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BOOK REVIEWS

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Blecher, D. P. and Le Merdy, C. *Operator algebras and their modules* (Oxford University Press, 2004), 396 pp, 0 198 52659 8 (hardback), £85.

The subject of operator spaces has undergone a rapid development over the past two decades. One may call it non-commutative functional analysis and explain it as follows. In standard functional analysis (the 'commutative' version) we consider normed spaces X and these may be realized as subspaces of the (commutative C^*-) algebra of continuous functions on a compact Hausdorff space. We may take the compact space to be the closed unit ball X_1^* of the dual space X^* of X, and embed X canonically in its double dual, a part of $C(X_1^*)$. This fact is a curiosity in the commutative theory, a step, perhaps, towards showing that $C(K)$ spaces are universal.

In the non-commutative theory we consider not only subspaces X of non-commutative C^*-algebras A, but also norms on matrix spaces $M_k(X) \subseteq M_k(A)$ for $k = 2, 3, \dots$. The matrix algebras $M_k(A)$ have a unique C^*-algebra norm. The morphisms of the theory are those linear operators between operator spaces X and Y that induce uniformly (operator norm) bounded linear operators on the matrix spaces ($M_k(X) \to M_k(Y)$). These are called *completely bounded* operators, and the adjective 'completely' is used throughout for properties that are uniformly valid on all matrix spaces.

The subject has its origins in work of Stinespring (1955), where complete positivity was first investigated, but a big breakthrough was the work of Arveson (1969) that established a Hahn–Banach-type extension theorem for completely positive (CP) maps valued in $\mathcal{B}(H)$ (the algebra of all bounded linear operators on a Hilbert space H). This was the start of a line of work that established CP maps as a central tool in C^*-algebra theory. As $*$-homomorphisms of C^*-algebras are very rigid, one aspect is that CP maps allow a richer class of morphisms. This resulted in consideration of the category of *operator systems* (unital $*$-closed subspaces of C^*-algebras) and CP maps.

The theory of operator spaces with completely bounded (CB) maps as the morphisms (or sometimes completely contractive maps) is the unordered version of the operator systems theory, and the results obtained from it amply justify the claim that it is a central tool. The book under review presents a selection of the successes of the theory of operator spaces, emphasizing the connection with algebraic aspects, as the title indicates.

So far, we have presented a brief sketch of the basic background, which is covered, in much more detail and at a gallop, in the first section of the book under review. The authors feel justified in summarizing the basics—skipping the proofs of some of the fundamental results that need lengthier arguments—because of the existence of other recent monographs presenting this material. For quite a number of years, the student interested in the developments in this area had a paucity of up-to-date presentations of the topic. The best available source for a time was the lecture notes of Paulsen [**4**] and there was significant extra information in the survey of Christensen and Sinclair [**2**]. In addition, Pisier had written several research monographs dealing with aspects of the theory that he had developed, so there was an ever-increasing literature for the student to try and digest. That situation has improved now with the books of Effros and Ruan [**3**], Paulsen's update [**5**] of his earlier notes and the account of Pisier [**6**]. All of these and the book under review take somewhat different directions. Their authors represent a significant fraction of the leading workers in the area.

The opening chapter considers not only the introductory theory mentioned briefly above, but also some less basic topics. An early breakthrough in the subject was the abstract characterization (due to Ruan) of operator spaces as normed spaces X equipped with norms on each $M_k(X)$ satisfying a short list of conditions. A result that seems astonishing at first is that the dual of an operator space is again an operator space in a useful way. Tensor products play a central role in the subject, and there are many of them to consider (minimal, projective and Haagerup tensor products, to name a few). In addition there are duality results for tensor products (so that the dual of the Haagerup tensor product is again of Haagerup type, for instance). The chapter ends, as all the chapters do, with very detailed notes and remarks pointing to original sources, further related work, and also to later sections of the book where some of the ideas are developed further. This aspect of forward referencing is valuable as a pointer to where things have progressed to, but it also tends to draw the reader into skipping forward to a later chapter.

The bibliography is extensive, with 446 items compared with 140 items in [**3**], 250 in [**5**] and approximately 400 in [**6**]. The book under review has what appears to be a good index. It does not have an index of notation (which is included in [**3**]) but it does have six condensed appendices drawing together background for the convenience of the reader. These appendices deal with basic operator theory, Banach spaces, Banach algebras, C^*-algebras and finally 'Modules and Cohen's factorization theorem'.

Chapter 2 (entitled 'Basic theory of operator algebras') starts to enter into the theme of the book in earnest. A key result, known as the Blecher–Ruan–Sinclair (or BRS) theorem after its authors, characterizes (unital) operator algebras A as Banach algebras with matricial norms such that the matrix spaces $M_k(A)$ are Banach algebras for $k = 1, 2, \dots$. The key is that one has to have both a Banach algebra structure and a compatible operator space structure. An application is that a quotient of an operator algebra is again an operator algebra. There are many other topics in this chapter, including unitizations; direct sums and other constructions such as ultraproducts, interpolation, direct limits, (minimal) tensor products; how uniform algebras fit into the operator space approach; second duals and Arens regularity; multiplier algebras; operator algebras that are dual spaces and have separately w^*-continuous multiplications.

Chapter 3 moves on to modules, again dealt with rather comprehensively, but perhaps one key result to mention is the CES (Christensen–Effros–Sinclair) theorem characterizing A–B-bimodules X that arise as $X \subset \mathcal{B}(K, H)$, where A can be nicely represented in $\mathcal{B}(H)$ and B in $\mathcal{B}(K)$. A section on 'function modules'—where the elements of the algebra act as multipliers on a Banach space in the sense of Cunningham, Alfsen and Effros—serves partly to lead to some characterizations of uniform algebras in operator space terms and also as a backdrop to operator space multipliers, which are discussed in Chapter 4. Dual operator modules are the topic of the final section of Chapter 3.

Chapter 4 introduces the important notion of injective envelope (for an operator space) and how this relates to other notions of envelope (C^*-envelope, triple envelope). Results on multipliers in the sense of operator spaces, including proofs of the BRS and CES theorems, are then discussed, followed by non-commutative M-ideals.

Chapter 5, on the 'Completely isomorphic theory of operator algebras', begins with a gem of the theory: Paulsen's result characterizing homomorphisms similar to completely contractive representations as those which are completely bounded. There are many examples discussed in this chapter, many of them classical. For example, results on Q-algebras (quotients of uniform algebras), algebras where the product is the Schur product, and a characterization due to Varopoulos of Banach algebras that are operator algebras are included.

Chapter 6, on 'Tensor products of operator algebras', deals in some detail with the maximal and normal tensor products, applications to dilation theory, and concludes with a discussion of nuclearity. Chapter 7 deals with criteria for an operator algebra to be self-adjoint and draws on several ideas including versions of nuclearity, the WEP, amenability and virtual diagonals. The final chapter, Chapter 8, is a long one at 63 pages and claims to require only material from Chapters 1–4. It deals with Hilbert C^*-modules and their W^* relatives, with more about ternary rings of operators (TROs) than is mentioned previously to deal with injective envelopes, and fruitful interactions between these topics.

What can we conclude from this volume? Certainly there is no lack of evidence for the success of the operator space approach both as a coherent theory and as a vehicle for tackling classical problems. The reader may wish to consult [**1**] (which is an expanded version of reference [73] of the book under review) for a more leisurely and complete account of the M-ideal theory mentioned in Chapter 4. The last chapter of [**1**] is entitled 'Future directions', and of course there is more recent literature than can be included in any account such as this. Perhaps the basic theory of operator spaces is now relatively well established as well as the common material from Blecher–Le Merdy and [**3**, **5**, **6**], though all of these sources do have somewhat different approaches, even for the basic major results. They also differ in the choices of applications and advanced topics they decide to include. As a rough guide, this book includes or assumes much of what is in [**5**] and [**3**]. In [**3**] we have a section on local reflexivity, a topic that is not covered in much detail in Blecher–Le Merdy, and [**3**] concludes with the BRS theorem. The approach of [**3**, **5**] seems more focused on finding a clean path to the main results, rather than on covering every aspect. In [**6**], there is rather more emphasis on topics that are closely related to Banach space concepts and areas in which there are precise constants that can be analysed. Moreover, [**6**] has a problem-oriented style and the last 49 pages are solutions to the exercises. Pisier has very little mention of CP maps and does not refer to the Hamana theory of injective envelopes. Perhaps because the injective envelope of an operator space X is usually related to the injective envelope of the related Paulsen operator system $\left(\begin{smallmatrix} \mathbb{C} & X \\ X^* & \mathbb{C} \end{smallmatrix}\right)$, [**6**] does not deal with this topic. Though the last part of [**6**] is potentially closest to Blecher–Le Merdy because it deals with non-self-adjoint operator algebras, it is a relatively short part and the topics treated are quite different despite a certain common core. The final chapter of [**6**] deals with the solution of the Halmos similarity problem (showing that a polynomially bounded $T \in \mathcal{B}(H)$ need not be similar to a contraction).

In summary, Blecher and Le Merdy have provided a great compendium of the many results and techniques in operator spaces that they regard as closest to the algebraic applications; they give a valuable road map to the literature, which is up to date, yet with a significantly new approach in comparison with the other recent accounts.

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