as above). Also $\beta\alpha$ is the unique factorisation as a maximal group image preserving homomorphism followed by an idempotent separating homomorphism with the domain of β being E-unitary.*

PROOF: Define ρ as above and let $\psi = \varphi|_{E(S)}$. Then, since $\varphi(s) \leqslant \rho(\sigma(s))$, (ψ, ρ) is compatible and $\varphi = \beta\alpha$ by Lemma 3.6.

It remains to prove the uniqueness statements. Suppose (ψ', ρ') is another compatible pair with $\varphi(s) = \psi'(ss^{-1})\rho'(\sigma(s))$; then $\psi = \psi'$ by Lemma 3.7. Now, for $g \in G$, $s \in \sigma^{-1}(g)$, $\varphi(s) = \psi(ss^{-1})\rho'(g) \leqslant \rho'(g)$. Thus

$$\rho(g) = \bigvee \varphi(\sigma^{-1}(g)) \leqslant \rho'(g)$$

whence $(\psi, \rho) \leqslant (\psi', \rho')$ as desired.

As to the uniqueness of α and β, it suffices to show that the congruence determined by α is unique. We proceed by examining the trace and kernel. Suppose $\equiv$ is a congruence such that the projection to $S/{\equiv}$ preserves the maximal group image, $S/{\equiv}$ is E-unitary, and the projection from $S/{\equiv}$ to T is idempotent separating. Then $\mathrm{tr}(\equiv) = \mathrm{tr}(\varphi)$. Suppose $s \in \ker{\equiv}$, then $s\ \sigma_S\ e \in E(S)$ since $\equiv\ \subseteq \sigma_S$. Conversely, if $s\ \sigma_S\ e \in E(S)$, then $s\ \sigma_{S/\equiv}\ e$ whence, since $S/{\equiv}$ is E-unitary, $s \in \ker{\equiv}$. Thus $\sigma^{-1}(1) = \ker{\equiv}$. But $\mathrm{tr}(\alpha) = \mathrm{tr}(\varphi)$ and $\ker\alpha = \sigma^{-1}(1)$, so $\equiv$ is induced by α. ∎

Theorem 1.2 now follows since $I(X)$ is complete.

3.3. PROOF OF THEOREM 1.1 We are now prepared to prove Theorem 1.1. That 4 implies 5 is Lemma 3.6 (since $\psi = \varphi|_{E(S)}$) while 5 implies 4 follows from the calculation $\varphi(s) = \psi(ss^{-1})\rho(\sigma(s)) \leqslant \rho(\sigma(s))$. That 5 implies 3 is the content of Corollary 3.3 once we show that the range of β is contained in T (viewed as a subsemigroup of $C(T)$). But since α is onto and $\beta\alpha = \varphi$, the image of β is contained in T; 3 implies 2 is obvious; 2 implies 1 follows from Lemma 3.4. For 1 implies 4, we view $\varphi : S \to T$ as a homomorphism $\varphi' : S \to C(T)$. Then, by Theorem 3.9, we can find a compatible pair $\psi : E(S) \to E(C(T))$ and $\rho : G(S) \to C(T)$ such that $\varphi(s) = \varphi'(s) = \psi(ss^{-1})\rho(\sigma(s)) \leqslant \rho(\sigma(s))$.

The uniqueness statements follow from Theorem 3.9. ∎

3.4. Proof of Corollary 1.3 Suppose $\varphi : I \to S$ is an E-unitary cover such that $G = G(I)$. Then, by Theorem 1.1 (4), there exists a dual prehomomorphism $\rho : G \to C(S)$ such that $\varphi(t) \leqslant \rho(\sigma(t))$. Since φ is onto, the result follows.

Suppose now that $\rho : G \to C(S)$ is a dual prehomomorphism such that, for each $s \in S$, $s \leqslant \rho(g)$ some $g \in G$. Define

$$I = \{(s, g) \in S \times G \mid s \leqslant \rho(g)\}. \tag{3.7}$$

Then, since $\rho(1)$ is idempotent, $E(I) = E(S) \times 1$ whence the projection to G is idempotent pure and the projection to S is idempotent separating. ∎

4. F-inverse Monoids

We now specialise our results to the case of F-inverse monoids. An inverse semigroup S is an F-inverse monoid if each σ_S-class has a maximum; such a semigroup must be a monoid and E-unitary.

Theorem 4.1. *Let $\varphi : S \to T$ be a homomorphism of inverse semigroups. Then the following are equivalent:*

1. *The image under φ of each σ_S-class of S has a maximum;*
2. *$\varphi = \beta\alpha$ with $\alpha : S \to I$ a surjective, maximal group image preserving homomorphism, $\beta : I \to T$ idempotent separating, and I an F-inverse monoid;*
3. *There is a compatible pair $\psi : E(S) \to E(T)$, $\rho : G(S) \to T$ such that $\varphi(s) = \psi(ss^{-1})\rho(\sigma(s))$, and, for each $g \in G(S)$, there exists $f \in \psi(E(S))$ which is maximum such that $g^{-1}f$ is defined.*
4. *There is a dual prehomomorphism $\rho : G(S) \to T$ such that $\varphi(s) \leqslant \rho(\sigma(s))$ all $s \in S$ and $\rho(g) \in \varphi(\sigma^{-1}(g))$.*

Moreover, α and β are unique, there is a unique minimal choice of (ψ, ρ), and I is the P-semigroup $P_{(\psi,\rho)}$ associated to (ψ, ρ).

Proof: For 4 implies 3, Lemma 3.6 gives that (ψ, ρ), where $\psi = \varphi|_{E(S)}$, is a compatible pair and $\varphi(s) = \psi(ss^{-1})\rho(\sigma(s))$. Suppose $\rho(g) = \varphi(s)$ with $s \in \sigma^{-1}(g)$; then if $g^{-1}f$ is defined, $f \leqslant \rho(g)\rho(g)^{-1} = \psi(ss^{-1})$ so the second condition of 3 is satisfied. For 3 implies 2, it suffices to show that $P_{(\psi,\rho)}$ is an F-inverse monoid. But if $f \in \psi(E)$ is maximum with $g^{-1}f$ defined, then (f, g) is maximum in the $\sigma_{P_{(\psi,\rho)}}$-class of g. For 2 implies 1, let A be a σ_S-class. Then, by Lemma 3.4 (1), $\alpha(A)$ is contained in a σ_I-class. But since α is surjective, maximal group image preserving, $\alpha(A)$ must, in fact, be a σ_I-class. Let $t = \max(\alpha(A))$; we claim $\beta(t) = \max(\varphi(A))$ (note: $\beta(t) \in \varphi(A)$). Indeed, if $r \in \varphi(A)$, then $r = \beta\alpha(s)$ with $s \in A$ and $\alpha(s) \leqslant t$ whence $r \leqslant \beta(t)$. For 1 implies 4, define $\rho(g) = \max(\varphi(\sigma^{-1}(g)))$.

The uniqueness statements follow from Theorem 1.1 and observing that the minimal choice of ρ is given by $\rho(g) = \max(\varphi(\sigma^{-1}(g)))$. ∎

We note that Theorem 4.1 applies if S is F-inverse since if $\theta : G(S) \to S$ is defined by $\theta(g) = \max(\sigma^{-1}(g))$, then $\varphi(s) \leqslant \rho(\sigma(s))$ where $\rho = \varphi\theta$ and $\rho(g) = \varphi(\theta(g))$. Thus every homomorphism from an F-inverse monoid factors as an idempotent pure homomorphism onto an F-inverse monoid followed by an idempotent separating homomorphism, a result of Munn and Reilly [8]. Since free inverse monoids are F-inverse, any homomorphism from a free inverse monoid can be coordinatised in terms of a semilattice homomorphism from the subsemilattice of elements represented by Dyck words and a dual prehomomorphism from a free group.

4.1. Proof of Corollary 1.4 Suppose $\varphi : I \to S$ is an F-inverse cover such that $G = G(I)$. Then, by Theorem 4.1, there exists a dual prehomomorphism $\rho : G \to S$ such that $\varphi(t) \leqslant \rho(\sigma(t))$. Since φ is onto, the result follows.

Suppose now that $\rho : G \to S$ is a dual prehomomorphism such that, for each $s \in S$, $s \leqslant \rho(g)$ for some $g \in G$ and let I be as in (3.7). Then, as in the proof of Corollary 1.3, I is an inverse semigroup and the projection to S is idempotent separating. But clearly $(\rho(g), g)$ is the maximum element projection to g, so S is an F-inverse monoid. ∎

4.2. Proof of Theorem 1.5 Suppose S is an X-generated inverse monoid and let $FG(X)$ be a free group on X. Define $\rho : FG(X) \to S$ by taking a reduced word w to its equivalence class in S. Since deleting subwords of the form xx^{-1} takes you up in the natural partial order, it easily follows that ρ is a dual prehomomorphism and that if $s \in S$ is represented by u, then $s \leqslant \rho(w)$ where w is the reduction of u. The result follows from Corollary 1.4. ∎

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