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THE FAILURE OF CANCELLATION LAWS FOR EQUIDECOMPOSABILITY TYPES

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ABSTRACT. Let **B** be a Boolean algebra and G a group of automorphisms of **B**. Define an equivalence relation $\sim$ on **B** by letting $x \sim y$ if there are $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n$ in **B** such that x is the disjoint union of the x_i, y is the disjoint union of the y_i, and for each i there is a member of G taking x_i to y_i. The equivalence classes under $\sim$ are called **equidecomposability types**. Addition of equidecomposability types is given by $(x)+(y) = (x \vee y)$ provided $x \wedge y = 0$. An example is given of a complete Boolean algebra **B** and a group G of automorphisms of **B** with $X, Y \in \mathbf{B}$ such that $(X)+(X) = (Y)+(Y)$ but $(X) \neq (Y)$, answering a question of Wagon (see [5 p. 231 problem 14]). Moreover **B** may be taken to be the algebra of Borel subsets of Cantor space modulo sets of the first category. It is also remarked that in this case equidecomposability types do not form a weak cardinal algebra.

1. Introduction. Let **B** be a Boolean algebra and G a group of automorphisms of **B**. We define an equivalence relation $\sim$ on **B** by letting $x \sim y$ if there are $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ in **B** such that x is the disjoint union of the x_i, y is the disjoint union of the y_i, and for each i there is a member of G taking x_i to y_i. It is readily seen that $\sim$ is an equivalence relation, and the equivalence classes are called **equidecomposability types** (with respect to **B** and G) after Tarski [**2**].

There is a natural partially defined operation of addition of equidecomposability types given by

$$(x)+(y) = (z)$$

whenever $x \wedge y = 0$ and $x \vee y = z$, where (x) is the equidecomposability type of x, etc. This is clearly independent of the choice of (disjoint) x and y from the types $(x), (y)$. Unfortunately + need not be total. A method for enlarging the class of equidecomposability types to a semigroup was given by Tarski [**2**] and is described in Chapter 8 of [**5**]. One construction (essentially equivalent to that given in [**5**]) is to consider the Boolean algebra **C** of infinite sequences of members of **B** under the co-ordinatewise operations. Note that **C** is complete (or countably complete) if and only if **B** is. The corresponding group H is taken to be the wreath product of G and Symm ω under its natural action on **C**. One may

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then check the following: (i) $\mathbf{B}$ may be naturally identified with $\{(b, 0, 0, \ldots) : b \in \mathbf{B}\} \subseteq \mathbf{C}$, (ii) under this identification, $b_1, b_2 \in \mathbf{B}$ are equidecomposable with respect to $\mathbf{B}$ and G if and only if they are equidecomposable with respect to $\mathbf{C}$ and H, (iii) addition of equidecomposability types with respect to $\mathbf{C}$ and H is totally defined. For this last it suffices to observe that $1_{\mathbf{C}} = (1_{\mathbf{B}}, 1_{\mathbf{B}}, 1_{\mathbf{B}}, \ldots)$ is equidecomposable with each of the disjoint elements $(1_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}}, \ldots)$ and $(0_{\mathbf{B}}, 1_{\mathbf{B}}, 0_{\mathbf{B}}, 1_{\mathbf{B}}, 0_{\mathbf{B}}, \ldots)$.

We shall therefore assume from now on that the family of equidecomposability types is closed under +. This means in particular that for each positive integer N, $x + x + \cdots + x = N.x$ is defined for equidecomposability types x. The question addressed in this paper is the validity of the cancellation laws

$$C(N) : N.x = N.y \longrightarrow x = y$$

for equidecomposability types x and y, principally in the case $N = 2$. This case is by far the easiest to handle.

THEOREM 1.1. *There is a complete Boolean algebra* $\mathbf{B}$ *and a group* G *of automorphisms of* $\mathbf{B}$ *such that the cancellation law* $C(2)$ *fails for equidecomposability types with respect to* $\mathbf{B}$ *and* G.

It seems likely that our methods can be adapted to discuss $N > 2$ also, but the technical details will be more involved. The goal would be to prove the following (as in [**3**, Theorem 6.2]).

CONJECTURE 1.2. *In the semigroup of equidecomposability types of a countably complete Boolean algebra* $\mathbf{B}$ *with respect to a group of automorphisms of* $\mathbf{B}$, $(\forall N \in Z) C(N) \longrightarrow C(M)$ *is provable for* $Z \cup \{M\}$ *a set of positive integers if and only if every prime factor of* M *is a factor of a member of* Z.

The question of the provability of $C(N)$ for equidecomposability types is a generalization of the corresponding question for arbitrary cardinal numbers. Of course in that case for the problem to have any significance we should ask whether $C(N)$ is provable without appeal to the axiom of choice. This was achieved by Tarski [**1**] and [**2**]. Guided by that case one would like equidecomposability types to behave as much like cardinals as possible. In particular one would like the Schröder-Bernstein Theorem to hold. It turns out (see [**5**]) that for this it suffices that $\mathbf{B}$ be countably complete, which we shall require to hold from now on. The Schröder-Bernstein Theorem may be formulated in this context as:

$$\text{if } x \leqq y \leqq z \text{ and } x \sim z \text{ then } x \sim y.$$

In our main result, that $C(2)$ is unprovable for equidecomposability types, we shall actually be able to arrange that $\mathbf{B}$ is complete, and it may be taken to be the category algebra (the family of Borel subsets of Cantor space, 2^{ω}, modulo the ideal of meagre sets).

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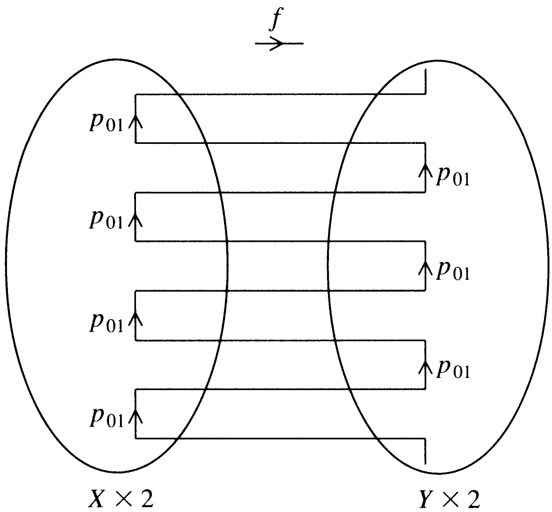

Figure 1. Illustration for the case $N = 2$.

Theorem 8.7 of [**5**] gives the result of König, Valkó and Kuratowski that $C(N)$ is provable when **B** is the power set Boolean algebra of a set and G is an arbitrary group of automorphisms of **B**. Notice that this implies that to find a counter-example we must look at a σ-field of subsets of an uncountable set, or else a quotient of such a σ-field. Moreover it is instructive to consider the contrast between their proof and that given by Tarski for cardinal numbers without choice, since this illustrates important features of our proof. Suppose that a 1-1 map f from $X \times N$ onto $Y \times N$ is given, where for simplicity we assume $X \cap Y = \emptyset$. For $0 \leqq i < j < N$ let p_{ij} be the permutation of $(X \cup Y) \times N$ given by

$$p_{ij}(t,k) = \begin{cases} (t,j) & \text{if } k = i \\ (t,i) & \text{if } k = j \\ (t,k) & \text{otherwise} \end{cases}$$

and let G be the (countable) group generated by $f \cup f^{-1}$ and $\{p_{ij} : 0 \leqq i < j < N\}$. Then any "effective" construction of a 1-1 map g from $X \times \{0\}$ onto $Y \times \{0\}$ should involve choosing $g(x,0)$ to lie in the G-orbit of $(x,0)$ for each $x \in X$. This is how the proofs mentioned above proceed. This is illustrated for the case $N = 2$ in Figure 1.

There is on each G-orbit a natural notion of "distance" defined by $d(s,t) =$ the smallest length of a word in the generators taking s to t. The rather complicated proof of Tarski's involved choices of $g(x,0)$ arbitrarily far from $(x,0)$ in this sense, whereas in the proof of König, Valkó and Kuratowski, $g(x,0)$ was taken at distance 1 from $(x,0)$ in each case. This difference was high-lighted in [**3**] in the

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study of properties $C(N)$ and $T(N)$, $C(N)$ the cancellation law given as above, and $T(N)$ a related but stronger form of "direct" cancellation law. It follows from what we show in §3 that if one can establish cancellation with $d(g(x,0),(x,0))$ bounded then it can also be established with distance 1 (provided **B** is countably complete). Thus Tarski's use of $d(g(x,0),(x,0))$ unbounded was essential in his case (where AC was not to be assumed).

In §2 we give the basic elements of the construction of the algebra **B**, $X, Y \in$ **B**, and group G such that $2.(X) = 2.(Y)$. In §3 the kernel of the proof, namely that $(X) \neq (Y)$, is presented. A key point here is the fact that **B** comprises the family of Borel subsets of a space homeomorphic to 2^ω, so that the Baire category theorem holds and any member of **B** has the property of Baire. At the same time we show how to deduce failure of cancellation for the category algebra. Finally in §4 we make some brief remarks on cardinal algebras and weak cardinal algebras. The failure of $C(2)$ immediately implies by [**2**] that equidecomposability types do not (necessarily) form a cardinal algebra. It is remarked that they do not even form a "weak cardinal algebra" in the sense of [**4**], since "approximate cancellation" fails.

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2. The Main Construction. The example in which $2.(X) = 2.(Y)$ and $(X) \neq (Y)$ will be constructed using a family of "undirected **Z**-cycles". A "**Z**-cycle" or "**Z**-sequence" would be a function with domain **Z** and would hence automatically have associated with it a direction, corresponding to the ordering of **Z**. Since we want to avoid this, and indeed any particular indexing of the entries by integers, we shall rather take an **undirected Z-cycle** to be a simple loop-free, circuit-free, connected graph in which every vertex has degree 2. To obtain the appropriate X and Y we then let T be the family of all undirected **Z**-cycles in which each vertex and edge is labelled so that

(i) the vertices are labelled 0 or 1, and adjacent vertices have different labels,

(ii) the edges are labelled by pairs (i,j) where $i,j \in \{0,1\}$,

(iii) if $(i,j),(k,l)$ are the labels on the edges incident with a vertex labelled 0 then $i \neq k$,

(iv) if $(i,j),(k,l)$ are the labels on the edges incident with a vertex labelled 1 then $j \neq l$.

We identify members of T which are isomorphic (by a label-preserving isomorphism). Of course any tree is bipartite, but (i) fixes an explicit bipartition. The import of (iii) is that from a vertex labelled 0, for each $i \in \{0,1\}$ we can pass to an adjacent vertex by an edge labelled (i,j) in precisely one way. Similarly for (iv) except with regard to the second co-ordinate. Let

$$X = \{(t,a) : t \in T \text{ and } a \text{ is a vertex of } t \text{ labelled } 0\} \text{ and}$$
$$Y = \{(t,a) : t \in T \text{ and } a \text{ is a vertex of } t \text{ labelled } 1\}.$$

Once again we identify members of X and Y which are isomorphic by a label and root-preserving isomorphism (where a is the **root** of (τ, a)). Strictly speaking

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then the members of X and Y are equivalence classes of labelled rooted trees. It is easier however to handle them using representative trees. Moreover this will be justified below in the sense that the equivalence classes having more than one member (the periodic case) form a meagre set in the topology to be introduced, so cause no essential problem. We further define

$$X_{ij} = \{(t,a) \in X : \text{ some edge incident with } a \text{ is labelled } (i,j)\} \text{ and}$$
$$Y_{ij} = \{(t,a) \in Y : \text{ some edge incident with } a \text{ is labelled } (i,j)\}.$$

Note that it follows from (iii) and (iv) that for each i, X is the disjoint union of X_{i0} and X_{i1} and Y is the disjoint union of Y_{0i} and Y_{1i}. We let f_{ij} be a bijection from $X_{ij} \cup Y_{ij}$ to itself given by $f_{ij}(t,a) = (t,b)$ where $\{a,b\}$ is an edge of t labelled (i,j). Then f_{ij} maps X_{ij} to Y_{ij} and Y_{ij} to X_{ij} and f_{ij}^2 is the identity. We extend f_{ij} to f'_{ij} on the whole of $X \cup Y$ by letting

$$f'_{ij}(u) = \begin{cases} f_{ij}(u) & \text{if } u \in X_{ij} \cup Y_{ij} \\ u & \text{otherwise.} \end{cases}$$

The group G of permutations of $X \cup Y$ is taken to be the group generated by $\{f'_{00}, f'_{01}, f'_{10}, f'_{11}\}$, and $\mathbf{B}$ will be the family of all Borel subsets of $X \cup Y$ under a topology shortly to be defined. Since each X_{ij}, Y_{ij} will be open (clopen in fact), we deduce that

$$\begin{aligned} 2.(X) &= [(X_{00}) + (X_{01})] + [(X_{10}) + (X_{11})] \\ &= [(Y_{00}) + (Y_{10})] + [(Y_{01}) + (Y_{11})] = 2.(Y). \end{aligned}$$

The appropriate topology is obtained by considering finite approximations to members of $X \cup Y$. Let T_1 be the family of finite connected subsets of members of T (with the induced labelling) and let $X_1 = \{(\tau, a) : \tau \in T_1$ and a is a vertex of τ labelled $0)\}$ and similarly for Y_1. If $(\tau, a) \in X_1$ we let $X(\tau, a) = \{(t, a') \in X : \tau$ is isomorphic to a connected subset of t by a label-preserving map taking a to $a'\}$, and similarly $Y(\tau, a)$ for $(\tau, a) \in Y_1$. The topology is that which is obtained by taking the family of all $X(\tau, a)$ for $(\tau, a) \in X_1$ and $Y(\tau, a)$ for $(\tau, a) \in Y_1$ as basic open sets.

By considering the $\tau \in T_1$ having two nodes joined by an edge labelled (i,j) we at once see that X_{ij} and Y_{ij} are open as remarked above. Moreover for any fixed k there are only finitely many members of T_1 having k vertices, (2^k in fact) and from this it easily follows that each basic $X(\tau, a), Y(\tau, a)$ is actually clopen (including X_{ij}, Y_{ij}). Moreover it is clear that $X \cup Y$ is a compact Hausdorff space with a countable base of clopen sets and no isolated points, so is homeomorphic to Cantor space 2^ω. An explicit homeomorphism is not hard to construct. More is true indeed. If we assign measures to the basic clopen sets by letting $\mu(X(\tau, a)) = 1/2^k$ if τ has k nodes, and similarly for $\mu(Y(\tau, a))$, then the homeomorphism with Cantor space may be chosen so as to carry μ to Lebesgue measure there. (We conjecture that this will form the basis of a proof

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that cancellation also fails in the measure algebra = the Borel subsets of 2^ω factored by the ideal of measure zero sets).

We deduce that the Baire category theorem holds in $X \cup Y$, and the usual properties of meagre and comeagre sets apply. In particular if we let **B** be the algebra of Borel subsets of $X \cup Y$ it follows that any member of **B** has the property of Baire, so is either meagre, or is comeagre on a non-empty basic clopen set.

Now the f'_{ij} were defined as permutations of $X \cup Y$. To show that they induce automorphisms of **B** it suffices to prove that they are homeomorphisms of the above topology, and since $(f'_{ij})^2 = 1$, that they take basic clopen sets to clopen sets. Now a basic clopen set $X(\tau, a)$ either has a as an interior vertex of τ, or is the union of two such basic clopen sets. But if a is interior, $f'_{ij}X(\tau, a)$ is either $Y(\tau, b)$ or $X(\tau, a)$, according as some edge $\{a, b\}$ incident with a is or is not labelled (i, j). Similarly for $Y(\tau, a)$. Hence any image of a basic clopen set under f'_{ij} is clopen, as required. It also follows that f'_{ij} induces an automorphism of **B** factored by the ideal of meagre sets.

As remarked above we have now done enough to establish $2.(X) = 2.(Y)$, and this holds in both cases, i.e. **B** and **B**/meagre sets. §3 will be devoted to proving that $(X) \neq (Y)$ in these algebras.

We conclude this section by remarking on some points of the construction. We have ensured that the orbits of G on $X \cup Y$ are clearly exhibited. Namely if $t \in T$, then $\{(t, a) : a \text{ is a vertex of } t\}$ is an orbit, and all orbits are of this form. Most of these orbits are infinite, but exceptionally some will be finite, where the labelling is periodic (remembering that we are identifying isomorphically labelled undirected **Z**-cycles). The finite orbits form a countable set however which is therefore meagre and so does not impede the proof. On an infinite orbit if (t, a) and (t, b) are two members there is a unique word w in the f_{ij} such that $w(t, a) = (t, b)$, with any two adjacent elements distinct. Although the definition of the f_{ij} (and f'_{ij}) is given separately on each orbit, it is nevertheless "uniform", as measured by the topology. Since there is a definability restriction on the members of **B**, this uniformity will enable us to show that no piecewise combination of members of G can take X to Y.

3. Proof of Non-Cancellation. The Boolean algebra **B** and its factor algebra (the category algebra) were defined in §2 and the group G given for which $2.(X) = 2.(Y)$. Here we shall show that $(X) \neq (Y)$. This means in the case of **B** that there is no bijection from X to Y which piecewise lies in G. The proof will at the same time establish the corresponding property for the category algebra.

If $(X) = (Y)$ then only finitely many members of G will be involved in showing this. Since each is represented by a reduced word in the generators there is a maximum distance N by which members of X are moved under the bijection. The proof proceeds by showing this is impossible by induction on N. Firstly we look at the basis case.

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LEMMA 3.1. *Suppose that S_{ij}, T_{ij} for $i, j \in \{0, 1\}$ are pairwise disjoint Borel subsets of $X \cup Y$ for which $S_{ij} \subseteq X_{ij}$, $T_{ij} \subseteq Y_{ij}$ and $f_{ij}S_{ij} = T_{ij}$. Then $X - \bigcup_{i,j<2} S_{ij}$ and $Y - \bigcup_{i,j<2} T_{ij}$ cannot both be meagre.*

Proof. Suppose otherwise for a contradiction. Firstly let $\tau \in T_1$ have an endpoint a labelled 0 and an edge $\{a, b\}$ incident with a labelled (i, j). We show that S_{ij} is not comeagre on $X(\tau, a)$. We form $\tau' \in T_1$ from two copies of τ, an additional vertex c and two new edges. Let c be labelled 1 and be joined to the two copies of vertex a by edges labelled $(1 - i, 0)$ and $(1 - i, 1)$. This ensures that $\tau' \in T_1$.

By hypothesis, $Y_{1-i0} - (T_{00} \cup T_{01} \cup T_{10} \cup T_{11})$ is meagre. Since $Y_{i0} \cap Y_{1-i0} = \emptyset$ and $T_{i0} \subseteq Y_{i0}$, $Y_{1-i0} - (T_{1-i0} \cup T_{01} \cup T_{11})$ is meagre. Similarly $Y_{1-i1} - (T_{1-i1} \cup T_{00} \cup T_{10})$ is meagre, from which we deduce that $(Y_{1-i0} \cap Y_{1-i1}) - (T_{1-i0} \cup T_{1-i1})$ is meagre. But $Y(\tau', c) \subseteq Y_{1-i0} \cap Y_{1-i1}$ and $Y(\tau', c)$ is non-meagre. Hence $Y(\tau', c) \cap T_{1-ik}$ is non-meagre for $k = 0$ or 1. Applying f_{1-ik} and noting that $X(\tau, a) \supseteq f_{1-ik}Y(\tau', c)$ we find that $X(\tau, a) \cap S_{1-ik}$ is non-meagre for $k = 0$ or 1. Hence S_{ij} cannot be comeagre on $X(\tau, a)$.

Now let $\tau \in T_1$ have a vertex a labelled 0 and an edge $\{a, b\}$ incident with a labelled (i, j). Suppose also that τ has an endpoint c on the opposite side of a from b labelled 0. We show by induction on the distance from a to c (necessarily even) that S_{ij} is not comeagre on $X(\tau, a)$.

If the distance is 0, $a = c$ and this is the case already covered.

Otherwise let d, e be the next vertices beyond a in the path to c. Then their labels are 1,0 respectively. Let the edges $\{a, d\}, \{d, e\}$ be labelled (k_1, l_1) and (k_2, l_2). By the induction hypothesis, $S_{k_2l_2}$ is not comeagre on $X(\tau, e)$. Applying $f_{k_2l_2}$, $T_{k_2l_2}$ is not comeagre on $Y(\tau, d)$. But $l_1 \neq l_2$ since d is labelled 1, so as above $Y_{k_1l_1} \cap Y_{k_2l_2} \subseteq T_{k_1l_1} \cup T_{k_2l_2}$. As $Y(\tau, d) \subseteq Y_{k_1l_1} \cap Y_{k_2l_2}$, $T_{k_1l_1}$ is not meagre on $Y(\tau, d)$. Applying $f_{k_1l_1}$, $S_{k_1l_1}$ is not meagre on $X(\tau, a)$ and so S_{ij} cannot be comeagre there.

But S_{ij} is Borel so that it has the property of Baire. Therefore if S_{ij} is non-meagre it is comeagre on some basic clopen set $X(\tau, a)$ where we may suppose that the endpoints of τ are labelled 0, contrary to what has just been shown. Hence S_{ij} is meagre for each i, j, so that $X - \bigcup_{i,j<2} S_{ij}$ is comeagre after all, giving the desired contradiction. □

Next we move towards the induction step. Let W be the set of ordered members of T_1, and V the set of members of W whose initial and final vertices are labelled 0,1 respectively. For $w \in W$ we define a bijection f_w. Let τ be the unordered version of w, and a and b its initial and final vertices. Then $\mathrm{dom} f_w = X(\tau, a)$ or $Y(\tau, a)$ (according as a is labelled 0 or 1) and range $f_w = X(\tau, b)$ or $Y(\tau, b)$, and $f_w(t, a) = (t, b)$, for $(t, a) \in \mathrm{dom} f_w$. Note that if the edges of τ in the order chosen by w are $\alpha_1, \alpha_2, \ldots, \alpha_K$ then f_w is just the composition $f_{\alpha_K} f_{\alpha_{K-1}} \ldots f_{\alpha_1}$ on the appropriate domain. Often we think of w just as a word in the α_i, neglecting the labels on the vertices, so that the definition just given amounts to letting $f_{\alpha_1\alpha_2\ldots\alpha_K} = f_{\alpha_K} f_{\alpha_{K-1}} \ldots f_{\alpha_1}$. We let $|w| = |\tau| = K =$ the number of edges of $w =$ the length of w as a word.

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Note that if $w \in V$, $\operatorname{dom} f_w = X(\tau, a)$ and range $f_w = Y(\tau, a)$.

LEMMA 3.2. *Suppose that $S_i \subseteq X, T_i \subseteq Y$ are pairwise disjoint Borel sets, w_i are distinct members of V, $1 \leqq i \leqq n$, and $N \geqq 3$, such that for $S = \bigcup_{1 \leqq i \leqq n} S_i$ and $T = \bigcup_{1 \leqq i \leqq n} T_i$,*

(i) $N = |w_1| \geqq |w_2| \geqq \ldots \geqq |w_n|$,

(ii) *if $w \in V$ and $|w| < N$ then $w = w_i$ for some i,*

(iii) $f_{w_i} S_i = T_i$,

(iv) *for any $g \in G$, $g(S \cup T) \subseteq S \cup T$,*

(v) *$S \cup T$ is a union of infinite G-orbits.*

Then there are pairwise disjoint Borel sets $S_i' \subseteq X, T_i' \subseteq Y$ such that $\bigcup_{1 \leqq i \leqq n} S_i' = S$, $\bigcup_{1 \leqq i \leqq n} T_i' = T$, $f_{w_i} S_i' = T_i'$ for each i, and $S_1' = T_1' = \emptyset$.

Proof. It is rather easier to think of the S_i, T_i, S_i', T_i' in terms of the bijections corresponding, which are piecewise combinations of members of G. We therefore let

$$\theta(x) = f_{w_i}(x) \text{ if } x \in S_i.$$

Thus θ is a bijection from S to T. From θ we shall define another bijection θ' between the two sets, and then S_i', T_i' will result by letting

$$S_i' = \{x \in S : \theta'(X) = f_{w_i}(x)\} \quad \text{and } T_i' = f_{w_i} S_i' \quad (= \theta' S_i').$$

This definition works because whenever u and v are in the same infinite G-orbit of $X \cup Y$ it is clear that there is a unique $w \in W$ such that $f_w(u) = v$, and by (v) the orbits under consideration **are** infinite.

As previously remarked it is easiest to think of θ and θ' as acting separately on each G-orbit in $S \cup T$ and this is essentially how our construction will go. On the other hand the definition of θ' in terms of θ has to be sufficiently "uniform" for the resulting sets S_i' and T_i' still to be Borel. The main point we have to aim for is that S_1' is empty. Rephrasing this, if $x \in S_1$ we must ensure that $\theta'(x) \neq \theta(x)$. This will be done by mapping x a little less far along its G-orbit than before. More precisely, if $w_1 = \alpha_1\alpha_2 \ldots \alpha_N$ where each $\alpha_i \in \{0, 1\}^2$ we shall let $\theta'(x) = f_{\alpha_1\alpha_2\ldots\alpha_{N-2}}(x)$ instead of $f_{\alpha_1\alpha_2\ldots\alpha_N}(x)$. This makes sense since by (i), $N = |w_1| \geqq 3$ and by (ii), $\alpha_1\alpha_2 \ldots \alpha_{N-2} = w_i$ for some i.

Since we propose making $\theta'(x)$ differ from $\theta(x)$ for $x \in S_1$, this will (or may) entail altering other values of $\theta(x)$. This has to be done with some care to ensure that if $\theta'(x) = f_{\alpha_1\alpha_2\ldots\alpha_K}(x)$ then $\alpha_1\alpha_2 \ldots \alpha_K = w_j$ for some $j \geqq 2$. In particular we must make sure that $K \leqq N$. Since we have a pictorial representation of the situation in mind, we make the following definitions.

Two members x_1 and x_2 of S are **adjacent** if for some $w \in W$ of length 2, $f_w(x_1) = x_2$. (This relation is symmetric since if $f_w(x_1) = x_2$ then $f_{w^R}(x_2) = x_1$ where w^R is the conversely ordered member of W). If x_1, x_2 are adjacent members of S then they are said to be **parallel** if there are $u, v \in W$ both of

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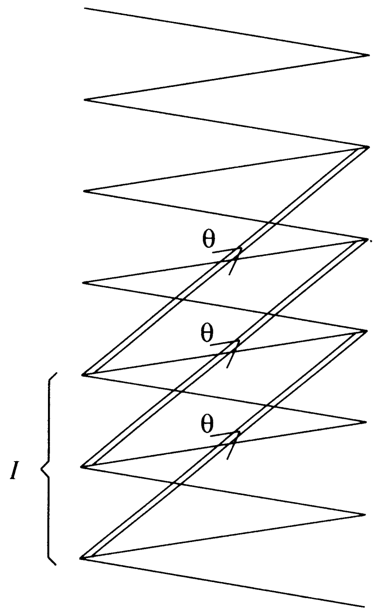

Figure 2

length 2 and i, j with $|w_i| = |w_j| = N$ such that $f_u(x_1) = x_2, x_1 \in S_i$, $x_2 \in S_j$ and either $f_{w_j} f_u(x_1) = f_v f_{w_i}(x_1)$ or $f_{w_i} f_{u^R}(x_2) = f_v f_{w_j}(x_2)$. The intuitive meaning of this is that x_1 and x_2 are both mapped in the same direction along their G-orbit by the greatest possible distance. An **interval** of S is a connected subset of S under the graph structure given by adjacency. A **parallel interval** is an interval I of S such that any two adjacent members of I are parallel, and some member of I is in S_1.

The terminology adopted is suggested by the natural mapping diagram, (see Figure 2).

Members of a parallel interval are mapped by the same distance and in the same direction. Since $\theta'(x) \neq \theta(x)$ for $x \in S_1$ this will entail altering the image of each member of a parallel interval.

LEMMA 3.3. *(i) If $x_0, x_1, \ldots, x_K$ are members of S such that x_i is adjacent to x_{i+1} for $0 \leqq i < K$ and $\{x_1, x_2, \ldots, x_{K-1}\}$ is a parallel interval with $K > N$ then $\{x_0, x_1, \ldots, x_K\}$ is also a parallel interval (i.e. any sufficiently large parallel interval can be extended arbitrarily far in both directions).*

(ii) Any finite maximal parallel interval has at most $N - 1$ members.

Proof. (i) Let $x_0 = (t, a_0)$ and let t have vertices $\{a_i : i \in \mathbf{Z}\}$ with $\{a_i, a_{i+1}\}$ an edge of t for each i, a_{2i} labelled 0 and a_{2i+1} labelled 1, and $x_i = (t, a_{2i})$ for $0 \leqq i \leqq K$. Let φ be the bijection from the set of even integers to the set of odd integers given by $\varphi(2i) = 2j + 1 \leftrightarrow \theta(t, a_{2i}) = (t, a_{2j+1})$. By definition

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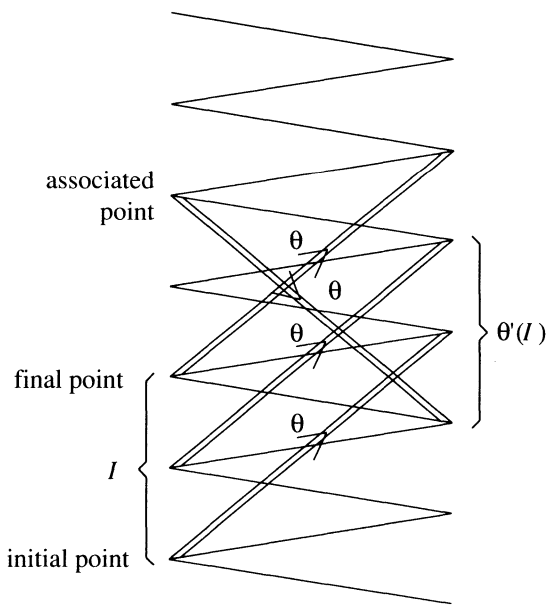

Figure 3

of "parallel interval", either $\varphi(2i) = 2i + N$ for all $i, 1 \leqq i \leqq K - 1$, or $\varphi(2i) = 2i - N$ for all such i. Suppose the former without loss of generality. To show that $\{x_0, x_1, \ldots, x_K\}$ is a parallel interval it suffices to see that $\varphi(0) = N$ and $\varphi(2K) = 2K + N$.

Suppose $\varphi(0) \neq N$. Then $\varphi^{-1}(N) \geqq 2$ since by assumption (i) of Lemma 3.2, $|\varphi(r) - r| \leqq N$ for all r. As $N \notin \varphi\{2, 4, \ldots, 2K - 2\}$, $\varphi^{-1}(N) \geqq 2K$. But $\varphi(\varphi^{-1}(N)) \geqq \varphi^{-1}(N) - N$ so $N \geqq 2K - N$ and $N \geqq K$ contrary to supposition.

Suppose $\varphi(2K) \neq 2K + N$. Then $\varphi(2K) < 2K + N$ and similarly we deduce that $\varphi(2K) \leqq N$. As $\varphi(2K) \geqq 2K - N$, again $N \geqq K$, a contradiction.

(ii) is immediate from (i). □

It follows from this lemma that any maximal parallel interval is either finite with $\leqq N - 1$ members, or is the whole of a G-orbit. It is obvious that distinct maximal parallel intervals cannot overlap, but we require a slightly stronger form of disjointness, namely that their θ'-images do not overlap. To make this more precise (since we haven't yet defined θ') we give the following definitions.

Let I be a finite maximal parallel interval. If $|I| = 1$ its one member is both its **initial** and **final point**. Otherwise it has two endpoints (t, a) and (t, b), $(t, a) \in S_i$, $(t, b) \in S_j$. Of these there is just one, (t, a) say, such that if $w_i = \beta_1\beta_2 \ldots \beta_N$ then $f_{\beta_2} f_{\beta_1}(t, a) \in I$, and we call (t, a) the **initial point** of I and (t, b) its **final point**. We call $\theta^{-1} f_{\beta_1\beta_2\ldots\beta_{N-2}}(t, a)$ the point **associated with** I. See Figure 3.

The idea is that as $f_{\beta_1\beta_2\ldots\beta_{N-2}}(t, a) \notin \theta(I)$, the point z associated with I lies outside I, but θ' will be so defined that $\theta'(I \cup \{z\}) = \theta(I \cup \{z\})$.

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LEMMA 3.4. *If I_1, I_2 are distinct finite maximal parallel intervals with associated points z_1, z_2 then $z_1 \neq z_2$.*

Proof. If I_1, I_2 are contained in different G-orbits, so are z_1 and z_2, so $z_1 \neq z_2$. So suppose they are both contained in the G-orbit $\{(t, a_i) : i \in \mathbf{Z}\}$ chosen as previously so that each $\{a_i, a_{i+1}\}$ is an edge of t, a_{2i} is labelled 0, a_{2i+1} is labelled 1, and let φ be given as before by $\varphi(2i) = 2j + 1 \Leftrightarrow \theta(t, a_{2i}) = (t, a_{2j+1})$. Let $J_j = \{i : (t, a_i) \in I_j\}$ for $j = 1, 2$.

The main point is to check that φ is order-preserving on $R = \{r : (t, a_r) \in S_1\}$. For suppose not and let $2r < 2s$ in R with $\varphi(2r) > \varphi(2s)$. Then $\varphi(2s) - 2s < \varphi(2r) - 2r$ so $\varphi(2s) - 2s = -N$ and $\varphi(2r) - 2r = N$. Therefore $\theta(t, a_{2r}) = (t, a_{2r+N})$ and $\theta(t, a_{2s}) = (t, a_{2s-N})$. Recalling that $w_1 = \alpha_1\alpha_2 \ldots \alpha_N$ it follows that $\{a_{2r+i-1}, a_{2r+i}\}$ and $\{a_{2s-i+1}, a_{2s-i}\}$ are both labelled α_i for $1 \leqq i \leqq N$. Now $0 < s - r = \frac{1}{2}[(\varphi(2s) + N) - (\varphi(2r) - N)] < N$. Putting $i = s - r$, we find that $\{a_{r+s-1}, a_{r+s}\}$ and $\{a_{r+s+1}, a_{r+s}\}$ both have the label α_i, contrary to stipulations (iii) and (iv) in the definition of T (which required that the two edges incident with a vertex of a member of T should have different labels.)

Now let the smallest and largest members of J_j be $2r_j$ and $2s_j$ respectively ($j = 1, 2$) and suppose that $2s_1 < 2r_2$. Since each of I_1 and I_2 contains a member of S_1, each of J_1 and J_2 contains a member of R, $2r$ and $2r'$ respectively. As $2r \leqq 2s_1 < 2r_2 \leqq 2r'$ and φ is order-preserving on R, $\varphi(2r) < \varphi(2r')$, from which it follows that $\varphi(2s) < \varphi(2s')$ for all $2s \in J_1, 2s' \in J_2$, and in particular that $\varphi(2s_1) < \varphi(2r_2)$. If $z_1 = z_2 = (t, a_m)$ then $\varphi(2m) = 2r_1 + N - 2$ or $2s_1 - N + 2$ (according as (t, a_{2r_1}) or (t, a_{2s_1}) is the initial point of I_1) and also $\varphi(2m) = 2r_2 + N - 2$ or $2s_2 - N + 2$. Since $r_1 \neq r_2$ and $s_1 \neq s_2$, either $2r_1 + N - 2 = 2s_2 - N + 2$ or $2r_2 + N - 2 = 2s_1 - N + 2$. The former implies

$$\varphi(2s_1) = 2s_1 + N > 2r_1 + N - 2 = 2s_2 - N + 2 > 2r_2 - N = \varphi(2r_2),$$

contrary to $\varphi(2s_1) < \varphi(2r_2)$, and the latter is also impossible, since

$$2s_1 - N + 2 < 2s_1 < 2r_2 < 2r_2 + N - 2. \qquad \square$$

We may now define θ' according to the following cases:

Case 1. If $x \in S_i$ lies in a parallel interval and $w_i = \beta_1\beta_2 \ldots \beta_N$ we let $\theta'(x) = f_{\beta_1\beta_2\ldots\beta_{N-2}}(x)$,

Case 2. if x does not lie in a parallel interval but is the point associated with a finite maximal parallel interval I having final point y then we let $\theta'(x) = \theta(y)$,

Case 3. if x does not lie in a parallel interval and is not the point associated with any finite maximal parallel interval, then $\theta'(x) = \theta(x)$.

We firstly remark that consideration of the function φ defined once we have represented a G-orbit in the form $\{(t, a_i) : i \in \mathbf{Z}\}$ shows that no point associated with a finite maximal parallel interval can lie in that interval or any other (one again uses the fact that φ is order-preserving on R). Thus θ' is well-defined. Next we see that for each G-orbit $\{(t, a_i) : i \in \mathbf{Z}\}$ indexed as before, θ' maps

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$\{(t, a_{2i}) : i \in \mathbf{Z}\} 1-1$ onto $\{(t, a_{2i+1}) : i \in \mathbf{Z}\}$. For this, since all points covered by Case 3 are mapped to the same place by θ and θ', it suffices to show that for each maximal parallel interval I, either I is the whole of the G-orbit (in which case $\theta'(I) = \theta(I)$ follows by Case 1), or I is a finite maximal parallel interval with associated point z and $\theta'(I \cup \{z\}) = \theta(I \cup \{z\})$, which follows by Cases 1 and 2.

Since θ is a bijection from S to T and θ' has the same effect setwise on G-orbits as θ, it follows that θ' is also a bijection from S to T. We may now deduce the values of the sets S_i' and T_i' as indicated above by

$$S_i' = \{x \in S : \theta'(x) = f_{w_i}(x)\} \quad \text{and } T_i' = f_{w_i}(S_i') = \theta' S_i'.$$

The facts that S is the disjoint union of the S_i' and T is the disjoint union of the T_i' follow since θ' is a bijection. We do need to remark that if $\theta'(x) = f_w(x)$ then $w = w_i$ for some i with $2 \leqq i \leqq n$. If x lies in some parallel interval then this follows from Case 1 and (ii) of Lemma 3.2, since w will be a proper initial segement of w_j for some j. If x is not associated with any finite maximal parallel interval then as $\theta'(x) = \theta(x)$, we have $\theta'(x) = f_{w_i}(x)$ for some i, and as any member of S_1 clearly lies in a parallel interval, $i > 1$. Suppose therefore that x is associated with a finite maximal parallel interval I. Then we may index the G-orbit of x in such a way that $I = \{(t, a_{2i}) : r \leqq i \leqq s\}$, $x = (t, a_{2m})$, and $\theta(t, a_{2i}) = (t, a_{2i+N})$ for $r \leqq i \leqq s$, with $s < m$. As $\theta(t, a_{2m}) = (t, a_{2r+N-2})$, $2r + N - 2 \geqq 2m - N$, so that

$$2m - N < 2r + N \leqq 2s + N < 2m + N.$$

Since $\theta'(t, a_{2m}) = (t, a_{2s+N}), x \in S_i'$ for some i with $1 \leqq i \leqq n$ (by (ii) of Lemma 3.2) and as $|w_i| < N, i > 1$.

The only other thing to check is that each S_i' is Borel. This is important, but though a little tedious to verify, is essentially straightforward, since θ' was defined in a fairly "natural" way.

LEMMA 3.5. *Let* $\bar{\alpha} = \alpha_1 \alpha_2 \ldots \alpha_K \in V$, $L = \frac{1}{2}(K - N + 2)$, *and let* $\bar{\imath} = (i_1, i_2, \ldots, i_L)$ *be a sequence of integers (necessarily between 1 and n) such that for* $1 \leqq j \leqq L$, $\alpha_{2j-1}\alpha_{2j} \ldots \alpha_{2j+N-2} = w_{i_j}$. *Then*

$$X_1^+(\bar{\alpha}, \bar{\imath}) = \{x \in S : x \in S_{i_1} \& f_{\alpha_2} f_{\alpha_1}(x) \\ \in S_{i_2} \& \ldots \& f_{\alpha_{K-N}} f_{a_{K-N-1}} \cdots f_{\alpha_1}(x) \in S_{i_L}\}$$

is Borel.

Proof. $X_1^+(\bar{\alpha}, \bar{\imath}) = S_{i_1} \cap f_{\alpha_1} f_{\alpha_2} S_{i_2} \cap \ldots \cap f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_{K-N}} S_{i_L}$ and as each f_α is a homeomorphism, this is Borel.

Similarly we see that if $\bar{\alpha} = \alpha_K \alpha_{K+1} \ldots \alpha_1 \alpha_2 \ldots \alpha_N \in V$ for odd $K \leqq 1$ and $\bar{\imath} = (i_M, i_{M+1}, \ldots, i_1)$ is a sequence of integers with $M = \frac{1}{2}(K+1)$ such

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that $\alpha_{2j-1}\alpha_{2j}\ldots\alpha_{2j+N-2} = w_{i_j}$ then $X_1^-(\bar{\alpha},\bar{\imath}) = \{x \in S : x \in S_{i_1}\&f_{\alpha_{-1}}f_{\alpha_0}(x) \in S_{i_0}\&\ldots\&f_{\alpha_K}f_{\alpha_{K+1}}\ldots f_{\alpha_0}(x) \in S_{i_M}\}$ is also Borel. □

LEMMA 3.6. *The following sets are Borel:*
(i) $X_1 = \{x :$ *x lies in a parallel interval*$\}$,
(ii) $X_2' = \{x :$ *x is the initial point of a finite maximal parallel interval*$\}$,
(iii) $X_2 = \{x :$ *x is the point associated with some finite maximal parallel interval*$\}$,
(iv) $X_3 = \{x \in S :$ *x does not lie in a parallel interval and is not the point associated with any finite maximal parallel interval*$\}$.

Proof. X_1 is the union of $X_1^+(\bar{\alpha},\bar{\imath})$ and $X_1^-(\bar{\beta},\bar{\jmath})$ for appropriate $\bar{\alpha},\bar{\beta},\bar{\imath},\bar{\jmath}$ so is Borel. Similarly the other sets may be obtained as Borel combinations of ones previously shown to be Borel or their homeomorphic images. □

We may now give a Borel expression for S_i' by

$$
\begin{aligned}
x \in S_i' \Leftrightarrow & (x \in X_1\&x \in S_j \text{ where } w_j \text{ is an end-extension of } w_i \\
& \quad \&|w_j| = |w_i| + 2) \\
& \text{or } (x \in X_2\& \text{ for some } \bar{\alpha} \in V, j \text{ and } \bar{i}, x \in S_j\& \\
& \quad f_{w_i}(x) = f_{\alpha_K}f_{\alpha_{K-1}}\ldots f_{\alpha_{N-1}}f_{w_j}(x)\& \\
& \quad f_{\alpha_1}f_{\alpha_2}\ldots f_{\alpha_{N-2}}f_{w_j}(x) \in X_1^+(\bar{\alpha},\bar{\imath}) \\
& \quad -\bigcup\{X_1^+(\bar{\alpha}\beta_1\beta_2,\bar{c}k_1k_2) : \alpha_1\ldots\alpha_K\beta_1\beta_2 \in V, 1 \leqq k_1,k_2 \leqq n\}) \\
& \text{or } (x \in X_3\&x \in S_i).
\end{aligned}
$$

The second clause corresponds to Case 2 of the definition of θ' and is illustrated in Figure 4 for $K = 9$, $N = 5$; note that the finite maximal parallel interval with which x is associated is

$$\{x', f_{\alpha_2}f_{\alpha_1}(x'), \ldots, f_{\alpha_{K-N}}\ldots f_{\alpha_1}(x')\}$$

with initial point

$$x' = f_{\alpha_1}f_{\alpha_2}\ldots f_{\alpha_{N-2}}f_{w_j}(x).$$

This completes the proof of Lemma 3.2. □

THEOREM 3.7. *X and Y are not equidecomposable with respect to* **B** *and G, even up to meagre sets. Hence the cancellation law C(2) fails in* (**B**, *G*) *and also in the category algebra (with the natural induced action of G).*

Proof. Suppose for a contradiction that X and Y are equidecomposable up to meagre sets, and let S_i, T_i be Borel sets for $1 \leqq i \leqq n$ such that $S_i \subseteq X, T_i \subseteq Y, S_i \cap S_j, T_i \cap T_j$ are meagre for $i \neq j, X - \bigcup_{i=1}^n S_i, Y - \bigcup_{i=1}^n T_i$ are meagre, and

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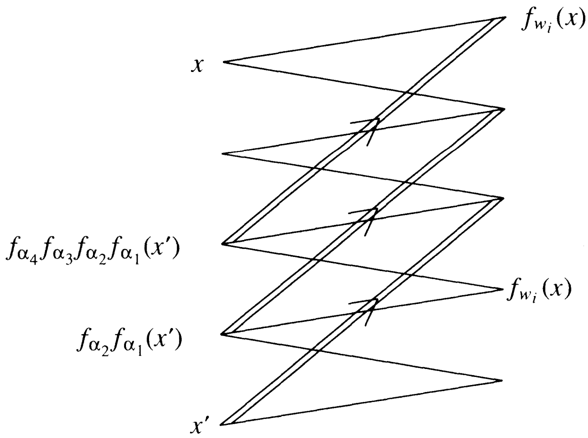

Figure 4

for some $g_i \in G, g_iS_i = T_i$. Now any group element can be represented by a word in the generators. Since each generator is an involution we only need to consider words of the form $h_1h_2 \dots h_m$ where h_j are generators and $h_j \neq h_{j+1}$ for each j.

Let g_i be so represented, and let N be the greatest length of a word appearing. Now X is the disjoint union of basic clopen sets of the form $X(t, a_0)$ where t has $2N + 1$ vertices with a_0 the middle one. By refining the partitions $\{S_i : 1 \leqq i \leqq n\}, \{T_i : 1 \leqq i \leqq n\}$ but not altering the group elements used to do the mapping, we may suppose that for each i, $S_i \subseteq X(t, a_0)$ for some such t and a_0. In addition we now suppose that each g_i is represented as a word of minimal length m in the generators. By choice of $N, m \leqq N$. Let us rewrite this word in the form $f'_{\alpha_m}f'_{\alpha_{m-1}} \dots f'_{\alpha_1}$. By the minimality of m, f'_{α_1} does not fix $X(t, a_0)$. Therefore an edge incident with a_0 must be labelled α_1 and $f'_{\alpha_1} = f_{\alpha_1}$ on $X(t, a_0)$. Let the other end of this edge be a_1. Similarly f'_{α_2} does not fix $X(t, a_1)$ and the other edge $\{a_1, a_2\}$ of t incident with a_1 is labelled α_2, and $f'_{\alpha_2} = f_{\alpha_2}$ on $X(t, a_1)$. Continuing in this way, using $m \leqq N$, we find consecutive vertices $a_0, a_1, a_2, \dots, a_m$ of t with $\{a_{j-1}, a_j\}$ labelled α_j and with $f'_{\alpha_j} = f_{\alpha_j}$ on $X(t, a_{j-1})$. Hence if w_i is the subtree of t with vertices $a_0, a_1, \dots, a_m$ in that order (with the induced labellings) g_i is equal to f_{w_i} on $X(t, a_0)$ and hence on S_i. Observe that as $S_i \subseteq X$ and $T_i \subseteq Y$, $w_i \in V$.

To sum up, we have Borel sets S_i, T_i and elements w_i of V such that $f_{w_i}S_i = T_i$ and up to meagre sets X, Y are respectively the disjoint unions of the S_i and T_i. By modifying these we may make the following also hold: (i) the w_i are all distinct (by lumping together those S_i, T_i with the same value of w_i), (ii) the S_i, T_i are pairwise disjoint (by removing appropriate meagre sets), (iii) any G-orbit of a member of $\bigcup S_i$ is infinite, (iv) if $g \in G$ and $x \in \bigcup_{1 \leqq i \leqq n}(S_i \cup T_i)$

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then $g(x) \in \bigcup_{1\leqq i\leqq n}(S_i \cup T_i)$ (by further removing a meagre set), (v) if N = the greatest length of some w_i (necessarily odd), then either $N = 1$, or $N \geqq 3$ and for every $w \in V$ of length less than N, $w = w_i$ for some i (by adding redundant, empty S_i's, T_i's as necessary, (vi) $|w_1| \geqq |w_2| \geqq \ldots \geqq |w_n|$ (by re-ordering), (vii) $\bigcup_{1\leqq i\leqq n}(S_i \cup T_i)$ is a union of G-orbits.

Examining Lemma 3.1 we firstly see that $N = 1$ is impossible. Let us now choose the S_i, T_i, w_i subject to all the above stipulations, firstly so that N is as small as possible, and secondly so that for that choice of N, n is as small as possible. We immediately find that Lemma 3.2 contradicts the choice of either N or n, establishing the theorem. □

4. Failure of Approximate Cancellation. "Weak cardinal algebras" were introduced in [4] in an attempt to derive as many properties of cardinal algebras as possible using only finitary addition +. The infinitary defining properties of a cardinal algebra were replaced by "finite refinement" and the following "approximate cancellation" law:

if $x + y = x + z$ there are p, q, r such that

$$x = x + p = x + q, \quad y = p + r, \quad z = q + r.$$

Summarizing what we know about weak cardinal algebras and cancellation laws:

(i) any cardinal algebra satisfies $C(N)$ for all $N \geqq 2$ [2],

(ii) any cardinal algebra is a weak cardinal algebra,

(iii) there is a weak cardinal algebra for which $C(N)$ fails ($N \geqq 2$) [3]; by (i) this cannot be made into a cardinal algebra however infinitary addition is defined,

(iv) surjective cardinals form a weak cardinal algebra; they also satisfy $C(N)$ for each $N \geqq 2$, [3]; it is unknown whether they form a cardinal algebra.

In view of the facts that the Schröder-Bernstein Theorem is provable for equidecomposability types (provided **B** is countably complete) and that the proofs for cardinal numbers of the Schröder-Bernstein Theorem and the approximate cancellation law are very similar one is led to enquire whether approximate cancellation is also provable there. The answer is "no" in our example above, (see Theorem 4.2 below), but in contrast to the König, Valkó and Kuratowski result, it isn't even provable for the case of power set Boolean algebras.

THEOREM 4.1. *There is a group G of permutations of ω such that the approximate cancellation law fails for equidecomposability types with respect to $P(\omega)$ and G.*

Proof. Instead of ω we use $T = \mathbf{Z} \times \omega$ and let G be generated by g where $g(n, i) = (n + 1, i)$ all $n \in \mathbf{Z}, i \in \omega$. Let $X = \{(n, i) \in T : |n| \leqq i\}, Y = \{(-i - 1, i) : i \in \omega\}$ and $Z = \{(i + 1, i) : i \in \omega\}$. Then X, Y, Z are pairwise disjoint, and $g(X \cup Y) = (X \cup Z)$. For approximation cancellation to hold

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for equidecomposability types there would have to be expressions for Y, Z as disjoint unions, $Y = P \cup R, Z = Q \cup R_1$, such that $R \sim R_1, X \sim X \cup P \sim X \cup Q$.

Since $R \sim R_1$ and any piecewise combination h of members of G must satisfy $h(n,i) = (m,i)$ where $|m - n|$ is bounded, R and R_1 are finite. Hence $P \neq \emptyset$. But if $(n,i) \in P$, some piecewise combination of members of G must then map $\{(n,i)\} \cup \{(m,i) : |m| \leqq i\}$1 − 1 onto $\{(m,i) : |m| \leqq i\}$, which is impossible. □

One may mimic this argument in many other instances. We give an outline in the case of our previous example.

THEOREM 4.2. *The approximate cancellation law fails for equidecomposability types for the algebra* **B** *and group* G *of Theorem 3.7.*

Proof. For $i \geqq 1$ we define $\tau_i \in T_1$ as follows. The vertices of τ_i are $\{a_i^j : -2i \leqq j \leqq 4i\}$ with a_i^j labelled 0 or 1 according as j is even or odd. The edges $\{a_i^j, a_i^{j+1}\}$ are labelled as follows:

$\{a_i^{2j}, a_i^{2j+1}\}$ is labelled (0, 0) and $\{a_i^{2j+1}, a_i^{2j+2}\}$ is labelled (1, 1) for $0 \leqq j < i$,

$\{a_i^{2j}, a_i^{2j+1}\}$ is labelled (0, 1) and $\{a_i^{2j+1}, a_i^{2j+2}\}$ is labelled (1, 0) for $-i \leqq j < 0$ and for $i \leqq j < 2i$.

We let $Z_i^j = X(\tau_i, a_i^{2j})$ for $0 \leqq j \leqq i$.

Observe that the Z_i^j are pairwise disjoint, since given a member of some Z_i^j we can recover the values of i and j by looking at the vertices reachable from the distinguished vertex only by edges labelled (0, 0) or (1, 1).

Let $A = \bigcup\{Z_i^0 : i \geqq 1\}$, $B = \bigcup\{Z_i^i : i \geqq 1\}$ and $C = \bigcup\{Z_i^j : 0 < j < i, i \geqq 1\}$. Then A, B, C are Borel and pairwise disjoint. Also $f_{11}f_{00}$ carries $A \cup C$ 1-1 onto $B \cup C$, so $A \cup C$ and $B \cup C$ are equidecomposable. If approximate cancellation held there would be pairwise disjoint Borel sets P, Q, R, R_1 with $A = P \cup R, B = Q \cup R_1, C \sim C \cup P \sim C \cup Q$ and $R \sim R_1$.

Suppose that $f_w(x) = y \neq x$ where $x \in Z_i^0, 2i \geqq |w| + 1$. Then w must be a proper initial segement of $((1,0),(0,1))^i$ or $((0,0),(1,1))^i$. But if $y = (t,a)$, in the former case the edges incident with a are labelled $(1,0),(0,1)$ and in the latter they are labelled $(0,0),(1,1)$, so that $y \notin B$. If N is the greatest length of a word w such that f_w is involved in establishing $R \sim R_1$ we therefore deduce that $R \subseteq \bigcup\{Z_i^0 : 2i < N + 1\}$, and hence that $P \supseteq \bigcup\{Z_i^0 : 2i \geqq N + 1\}$.

Now let M be the greatest length of a word w such that f_w is involved in establishing $C \sim C \cup P$ and let $i \geqq \frac{1}{2}(N + 1), \frac{1}{2}(M + 1)$. Let $x \in \bigcup_{0 \leqq j < i} Z_i^j$. Then if f is the given bijection from $C \cup P$ to C and $f(x) = f_w(x)$, w must be of the form $((0,0),(1,1))^k$ or $((1,1),(0,0))^k$, $k < i$, as $f(x) \in C$, since f_w preserves $\bigcup\{Z_i^j : j \in \mathbf{Z}\}$ and all edges between x and $f(x)$ are labelled $(0,0)$ or $(1,1)$. In particular it follows that $f(x) \in \bigcup_{0<j<i} Z_i^j$. Therefore if $y \in Z_i^0, f$ maps

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$\{(f_{11}f_{00})^j y : 0 \leqq j < i\}$ 1 − 1 onto $\{(f_{11}f_{00})^j y : 0 < j < i\}$, which is impossible since these are finite sets. □

By modifying the argument a little, we find that approximate cancellation also fails when we factor out either the ideal of meagre or of measure zero sets.

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