ε-ENTROPY OF THE BROWNIAN MOTION WITH THE MULTI-DIMENSIONAL SPHERICAL PARAMETER

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§ 1. Introduction

M.S. Pinsker [3] has given a general method of calculating the ε-entropy of a Gaussian process and obtained, for example, an exact proof of the estimate for the ε-entropy of the ordinary Brownian motion $B(t)$, $0 \leqq t \leqq 1$, which was presented without proof by A.N. Kolmogorov [1].

In this article, we estimate the ε-entropy of the *Brownian motion with the multidimensional spherical parameter*, by using the expansion of the Brownian motion with a multidimensional parameter by H.P. McKean [4] and by generalizing the Pinsker's method of calculating the ε-entropy.

Let $X(A, \omega)$, $A \in E^d$ (d-dimensional Euclidean space), $\omega \in \Omega(P)$, be a Brownian motion with a parameter space E^d, that is, $\{X(A), A \in E^d\}$ forms a Gaussian system and

1) $E[X(A)] = 0$ for every A,

2) $X(O) = 0$, where O is the origin of E^d,

3) $E[(X(A) - X(B))^2] = \mathrm{dis}(A, B)$, where $E(X)$ and $\mathrm{dis}(A, B)$ denote the expectation of a random variable X and the Euclidean distance between A and B, respectively.

We shall call $X(A)$ when the parameter A is restricted to the unit sphere[1] S^{d-1} in E^d the *Brownian motion with the d-dimensional spherical parameter* and denote it, as in the preceding case, by $X(A)$, $A \in S^{d-1}$.

The ε-entropy $H_\varepsilon(X)$ of the process $X(A)$ is defined as follows: Let $\varepsilon > 0$ be arbitrarily fixed, and consider an approximating process $X'(A)$ for the process $X(A)$ on S^{d-1} satisfying the *condition of reproducing accuracy*,

$$(1) \qquad \int_{S^{d-1}} E[(X'(A) - X(A))^2]\, d\sigma(A) \leqq \varepsilon^2$$

Received March 8, 1967.

1) Without loss of generality we may consider the unit sphere only.

where $d\sigma$ is the uniform probability measure on S^{d-1}. Then, the ε-entropy of the process $X(A)$ is defined as

$$H_\varepsilon(X) = \inf I(X', X), \tag{2}$$

where $I(X', X)$ is the amount of information contained in a process X' with respect to the process X and the infimum is taken for all processes X' satisfying the condition (1).

Our aim is to prove that the ε-entropy of the Brownian motion on S^{d-1} is of order $\varepsilon^{-2(d-1)}$ (Theorem 2);

$$H_\varepsilon(X) = O(\varepsilon^{-2(d-1)}). \tag{3}$$

It seems to be interesting to note that the ε-entropy (in Kolmogorov-Tihomirov's sense, cf. Kolmogorov-Tihomirov [2]) of the space of $\frac{1}{2}$-Hölder continuous functions of $(d-1)$-variables with the sup-norm has the same order $O(\varepsilon^{-2(d-1)})$.

The author is greatly indebted to Professors T. Hida and N. Ikeda for their kind suggestions and constant encouragement.

§ 2. The generalization of Pinsker's method

Pinsker's method of calculating the ε-entropy of a Gaussian process with one dimensional parameter is as follows: Let $X(t)$, $0 \leqq t \leqq T$, be a Gaussian process with mean 0 whose covariance function $r(s, t) = E[X(s)X(t)]$ is continuous in (s, t). Then the ε-entropy $H_\varepsilon(X)$ of the process $X(t)$ is given by the formula

$$H_\varepsilon(X) = \frac{1}{2} \sum_{\lambda_i > \theta^2} \log \frac{\lambda_i}{\theta^2}, \tag{4}$$

where λ_i $(i = 1, 2, \cdots)$ are the eigen-values of the integral operator with the kernel $r(s, t)$ in $L^2[0, T]$, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq 0$, and θ is determined (uniquely) by the equation

$$\sum_{i=1}^{\infty} \min(\theta^2, \lambda_i) = \varepsilon^2 . \tag{5}$$ [2)]

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2) By Mercer's theorem

$$\sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \lambda_i \int_0^T [\varphi_i(t)]^2 dt = \int_0^T \sum_{i=1}^{\infty} \lambda_i [\varphi_i(t)]^2 dt = \int_0^T r(t, t) dt < \infty.$$

The right-hand side of the relation (4) also equals to the ε-entropy of the infinite dimensional Gaussian random variable $X^* = (X_1^*, X_2^*, \cdots)$[3]:

$$X_i^* = \int_0^T \varphi_i(t)X(t)dt^{\,4)} \qquad (i = 1, 2, \cdots) \tag{6}$$

where $\varphi_i(t)$ is the eigen-function of the integral operator corresponding to the eigenvalue λ_i and $E[X_i^* X_j^*] = \lambda_i \delta_{ij}$.

As an example, if in particular the sequence $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq 0$ of the eigen-values of the integral operator with the kernel corresponding to a Gaussian process takes the form: $\lambda_k = ck^{-s} (s > 1; k = 1, 2, \cdots)$, then, the ε-entropy of the process is

$$H_\varepsilon(X) = O(\varepsilon^{-\frac{2}{s-1}}). \tag{7}$$

Now, we proceed to a Gaussian process $X(A)$, $A \in S^{d-1}$, with mean 0. Assume the continuity of the covariance function $r(A, B) = E[X(A)\,X(B)]$ in $S^{d-1} \times S^{d-1}$, so $\sum_{i=1}^{\infty} \lambda_i$ is finite (see the discussion in the footnote 2)) where λ_i, $i = 1, 2, \cdots$, are the eigenvalues of the integral operator with the kernel $r(A, B)$ in $L^2(S^{d-1}, d\sigma)$. Then, the following entirely analogous result holds, and we state it as a theorem.

THEOREM 1. *The ε-entropy $H_\varepsilon(X)$ of the above Gaussian process $X(A)$, $A \in S^{d-1}$ is*

$$H_\varepsilon(X) = \frac{1}{2} \sum_{\lambda_i > \theta^2} \log \frac{\lambda_i}{\theta^2} \tag{4'}$$

where $\lambda_i (i = 1, 2, \cdots)$ with $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq 0$ are eigen-values of the integral operator and θ is determined by the equation (5). *The right-hand side of the relation* (4′) *equals also to the ε-entropy of the infinite dimensional Gaussian random variable $X^* = (X_1^*, X_2^*, \cdots)$:*

$$X_i^* = \int_{S^{d-1}} \varphi_i(A)X(A)d\sigma(A) \qquad (i = 1, 2, \cdots) \tag{6'}$$

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3) The ε-entropy of X^* is defined as $H_\varepsilon(X^*) = \inf I(\widetilde{X}^*, X^*)$ where the infimum is taken for all infinite dimensional approximating random variables $\widetilde{X}^* = (\widetilde{X}_1^*, \widetilde{X}_2^*, \cdots)$ satisfying the condition: $\sum_{i=1}^{\infty} E[(\widetilde{X}_i^* - X_i^*)^2] \leqq \varepsilon^2$.

4) This (Bochner) integral is determined as an element of $L^2(\Omega)$.

https://doi.org/10.1017/S0027763000026581 Published online by Cambridge University Press

where $\varphi_i(A)$ *is the eigen-function of the integral operator corresponding to the eigen-value* λ_i, *and* $E[X_i^* X_j^*] = \lambda_i \delta_{ij}$.

Proof. The proof is quite similar to the proof for one dimensional parameter case dealt by M.S. Pinsker [3], except for the construction of the process $\dot{\xi}$ ([3], formula (132)). The proof, however, can be carried out by using the extension theorem of Urysohn, so that we shall not continue the proof further.

§ 3. The main result

We are now in a position to prove our main result.

THEOREM 2. *The* ε*-entropy of the Brownian motion with the* d*-dimensional spherical parameter is of order* $\varepsilon^{-2(d-1)}$;

$$H_\varepsilon(X) = O(\varepsilon^{-2(d-1)}). \tag{8}$$

Proof. According to H.P. McKean [4] the Brownian motion with the d-dimensional parameter can be expanded as a sum of mutually independent Gaussian processes associated with spherical harmonics. We state this expansion and some related results with the Gaussian process $X(A)$, $A \in S^{d-1}$.

$$X(A) = \sum_{n \geqq 0} \sum_{l=1}^{D(n)} x_n^l(1)\, h_n^l(A), \quad A \in S^{d-1} \tag{9}$$

where $h_n^l(A)$ is a spherical harmonics of degree n satisfying

$$\int_{S^{d-1}} h_n^l(A) h_m^k(A) d\sigma(A) = \begin{cases} 1, & \text{if } l = k,\ n = m \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

$D(n)$ is the dimension of the vector space spanned by all the spherical harmonics of degree n,

$$D(n) = (2n - 2 + d)\frac{(n-3+d)!}{(d-2)!\, n!} \qquad (d \geqq 2,\ n \geqq 0)^{5)} \tag{11}$$

and $x_n^l(1)$ $(n \geqq 0,\ 1 \leqq l \leqq D(n))$ are mutually independent Gaussian random variables which can be expressed in the form

$$x_n^l(1) \equiv x_n^l = C(d) \int_0^1 C_n(u) dB_n^l(u). \tag{12}$$

5) For $d=2$ and $n=0$, $D(n)=1$.

The processes $B_n^l(u)$ $(n \geqq 0,\ 1 \leqq l \leqq D(n))$ appeared in the above expression are mutually independent standard Brownian motions and

$$(13) \qquad C_n(u) = \frac{\int_0^{\cos^{-1} u} p_n(\cos\theta) \sin^{d-2}\theta d\theta}{\int_0^{\pi} \sin^{d-2}\theta d\theta}, \quad n \geqq 0$$

with $p_n(\cos\theta) = C_n^{\frac{d-2}{2}}(\cos\theta) \Big/ C_n^{\frac{d-2}{2}}(1)$, where $C_n^{\nu}(\cdot)$ is the Gegenbauer polynomial and $C(d)$ is a constant depending only on d.

By the expansion (9) and by the independence of the random variables x_n^l with $E[x_n^l] = 0$ $(n \geqq 0,\ 1 \leqq l \leqq D(n))$ we easily see that the covariance function of the process $X(A)$ is expressed in the form

$$(14) \qquad r(A, B) = \sum_{n \geqq 0} \sum_{l=1}^{D(n)} E[(x_n^l)^2]\, h_n^l(A)\, h_n^l(B).$$

Using this, Mercer's expansion theorem shows us that the eigen-values $\lambda_n^l(n \geqq 0, 1 \leqq l \leqq D(n))$ of the integral operator with the kernel $r(A, B)$ are equal to $E[(x_n^l)^2]$. Therefore, if we know the amount $E[(x_n^l)^2]$ we can obtain the ε-entropy of the Brownian motion with the parameter space S^{d-1} by the formula (4$'$). In fact, we can prove in the following that for large n, $E[(x_n^l)^2] = O(n^{-d})$, $1 \leqq l \leqq D(n)$, holds. Once the result is shown, then just by renumbering the double sequence of random variables x_0^1, x_1^1, x_1^2, $\cdots$, $x_1^{D(1)}$, $x_2^1, \cdots$ into the ordinary sequence x_1', $x_2', \cdots$, while keeping the original order, we can easily apply Theorem 1 in §2. If x_k', for large k, corresponds to the original random variable $x_N^M (1 \leqq M \leqq D(N))$, then by the relation $\sum_{n=0}^{N} n^{d-2} = O(N^{d-1})$ (this nearly equals to k) and by the formula (11) ($D(n) = O(n^{d-2})$ for large n), we obtain $N = O(k^{\frac{1}{d-1}})$, so that $E[(x_k')^2] = O\left((k^{\frac{1}{d-1}})^{-d}\right) = O(k^{-\frac{d}{d-1}})$. Then, by this and the formula (7), follows the desired result $H_\varepsilon(X) = O(\varepsilon^{-\frac{2}{\frac{d}{d-1}-1}}) = O(\varepsilon^{-2(d-1)})$.

Therefore, in the following, we are to prove that

$$(15) \qquad E[(x_n^l)^2] = O(n^{-d}), \qquad 1 \leqq l \leqq D(n)$$

holds for large n.

First of all, we show the formula (15) in case the dimension $d=2$ and 3, and then, generalizing it, we proceed to prove the formula (15) for $d \geqq 4$, that is, (I) in case d is an even integer and (II) when d is odd.

In case $d=2$, $p_n(\cos\theta)$ in the expression (13) turns out to be $\cos n\theta$, so that $C_n(u) = \dfrac{1}{n\pi}\sin(n\cos^{-1}u)$. From this we have,

$$\begin{aligned} E[(x_n^l)^2] &= \frac{1}{n^2\pi}\int_0^1 \sin^2(n\cos^{-1}u)du \\ &= \frac{1}{n^2\pi}\int_0^{\frac{\pi}{2}} \sin^2 n\theta \ \sin\theta\, d\theta \\ &= O(n^{-2}). \end{aligned}$$

While in case $d=3$, $p_n(\cos\theta) = P_n(\cos\theta)$, hence we have $C_n(u) = \dfrac{1}{2}\dfrac{P_{n-1}(u) - P_{n+1}(u)}{2n+1}$ where $P_n(\cdot)$ is the n-th Legendre polynomial. Then, by the orthogonality of the Legendre polynomials, we obtain

$$\begin{aligned} E[(x_n^l)^2] &= \frac{1}{(2n+1)^2}\left\{\int_0^1 (P_{n+1}(u))^2\, du + \int_0^1 (P_{n-1}(u))^2 du\right\} \\ &= O(n^{-3}). \end{aligned}$$

In case $d \geqq 4$, by the formula (12), we have

$$\begin{aligned} E[(x_n^l)^2] &= (C(d))^2 \int_0^1 (C_n(u))^2 du \\ &= (\text{a constant depending on } d \text{ only}) \times \left\{C_n^{\frac{d-2}{2}}(1)\right\}^{-2} \\ &\quad \times \int_0^1 \left\{\int_0^{\cos^{-1}u} C_n^{\frac{d-2}{2}}(\cos\theta)\sin^{d-2}\theta d\theta\right\}^2 du \end{aligned}$$

and this expression becomes,

$$O(n^{-2d+6}) \times \int_0^1 \left\{\int_0^{\cos^{-1}u} C_n^{\frac{d-2}{2}}(\cos\theta)\sin^{d-2}\theta\, d\theta\right\}^2 du$$

for large n, since $C_n^{\frac{d-2}{2}}(1) = \dfrac{\Gamma(n+d-2)}{n!\,\Gamma(d-2)} = O(n^{d-3})$.

To prove $E[(x_n^l)^2] = O(n^{-d})$, we must show that the above integral (we denote it by I_d) is of order $O(n^{d-6})$.

(I) The proof of the fact that $I_d = O(n^{d-6})$ for $d = 2p + 2$ ($p \geqq 1$, integer).

First we estimate the integrand of the above integral. Let the following integral be denoted by $I_p(u)$,

$$I_p(u) = \int_0^{\cos^{-1}u} C_n^{\frac{d-2}{2}}(\cos\theta)\sin^{d-2}\theta d\theta = \int_0^{\cos^{-1}u} C_n^p(\cos\theta)\sin^{2p}\theta d\theta.$$

The integrand $C_n^p(\cos\theta)\sin^{2p}\theta$ of the above integral becomes, by using the recurrence formula for the Gegenbauer polynomials

$$\text{(16)}\qquad \sin^2\theta C_n^{\nu+1}(\cos\theta) = \frac{1}{2\nu}\left\{(n+2\nu)C_n^{\nu}(\cos\theta) - (n+1)\cos\theta\, C_{n+1}^{\nu}(\cos\theta)\right\}$$

and the formula $\sin\theta\, C_n^1(\cos\theta) = \sin(n+1)\theta$,

$$\begin{aligned} C_n^p(\cos\theta)\sin^{2p}\theta &= \sin^2\theta C_n^p(\cos\theta)\sin^{2(p-1)}\theta \\ &= \frac{1}{2(p-1)}\left\{(n+2(p-1))C_n^{p-1}(\cos\theta)\sin^{2(p-1)}\theta - (n+1)\cos\theta C_{n+1}^{p-1}(\cos\theta)\sin^{2(p-1)}\theta\right\} \\ &= \frac{1}{2^{p-1}(p-1)!}\left\{A_1^p(n)\sin\theta\sin(n+1)\theta + A_2^p(n)\cos\theta\sin\theta\sin(n+2)\theta\right. \\ &\qquad \left. + A_3^p(n)\cos^2\theta\sin\theta\sin(n+3)\theta + \cdots + A_p^p(n)\cos^{p-1}\theta\sin\theta\sin(n+p)\theta\right\} \end{aligned}$$

where $A_1^p(n), A_2^p(n), \cdots, A_p^p(n)$ are polynomials of n of order $(p-1)$. Noticing that $\sin\theta\sin(n+1)\theta$, $\cos\theta\sin\theta\sin(n+2)\theta$, $\cdots$ and $\cos^{p-1}\theta\sin\theta\sin(n+p)\theta$ are all expressed as the linear combinations of $\cos n\theta$, $\cos(n+2)\theta$, $\cdots$, $\cos(n+2p)\theta$, we can show that the integral becomes

$$\text{(17)}\qquad I_p(u) = \sum_{k=0}^{p} \frac{B_k^p(n)}{n+2k}\sin(n+2k)\alpha, \quad \alpha = \cos^{-1}u$$

where B_k^p, $k = 0, 1, \cdots, p$, are polynomials of n of order at most $(p-1)$. Therefore, changing the variable of integration into α, and making use of the fact

$$\int_0^{\frac{\pi}{2}} \sin(n+2k)\alpha\sin(n+2l)\alpha\sin\alpha d\alpha = \frac{1}{2}\left\{\frac{1}{1-4(k-l)^2} + O(n^{-2})\right\},$$

we have

$$I_d = \int_0^1 \{I_p(u)\}^2 du = \int_0^{\frac{\pi}{2}}\left\{\sum_{k=0}^{p}\frac{B_k^p(n)}{n+2k}\sin(n+2k)\alpha\right\}^2 \sin\alpha d\alpha$$

$$= \sum_{k,l=0}^{p} \frac{B_k^p(n) B_l^p(n)}{(n+2k)(n+2l)} \int_0^{\frac{\pi}{2}} \sin(n+2k)\alpha \sin(n+2l)\alpha \sin\alpha d\alpha$$
$$= O(n^{2p-4}) \; (= O(n^{d-6})) .$$

The last estimation is valid if the coefficient of the term n^{2p-4} never vanishes, that is, if at least one of the coefficients of the term n^{p-1} of the polynomials $B_k^p(n)$ $(k = 0, 1, \cdots, p)$ does not vanish. But this is true, for example, $B_0^p(n)$ has non zero coefficient of n^{p-1}.

(II) The proof of the fact that $I_d = O(n^{d-6})$ for $d = 2p+3$ $(p \geqq 1$, integer).

Similarly to (I), we denote the following integral by $I_p(u)$,

$$I_p(u) = \int_0^{\cos^{-1}u} C_n^{\frac{d-2}{2}}(\cos\theta) \sin^{d-2}\theta d\theta = \int_0^{\cos^{-1}u} C_n^{p+\frac{1}{2}}(\cos\theta) \sin^{2p+1}\theta d\theta$$

then, by the relation

$$C_n^{p+\frac{1}{2}}(\cos\theta) = \frac{2^p p!}{(2p)! \sin^p\theta} P_{n+p}^p(\cos\theta) \tag{18}$$

for the half-integer Gegenbauer polynomial $C_n^{p+\frac{1}{2}}$ and the associated Legendre polynomial P_{n+p}^p, we have

$$I_p(u) = c(d) \int_0^{\cos^{-1}u} P_{n+p}^p(\cos\theta) \sin^{p+1}\theta d\theta$$

where $c(d)$ is a constant depending on d. By definition,

$$P_{n+p}^p(x) = (1-x^2)^{\frac{p}{2}} \frac{d^p}{dx^p} P_{n+p}(x)$$

and by changing the variable of integration into $x = \cos\theta$, we get

$$\frac{1}{c(d)} I_p(u) = \int_u^1 \frac{d^p}{dx^p} P_{n+p}(x) (1-x^2)^p dx$$
$$= -(1-u^2)^p \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u) + 2p \int_u^1 x(1-x^2)^{p-1} \frac{d^{p-1}}{dx^{p-1}} P_{n+p}(x) dx$$

From this, the desired integral I_d is

$$[c(d)]^2 \cdot I_d = [c(d)]^2 \int_0^1 \{I_p(u)\}^2 du = \int_0^1 \left\{(1-u^2)^p \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u)\right\}^2 du$$
$$- 4p \int_0^1 (1-u^2)^p \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u) \left\{\int_u^1 x(1-x^2)^{p-1} \frac{d^{p-1}}{dx^{p-1}} P_{n+p}(x) dx\right\} du \tag{19}$$

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$$+ 4p^2 \int_0^1 \left\{ \int_u^1 x(1-x^2)^{p-1} \frac{d^{p-1}}{dx^{p-1}} P_{n+p}(x)dx \right\}^2 du.$$

To estimate these integrals, we first express $(1-u^2)^p \dfrac{d^p}{du^p} P_n(u)$ in terms of $P_n(u)$ and $P_{n-1}(u)$. For this purpose, we make use of the recurrence formula of the Legendre polynomials $(1-x^2)P_n'(x) = n(P_{n-1}(x) - xP_n(x))$ and the differential equation derived from the Legendre's differential equation

$$(20) \qquad (1-x^2) \frac{d^k}{dx^k} P_n(x) - 2(k-1)x \frac{d^{k-1}}{dx^{k-1}} P_n(x)$$
$$+ (n + (k-1))(n - (k-2)) \frac{d^{k-2}}{dx^{k-2}} P_n(x) = 0, \quad (k \geqq 2).$$

For any $p \geqq 1$, we have

$$(21) \qquad (1-u^2)^p \frac{d^p}{du^p} P_n(u) = P_{n-1}(u)Q_{n-1,p}(u) + P_n(u)Q_{n,p}(u)$$

where $Q_{n-1,p}(u)$ and $Q_{n,p}(u)$ are polynomials of u of the form

$$(22) \qquad Q_{n-1,p}(u) = \sum_{k=0}^{p-1} C_k(n)u^k, \quad Q_{n,p}(u) = \sum_{k=0}^{p} D_k(n)u^k.$$

The coefficients $C_0(n), C_1(n), \cdots, C_{p-1}(n), D_0(n), D_1(n), \cdots, D_p(n)$ have the following properties: (i) $C_{p-1}(n) \neq 0$, $D_p(n) \neq 0$ (ii) they are the polynomials of n with the order at most p (iii) if p is an even integer, then $D_0(n)$ is the polynomial of order p and if p is odd, $C_0(n)$ is the polynomial of order p. By these facts and by the property of the Legendre polynomial: $\int_0^1 \{P_n(x)\}^2 dx = O(n^{-1})$ for large n, we can easily show that the first integral of the right-hand side of the equality (19) becomes,

$$\int_0^1 \left\{ (1-u^2)^p \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u) \right\}^2 du = \int_0^1 (1-u^2)^2 \left\{ (1-u^2)^{p-1} \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u) \right\}^2 du$$
$$= O(n^{2(p-1)}) \cdot O(n^{-1}) = O(n^{d-6}).$$

For the second integral of the right-hand side of (19), we have

$$\left| \int_0^1 (1-u^2)^p \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u) \left\{ \int_u^1 x(1-x^2)^{p-1} \frac{d^{p-1}}{dx^{p-1}} P_{n+p}(x)dx \right\} du \right|$$
$$\leqq \left\{ \int_0^1 \left\{ (1-u^2)^p \frac{d^{p-1}}{du^{p-1}} P_{n+p}(u) \right\}^2 du \right\}^{1/2} \cdot \left\{ \int_0^1 \left\{ \int_u^1 x(1-x^2)^{p-1} \frac{d^{p-1}}{dx^{p-1}} P_{n+p}(x)dx \right\}^2 du \right\}^{1/2}$$

The first term of the product on the right-hand side of the inequality, by the above result, has the order $O(n^{\frac{d-6}{2}})$ and the integrand of the second term can be evaluated as follows:

$$\left\{\int_u^1 x(1-x^2)^{p-1}\frac{d^{p-1}}{dx^{p-1}}P_{n+p}(x)dx\right\}^2 \leqq \int_u^1 x^2dx \cdot \int_u^1\left\{(1-x^2)^{p-1}\frac{d^{p-1}}{dx^{p-1}}P_{n+p}(x)\right\}^2dx$$
$$< \int_0^1 x^2dx \cdot \int_0^1\left\{(1-x^2)^{p-1}\frac{d^{p-1}}{dx^{p-1}}P_{n+p}(x)\right\}^2dx = O(n^{d-6}).$$

Hence the second integral is at most of order $O(n^{d-6})$. As for the last integral of the equality (19), by a similar approach, we estimate it to be at most of order $O(n^{d-6})$. This proves the desired result for $d=2p+3\,(p\geqq 1)$, and thus we have proved the theorem completely.

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