ON A TYPE OF *K*-CONTACT RIEMANNIAN MANIFOLD

Dedicated to Professor R. N. Sen on his 75-th birthday

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Introduction

Let M be an n-dimensional ($n = 2m+1, m \geqq 1$) real differentiable manifold. If on M there exist a tensor field ϕ^i_j, a contravariant vector field ξ^i and a covariant vector field η_i such that

$$(1)\qquad \begin{cases} \xi^i\eta_i = 1, \text{rank}\,(\phi^i_j) = n-1, \ \phi^i_j\,\xi^j = 0, \phi^i_j\,\eta_i = 0 \\ \phi^i_j\phi^j_k = -\delta^i_k+\xi^i\eta_k \end{cases}$$

then M is said to have an *almost contact structure* with the structure tensors (ϕ, ξ, η) [1], [2]. Further, if a positive definite Riemannian metric g satisfies the conditions

$$(2)\qquad \eta_i = g_{ij}\xi^j$$

$$(3)\qquad g_{ij}\phi^i_h\phi^j_k = g_{hk}-\eta_h\eta_k$$

then g is called an *associated Riemannian metric* to the almost contact structure and M is then said to have an *almost contact metric structure*. On the other hand, M is said to have a *contact structure* [2], [4] if there exists a 1-form η over M such that $\eta \wedge (d\eta)^m \neq 0$ everywhere over M where $d\eta$ means the exterior derivation of η and the symbol $\wedge$ means the exterior multiplication. In this case M is said to be a contact manifold with contact form η. It is known [2, Th. 3,1] that if $\eta = \eta_i dx^i$ is a 1-form defining a contact structure, then there exists a positive definite Riemannian metric g_{ij} in M such that $\phi^h_i = g^{hr}\phi_{ri}$ and $\xi^i = g^{ir}\eta_r$ define an almost contact metric structure with η_i and g_{ij} where

$$(4)\qquad \phi_{ij} = \tfrac{1}{2}(\partial_i\eta_j-\partial_j\eta_i),$$

the symbol ∂_i standing for $\partial/\partial x^i$.

In this case M is said to be a contact Riemannian manifold with contact form η, associated vector field ξ, (1,1) tensor field ϕ and the associated Riemannian metric g. If ξ is a Killing vector field with respect to g, then M is called a

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K-contact Riemannian manifold where the adjective K means Killing. Further, if the relation

$$\eta_r R^r_{jkl} = g_{jk}\eta_l - g_{jl}\eta_k \tag{5}$$

holds, then M is called a Sasakian manifold [2, Th. 11.3].

The present paper deals with a type of K-contact Riemannian manifold of dimension $n(n = 2m+1, m > 1)$ for which

$$\nabla_l C^h_{ijk} = 0 \tag{6}$$

where ∇ denotes covariant differentiation with respect to g and

$$\begin{aligned} C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j) \\ + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \end{aligned} \tag{7}$$

is the conformal curvature tensor. It is proved that such a K-contact Riemannian manifold is Sasakian and has constant curvature 1.

1. Some formulas in a K-contact Riemannian manifold

Let us consider an n-dimensional $(n = 2m+1, m \geqq 1)$ K-contact Riemannian manifold with a contact form η, the associated vector field ξ, (1,1) tensor field ϕ and the associated Riemannian metric g. Then in such a manifold, besides the relations (1), (2), (3), and (4), the following formulas hold [2, Th. 10.21; 3, (27.15)].

$$\nabla_j \eta_i = -\phi_{ij} \tag{1.1}$$

$$\nabla_j \xi^i = -\phi^i_j \tag{1.2}$$

$$\nabla_k \phi^i_j = R^i_{jkl}\xi^l \tag{1.3}$$

$$R_{ij}\xi^j = (n-1)\eta_i \tag{1.4}$$

Further, since ξ is a Killing vector field, R_{ij} and R remain invariant under it, that is,

$$\mathscr{L}R_{ij} = 0 \tag{1.5}$$

and

$$\mathscr{L}R = 0 \tag{1.6}$$

where $\mathscr{L}$ denotes Lie derivation with respect to the vector field ξ^i.

2. Locally conformally symmetric K-contact Riemannian manifold

A Riemannian manifold $M_n(n > 3)$ satisfying (6) shall be called locally conformally symmetric [5]. Let us now suppose that an n-dimensional $(n = 2m+1,$

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$m > 1$) K-contact Riemannian manifold is locally conformally symmetric. From (7) it follows that

$$(2.1)\qquad \nabla_h C^h_{ijk} = \frac{n-3}{n-2} R_{ijk} \quad [6, (28.16)]$$

where

$$(2.2)\qquad R_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(g_{ik}\nabla_j R - g_{ij}\nabla_k R).$$

In virtue of (6), it follows from (2.1) and (2.2) that

$$(2.3)\qquad \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(g_{ik}\nabla_j R - g_{ij}\nabla_k R) = 0$$

Again from (1.5) and (1.6) we get

$$(2.4)\qquad (\nabla_l R_{ij})\xi^l + R_{lj}\nabla_i \xi^l + R_{il}\nabla_j \xi^l = 0$$

and

$$(2.5)\qquad \xi^l \nabla_l R = 0.$$

Using (1.2), (2) and (2.5) it follows from (2.3) and (2.4) that

$$(2.6)\qquad R_{rj}\phi^r_i + R_{ir}\phi^r_j - (\nabla_j R_{ir})\xi^r + \frac{1}{2(n-1)}\eta_i \nabla_j R = 0$$

With the help of (1.4) the relation (2.6) reduces to

$$(2.7)\qquad -(n-1)\phi_{ij} - R_{rj}\phi^r_i = \frac{1}{2(n-1)}\eta_i \nabla_j R$$

Transvecting (2.7) with ϕ^i_t we get

$$R_{tj} = (n-1)g_{tj}$$

Hence

$$\nabla_l R_{tj} = 0.$$

Therefore from (7) it follows that $\nabla_l R^h_{ijk} = 0$. Hence the manifold is locally symmetric.

It has been proved by Tanno [7] that a locally symmetric K-contact Riemannian manifold is Sasakian and has constant curvature 1. We can therefore state the following theorem.

THEOREM 1. *Any locally conformally symmetric K-contact Riemannian manifold is Sasakian and has constant curvature* 1.

As an immediate consequence of this we have another theorem which can be stated as follows:

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THEOREM 2. *Any complete, simply connected and locally conformally symmetric K-contact Riemannian manifold is globally isometric with a unit sphere.*

We conclude by mentioning that the above theorems remain valid when the words *locally conformally symmetric* in their statements are replaced by the words *locally projectively symmetric*.

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