

ON GENERALIZED AVERAGING OPERATORS

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1. Introduction. Let

$$\nabla_h f(z) = \frac{1}{2}[f(z+h) + f(z)].$$

Sumner (4) has discussed $\nabla_h^\lambda f(z)$ for arbitrary real λ and h , where $f(z)$ is an entire function of exponential type $\tau < \pi/|h|$. I shall show that in this case an alternative definition of ∇_h^λ , which leads to Sumner's results more quickly, is equivalent to Sumner's. (However, Sumner's definition is, in principle, applicable to a wider class of functions.) I also make some remarks on Sumner's question about the existence of nontrivial solutions of $\nabla_h^\lambda f = 0$. In particular, I shall show that in a certain sense there are such solutions for every positive λ .

2. Definitions. Let $f(z)$ be an entire function of exponential type τ , let S be its conjugate indicator diagram, and let $F(w)$ be the Borel-Laplace transform of f . (For terminology, see, for example, (1).) Then we have the Pólya representation

$$(1) \quad f(z) = (2\pi i)^{-1} \int_C F(w) e^{zw} dw,$$

where C is a contour surrounding S ; since S is a subset of the disk $|w| \leq \tau$, C can in particular be the circumference $|w| = \tau + \epsilon$, $\epsilon > 0$.

Let $\phi(w)$ be regular on S , hence on contours C which are sufficiently close to the boundary of S . If D stands for d/dz , we can define the operator $\phi(D)$ by

$$(2) \quad \phi(D)f(z) = (2\pi i)^{-1} \int_C F(w) \phi(w) e^{zw} dw,$$

where the definition is independent of the particular contour C that is employed provided that we admit only contours lying within the domain of regularity of ϕ . If $K[S]$ denotes the class of entire functions of exponential type whose conjugate indicator diagrams are subsets of S , the operator $\phi(D)$ applies to all elements of $K[S]$ and transforms them into elements of $K[S]$. (If ϕ has poles in S we can still define $\phi(D)$ by (2) but we get, in general, different values for $\phi(D)f$ according to which C we take, if there are poles of ϕ not in the conjugate indicator diagram of f .)

If ψ is regular over the range of $\phi(w)$ for w in S , we have

$$\psi[\phi(D)]f = (2\pi i)^{-1} \int_C F(w) \psi[\phi(w)] e^{zw} dw,$$

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so in particular

$$(3) \quad [\phi(D)]^\lambda f = (2\pi i)^{-1} \int_C F(w) [\phi(w)]^\lambda e^{zw} dw$$

for any positive integral λ . If $\phi(w)$ is zero-free on S , (3) holds for arbitrary (real or imaginary) λ ; we must of course specify which regular branch of $[\phi(w)]^\lambda$ is to be taken. If, for example, $[\phi(w)]^\lambda = e^{\lambda \log \phi(w)}$ with the same branch of $\log \phi(w)$ for all λ , it is then immediate that

$$(4) \quad [\phi(D)]^\lambda \{ [\phi(D)]^\mu f(z) \} = [\phi(D)]^{\lambda+\mu} f(z).$$

As a special case, we see that $[\phi(D)]^{-\lambda}$ inverts $[\phi(D)]^\lambda$.

For example, if ϕ is regular and zero-free in a region containing $w = a$, we have

$$(5) \quad [\phi(D)]^\lambda e^{az} = [\phi(a)]^\lambda e^{az}.$$

An alternative representation is obtained by expanding $\phi(w)e^{w(z-t)}$ in powers of w :

$$\phi(w)e^{w(z-t)} = \sum_{n=0}^\infty w^n p_n(z-t).$$

The $p_n(z)$ are the Appell polynomials (3) generated by $\phi(w)$, and (2) can be written

$$(6) \quad \phi(D)f(z) = \sum_{n=0}^\infty f^{(n)}(t)p_n(z-t),$$

where the series converges if S is inside the largest circle of regularity of $\phi(w)$ with center at 0, and is Mittag-Leffler summable (2, p. 79) if S is inside the Mittag-Leffler star of ϕ with respect to 0. In particular, if $t = z$ and

$$\phi(w) = \sum_{n=0}^\infty c_n w^n,$$

(6) becomes

$$\phi(D)f(z) = \sum_{n=0}^\infty c_n f^{(n)}(z) = \sum_{n=0}^\infty c_n D^n f(z),$$

which is a very natural interpretation of $\phi(D)f$.

In Sumner's case $\phi(D) = \nabla_h = \frac{1}{2}(e^{hD} + 1)$, so that $\phi(w) = \frac{1}{2}(e^{hw} + 1)$. If h is any non-zero complex number, the zeros of ϕ closest to 0 are at $w = \pm i\pi/h$, so that (3) is valid (in the sense of transforming an element of $K[S]$ into an element of $K[S]$) if f is an entire function of exponential type less than $\pi/|h|$, and more generally if S avoids the points $(2k+1)i\pi/h$. For example, (3) defines ∇_h^λ for all λ if h is real and $f(z)$ is an exponential sum

$$\sum_{j=0}^n a_j e^{b_j z}$$

with real b_j .

If we choose $\log \{ \frac{1}{2}(e^{hw} + 1) \}$ so that it reduces to 0 at $w = 0$, we then have

$$\lim_{h \rightarrow 0} \nabla_h^\lambda f(z) = f(z),$$

uniformly on compact sets, if f is an entire function of exponential type (the left-hand side being defined when $|h|$ is sufficiently small). We also have

$$\lim_{\lambda \rightarrow 0} \nabla_h^\lambda f(z) = f(z)$$

when $f(z)$ is of exponential type less than $\pi/|h|$.

3. Equivalence of the definitions. We now compare our definition of ∇_h^λ with Sumner's. Since both versions satisfy (4), and coincide for integral λ , and Sumner's was given only for real λ , it is enough to consider the range $0 < \lambda < 1$; for convenience we take $h > 0$.

For this range, Sumner's definition is

$$(S) \nabla_h^\lambda f(z) = \frac{1}{2^\lambda \Gamma(1 - \lambda) \cdot 2\pi i} \int_{b-i\infty}^{b+i\infty} \{f(z - hw) + f(z + h - hw)\} \times \Gamma(w) \Gamma(1 - \lambda - w) dw,$$

where $0 < b < 1 - \lambda$ and $f(z)$ is of exponential type less than π/h . Representing $f(z)$ by (1), we have

$$(7) \quad (S) \nabla_h^\lambda f(z) = \frac{1}{2^\lambda \Gamma(1 - \lambda) (2\pi i)^2} \int_{b-i\infty}^{b+i\infty} \Gamma(w) \Gamma(1 - \lambda - w) dw \times \int_C (e^{zt} + e^{(z+h)t}) e^{-hw t} F(t) dt.$$

If we change the order of integration, this becomes

$$(S) \nabla_h^\lambda f(z) = \frac{1}{2^\lambda \Gamma(1 - \lambda) (2\pi i)^2} \int_C F(t) (e^{zt} + e^{(z+h)t}) dt \times \int_{b-i\infty}^{b+i\infty} e^{-hw t} \Gamma(w) \Gamma(1 - \lambda - w) dw = \frac{1}{2^\lambda \cdot 2\pi i} \int_C e^{zt} (1 + e^{ht}) F(t) (1 + e^{ht})^{\lambda-1} dt,$$

which coincides with (3) when $\phi(w) = \frac{1}{2}(e^{hw+1})$. Here we have used a definite integral quoted by Sumner (4, p. 438).

It remains to justify the change of order of integration. It is sufficient to verify that the iterated integral is absolutely convergent. Now, for fixed z and h , $F(t)(e^{zt} + e^{(z+h)t})$ is bounded on C , so it is enough to show that

$$(8) \quad \int_{b-i\infty}^{b+i\infty} |e^{-hw t} \Gamma(w) \Gamma(1 - \lambda - w) dw|$$

converges uniformly for t on C , where $0 < b < 1 - \lambda$. We can take C to be a circumference $|t| = (\pi/h) - \epsilon$, $\epsilon > 0$, so that

$$|e^{-hw t}| < \exp\{h|y|[(\pi/h) - \epsilon]\} = O(e^{\sigma|y|})$$

with $\sigma < \pi$ and independent of t . Now we have

$$\Gamma(1 - \lambda - w) = \frac{\pi \csc \pi(1 - \lambda - w)}{\Gamma(\lambda + w)},$$

so that

$$|\Gamma(w)\Gamma(1 - \lambda - w)| \leq \pi |\csc \pi(1 - \lambda - w)| |\Gamma(w)/\Gamma(\lambda + w)|.$$

Since $\csc \pi(1 - \lambda - w) = O(e^{-\pi|y|})$ and

$$\Gamma(w)/\Gamma(\lambda + w) = O(|w^{-\lambda}|) = O(|y|^{-\lambda})$$

as $w = b + iy \rightarrow \infty$, the integrand in (8) is $O(e^{-\delta|y|})$ with $\delta > 0$ and independent of t . Hence (8) does converge uniformly for t on C and the change of order of integration in (7) is legitimate.

4. The existence of eigenvalues. Sumner noted that there are nontrivial solutions of $\nabla_h^\lambda f(z) = 0$ which are entire functions of exponential type, although not of type less than $\pi/|h|$, when λ is a positive integer. It is easy to show that there are no nontrivial solutions which are of exponential type less than $\pi/|h|$. More generally, if $\phi(D)f(z)$ is defined by (2), where $\phi(w)$ is regular on S and $F(w)$ is regular outside S , we have, if $\phi(D)f(z) \equiv 0$,

$$\int_C F(w)\phi(w)e^{zw}dw \equiv 0.$$

This implies, by a lemma of Pólya's (1, p. 110), that $F(w)\phi(w)$ is regular inside C . If $\phi(w)$ has no zeros inside C , $F(w)$ must be regular inside C , and so everywhere; hence, since F vanishes at ∞ , $F(w) \equiv 0$ and so $f(z) \equiv 0$.

On the other hand, if $\phi(w)$ has a zero at $w = a \in S$, (2) shows that $\phi(D)$ (and all its positive integral powers) annul the function f whose Borel-Laplace transform F is $F(w) = 1/(w - a)$, namely $f(z) = e^{az}$. Moreover, $[\phi(D)]^2$ also annuls the inverse Laplace transform of $1/(w - a)^2$, namely $f(z) = ze^{az}$; and so on.

In particular, $\nabla_h^\lambda f(z) = 0$ for positive integral λ if $f(z) = e^{\pm i\pi z/h}$ (cf. (5)), or $\sin(\pi z/h)$ or $\cos(\pi z/h)$; $\nabla_h^\lambda f(z) = 0$ for $\lambda = 2, 3, \dots$ if $f(z) = z \sin(\pi z/h)$ or $z \cos(\pi z/h)$; and so on.

Sumner raised the question of whether there is a set of values of λ which are eigenvalues for ∇_h^λ in the sense that for these, and only for these, there are nontrivial solutions (eigenfunctions) of $\nabla_h^\lambda f(z) = 0$. We have, of course, defined $\nabla_h^\lambda f(z)$, when λ is not a positive integer, only when f is of exponential type less than $\pi/|h|$, but it would be natural to extend the definition as follows. If $|k| < |h|$, and f is of exponential type $\pi/|h|$, $\nabla_k^\lambda f(z)$ is defined and we take

$$(9) \quad \nabla_h^\lambda f(z) = \lim_{k \rightarrow h} \nabla_k^\lambda f(z) \quad (|k| < |h|),$$

if the limit exists. We may, of course, lose property (4) with the extended definition.

Now by (5) we have, if $|k| < |h|$,

$$\nabla_k^\lambda e^{\pm i\pi z/h} = \left\{ \frac{1}{2}(e^{\pm i\pi k/h} + 1) \right\}^\lambda e^{\pm i\pi z/h},$$

and if $\Re(\lambda) > 0$, this approaches 0 as $k \rightarrow h$. Thus if ∇_h^λ is defined by (9), every λ of positive real part is an eigenvalue and $e^{\pm i\pi z/h}$ are eigenfunctions,

just as they were for positive integral λ . Furthermore, $ze^{\pm i\pi z/h}$ are also eigenfunctions when $\Re(\lambda) > 1$, and so on.

It is interesting to note that there are still other eigenfunctions. To see this, suppose that the Borel-Laplace transform $F(w)$ of $f(z)$ is such that $F(w)(w \pm i\pi/k)^\lambda$ is uniformly dominated by an integrable function in an annulus $\pi/|h| < |w| < a$. We can then shrink the contour C in (2) to $|w| = \pi/|h|$ and obtain

$$\nabla_k^\lambda f(x) = (2\pi i)^{-1} \int_{|w|=\pi/|h|} F(w) \left\{ \frac{1}{2}(e^{kw} + 1) \right\}^\lambda e^{zw} dw;$$

then we let $k \rightarrow h$, and obtain finally

$$(10) \quad \nabla_h^\lambda f(x) = (2\pi i)^{-1} \int_{|w|=\pi/|h|} F(w) \left\{ \frac{1}{2}(e^{hw} + 1) \right\}^\lambda e^{zw} dw,$$

where the branch of the power is one that is regular in the plane cut from $\pm i\pi/h$ to ∞ .

Now consider (for the sake of simplicity) positive values of h , and apply (10) to $f(z) = J_0(\pi z/h)$. Then $F(w) = (w^2 + \pi^2/h^2)^{-\frac{1}{2}}$ for $|w| > h$, and satisfies the hypotheses required for (10) if $\lambda = \frac{1}{2}$. Thus

$$(11) \quad \nabla_h^{\frac{1}{2}} f(z) = (2\pi i)^{-1} \int_{|w|=\pi/h} \{w^2 + \pi^2/h^2\}^{-\frac{1}{2}} \left\{ \frac{1}{2}(e^{hw} + 1) \right\}^{\frac{1}{2}} e^{zw} dw.$$

Here the first square root is defined in the plane cut from $i\pi/h$ to $-i\pi/h$. However, continuing either square root around $\pm i\pi/h$ replaces it by its negative, so the integrand in (11) is regular in the closed disk $|w| \leq \pi/h$. Hence the integral in (11) is zero. Thus $J_0(\pi z/h)$ is still another eigenfunction corresponding to $\lambda = \frac{1}{2}$.

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