PART V

TRAPPED MOTION
IN THE THREE-BODY PROBLEM

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

ASYMPTOTIC APPROACH TO MIRROR CONDITIONS AS A TRAPPING MECHANISM IN N-BODY HIERARCHICAL DYNAMICAL SYSTEMS

Archie E. Roy

Department of Astronomy,
Glasgow University,
Glasgow, U.K.

ABSTRACT

The consequences for hierarchical stability of almost circular almost coplanar, low perturbation orbits in an n-body hierarchical dynamical system is discussed. It is shown that frequent close approaches to mirror conditions with subsequent reversing of perturbations is ensured by such properties. The part played by near commensurabilities in mean motion is also discussed, the Sun-Jupiter-Saturn case being taken as an example.

1. INTRODUCTION

The Solar System, comprised of the planetary system and the satellite systems, exhibits a hierarchical structure in that the orbits can be ordered in size in each system. The main bodies in each system also exhibit hierarchical stability in that the ordering in size does not change in a time long compared with the longest period of revolution in that system. In addition we see that for the most part the orbits are almost circular and almost coplanar.

A question worth examining is: how did the solar system come to be trapped into such a situation and what maintains it in the trap? Indeed a further related question is: what is the nature of the trap?

With respect to n-body hierarchical systems with $n > 3$, there would appear to be no analytical criterion of stability analogous to the criterion based on the product of the square of the angular momentum and the energy of a hierarchical three-body system, drawn attention to in recent years by a number of authors (Easton 1971; Marchal 1971; Marchal and Saari 1975; Smale 1970; Saari 1974; Zare 1976, 1977).

V. V. Markellos and Y. Kozai (eds.), Dynamical Trapping and Evolution in the Solar System, 277–287.

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

Nevertheless, although in what follows we are interested in systems with $n > 3$, we first of all consider the general three-body hierarchical problem before turning our attention to cases where $n > 3$. As a particular example of this problem we will consider the Sun-Jupiter-Saturn case though it will become evident that many examples within the solar system could have been taken.

2. THE MIRROR THEOREM

Consider a general three-body system of the hierarchical type with bodies P_1, P_2, and P_3 of masses m_1, m_2 and m_3 such that P_1 and P_2 form a binary and P_3 is in orbit about the centre of mass C_{12} of P_1 and P_2. Let $m_2 < m_1$.

Let the osculating semimajor axes and eccentricities of the binary and the third body orbits be a_2, a_3, e_2 and e_3, suffix two referring to the binary orbit. Let $a_3 > a_2$ and let the system for simplicity be coplanar.

Then the mutual attractions of the bodies will perturb the values of a_2, a_3, e_2 and e_3.

$$\text{If} \quad a_2 (1+e_2) \left(\frac{m_1}{m_1 + m_2} \right) < a_3(1-e_3) , \tag{1}$$

the pericentre distance of P_3 from C_{12} will be greater than the apocentre distance of P_2 from C_{12} and the orbits will not cross.

Let us define two synodic periods S_{23} and Π_{23}. The first is the synodic period of the bodies P_2 and P_3, being the time between successive similar configurations of P_1, P_2 and P_3, for example between conjunctions P_1 P_2 P_3. If n_2 and n_3 are the mean motions of P_2 and P_3 about C_{12} while T_2 and T_3 are their periods of revolution, then

$$n_2 = \frac{2\Pi}{T_2} \quad ; \quad n_3 = \frac{2\Pi}{T_3} \tag{2}$$

and

$$\frac{1}{S_{23}} = \frac{1}{T_2} - \frac{1}{T_3} . \tag{3}$$

The second synodic period Π_{23} is the synodic period of the apses of the orbits. It is the time between successive similar configurations of C_{12} with Π_2 and Π_3, the pericentres of the inner and outer orbits respectively. For example such a configuration may be a conjunction C_{12} Π_2 Π_3. If $\dot{\omega}_2$ and $\dot{\omega}_3$ are the mean secular motions of Π_2 and Π_3 about C_{12}, while τ_2 and τ_3 are their periods of revolution, then

$$\dot{\omega}_2 = \frac{2\Pi}{\tau_2} \quad ; \quad \dot{\omega}_3 = \frac{2\Pi}{\tau_3} \tag{4}$$

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

and

$$\frac{1}{\Pi_{23}} = \frac{1}{\tau_2} - \frac{1}{\tau_3} \qquad (5)$$

Consider for example the system Sun-Jupiter-Saturn, neglecting the inclinations of the orbits. Then $m_1 = 1$; $m_2 = 0.0009551$; $m_3 = 0.0002680$ in solar mass units.

Also, at the present time,

$a_2 = 5.203$ AU	$a_3 = 9.539$ AU
$e_2 = 0.048$	$e_3 = 0.056$
$n_2 = 0.2649$ rad/yr	$n_3 = 0.1066$ rad/yr
$T_2 = 11.86$ yr	$T_3 = 29.46$ yr
$\dot{\omega}_2 = 2.053 \times 10^{-5}$ rad/yr	$\dot{\omega}_3 = 1.366 \times 10^{-4}$ rad/yr
$\tau_2 = 306{,}000$ yr	$\tau_3 = 46{,}000$ yr.

Then

$S_{23} = 19.852$ years.

and

$\Pi_{23} = 54{,}138$ years.

In all reasonably durable systems $S_{23} \ll \Pi_{23}$.

The Roy-Ovenden mirror theorem (1955) states that if, in an n-body dynamical system, the mutual radius vectors are all perpendicular to the mutual velocity vectors at any time, then the behaviour of the system after that time is a mirror image of the behaviour before that time.

In the coplanar three-body system under discussion, if the bodies' configuration satisfies any of the following eight cases, a mirror configuration occurs.

$P_1P_2P_3$ in line; P_2 and P_3 at pericentre
$P_1P_2P_3$ in line; P_2 at pericentre, P_3 at apocentre
$P_1P_2P_3$ in line; P_2 at apocentre, P_2 at pericentre
$P_1P_2P_3$ in line; P_2 and P_3 at apocentre
$P_2P_1P_3$ in line; P_2 and P_3 at pericentre
$P_2P_1P_3$ in line; P_2 at pericentre, P_3 at apocentre
$P_2P_1P_3$ in line; P_2 at apocentre, P_3 at pericentre
$P_2P_1P_3$ in line; P_2 and P_3 at apocentre.

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

3. APPROXIMATE MIRROR CONFIGURATIONS

We now consider the conditions under which approximations of particular degree of accuracy to mirror configurations occur.

Let the apse lines of the orbits coincide at time t_0. It is extremely unlikely that a conjunction of the bodies takes place at this moment and even more unlikely that if it does, the bodies will be on the common apse line.

The apse lines separate at a rate of $2\pi/\pi_{23}$. Let the first conjunction of the bodies occur at time $t_1 > t_0$. Then the angle θ between the apses at t_1 will be given by

$$\theta = \frac{2\pi}{\pi_{23}}(t_1 - t_0) = \theta_0, \quad \text{say.}$$

Now $t_1 - t_0 \lesssim S_{23}$ otherwise the previous conjunction must have been nearer the apse line. This point will be re-examined later more carefully.

Hence

$$\theta_0 \lesssim \frac{2\Pi \ S_{23}}{\Pi_{23}} \quad . \tag{6}$$

The angle advanced through by the conjunction line of the bodies in one synodic period S_{23} is Φ, given by

$$\Phi = n_3 \ S_{23} \tag{7}$$

since the radius vector of the body P_2 has to advance 2π radians with respect to the radius vector of the body P_3.

Let the angle between the common apse line at t_0 and the conjunction line at t_1, be α_0 radians. Then at t_1, the angles between the conjunction line at t_1 and the positions of the apses at t_1 will be given by

$$\beta_2 = \alpha_0 - \dot{\omega}_2(t_1 - t_0), \ \beta_3 = \alpha_0 - \dot{\omega}_3(t_1 - t_0) = \beta_2 + \theta_0 \ . \tag{8}$$

The angles β_2 and β_3 are the true anomalies of P_2 and P_3 respectively when the conjunction takes place.

For a mirror configuration to take place, the angles between the velocity vectors and the mutual radius vectors should be $\pi/2$ radians. We now consider what is the size of the angle γ between the velocity vector in an elliptical orbit and the radius vector for a given pair of values of the true anomaly f and eccentricity e.

The angle γ may easily be shown to be given by the relations:

$$\sin\gamma = \frac{1+e\cos f}{(1+2e\cos f+e^2)^{\frac{1}{2}}}\ ; \quad \cos\gamma = \frac{-e\sin f}{(1+2e\cos f+e^2)^{\frac{1}{2}}} \qquad (9)$$

It may be noted that for e equal to zero, $\gamma = \pi/2$ while for $f = 0$ or π, $\gamma = \pi/2$.

From (9) we have

$$\frac{\partial\gamma}{\partial f} = \frac{e(\cos f + e)}{(1+2e\cos f + e^2)^{\frac{1}{2}}} \qquad (10)$$

so that for a given value of e, the maximum value of γ, namely γ_{max} is given by putting

$$\cos f = -e \qquad (11)$$

in (9). If we do so, we find that

$$\sin\gamma_{max} = \sqrt{1-e^2}\ ; \quad \cos\gamma_{max} = -e. \qquad (12)$$

As $e \to 1$, $\gamma_{max} \to \pi$.

Also, remembering that the radius vector r is given by

$$r = \frac{a(1-e^2)}{1+e\cos f},$$

we have, when $\cos f = -e$,

$$r = a.$$

In other words, maximum γ occurs when the orbiting body lies at the end of the semiminor axis.

In the case of Jupiter, γ_{max} is found to be 92.°75 while in the case of Saturn, γ_{max} has the value of 93.°21.

These values occur at true anomalies 92.°75 and 93.°21 respectively. The approach of γ towards 90° for the orbit of Jupiter and Saturn as the true anomaly f decreases from f given by $\cos f = -e$ is shown in Table 1 below:

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

v / f	Jupiter $e = .048$	Saturn $e = .056$
90°	$92^\circ.7$	$93^\circ.2$
80	92.7	93.1
50	92.0	92.4
20	90.9	91.0
10	90.5	90.5
5	90.2	90.3
1	90.05	90.05

Table 1

Returning to the three-body system of Sun-Jupiter-Saturn we see that by (6), the maximum separation θ_o of the apses (after they have coincided at t_o) before the first conjunction of the planets occurs at t_1 is only about 4 arc minutes at most.

The poorest approximation to a satisfaction of a mirror condition will therefore occur if the first conjunction of the bodies after apse coincidence occurs with α of order 90^o. If however, a conjunction takes place thereafter, with true anomalies much nearer zero, before the apses have separated appreciably, then a much better approximation to a mirror configuration will occur. Even in the case of Jupiter and Saturn, it is seen that for $f < 20^o$, the value of γ is within one degree of 90^o. The implications of this for stability are worth listing.

For systems with perturbations of the Keplerian orbits small enough to ensure that $\pi_{23} >> S_{23}$, each apse conjunction epoch will be preceded and followed by a time interval in which the apse separation angle θ is very small; this time will itself be large compared with the bodies' synodic period S_{23} so that a number of conjunctions scattered round the orbits can take place in this time interval. A good chance will exist that one of them will occur at small true anomalies so providing a good approximation to a mirror condition with almost complete reversal of the previous build-up of perturbations.

If the perturbations are large, however, this time interval will not only be much smaller because Π_{23} will be smaller, but the orbital eccentricities will be larger. In order to produce as good an approximation to a mirror condition as before, the system will have to find a conjunction <u>in that smaller time interval</u> at much smaller true anomalies than before.

For example, if the eccentricities of Jupiter and Saturn's orbits were as large as 0.2, say, then for γ to be within one degree of 90°, the true anomaly would have to be less than 6° instead of less than 20°. And if the masses of Jupiter and Saturn were increased by a factor of 20, say, then Π_{23} would be decreased by a corresponding factor.

4. THE EFFECT OF NEAR COMMENSURABILITY IN MEAN MOTION

Now let us consider the effect of the well-known commensurability in mean motion in theSun-Jupiter-Saturn case. We have:

$$n_2 = 0.2649 \text{ rad./yr} \quad ; \quad n_3 = 0.1066 \text{ rad./yr},$$

so that

$$2n_2 - 5n_3 = - 0.0032 \text{ rad./yr}.$$

The conjunction system of lines may therefore be looked upon as a three-spoke wheel slowly rotating. The angle between two consecutive conjunction lines or spokes is given by (7); viz.

$$\Phi = n_3 S_{23} = 4.23244 \text{ rad.} = 242^{\circ}.5$$

so that in this case $3\Phi = 727^{\circ}.5$ or $7^{\circ}.5$.

The fourth conjunction line therefore lies $7^{\circ}.5$ ahead of the first and the wheel may be looked upon as rotating at an angular speed of $7^{\circ}.5$ /3 S_{23} or $2^{\circ}.5$ /synodic period of the bodies.

Let us suppose that the apse lines come together at t_o and that the first conjunction of the bodies occurs at time t_1 so that by (6)

$$\theta_o = \frac{2\pi}{\Pi_{23}} (t_1 - t_o) < \frac{2\pi}{\Pi_{23}} S_{23}$$

which as we have seen is less than 4 arc minutes.

The angle θ between the apse lines after a further number k of synodic periods has elapsed is given by

$$\theta = \frac{2\pi}{\Pi_{23}} (t_1 - t_o + kS_{23}) = \theta_o + \frac{2k\pi S_{23}}{\Pi_{23}} \tag{13}$$

or

$$\theta < \frac{2\pi(k+1) S_{23}}{\Pi_{23}} . \tag{14}$$

Suppose the angle α between the directions of the apse line conjunction at t_o and the bodies' conjunction at t_1 , is α_o . Suppose further that it is the worst possible case, in that the conjunction occurs where $\alpha_o \sim 90^{\circ}$, so making the true anomalies

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

f at the time of conjunction also $\sim 90^{\circ}$. Then considering only this particular conjunction line spoke, we know that since the conjunction line wheel rotates at a rate of $2^{\circ}.5$ /synodic period, there must have been a conjunction within $7^{\circ}.5$ of the apse line conjunction direction at t_o at a time$(90^{\circ}/2^{\circ}.5)S_{23}$ before t_1 that is 715 years before t_1. In this time interval the apse lines would have separated by about $5^{\circ}.15$ the faster moving apse of Saturn's orbit having contributed most of this $5^{\circ}.15$. The true anomalies at this time must therefore be of order 5° at most. From Table 1 we see that there is a mirror condition to within $0^{\circ}.2$ accuracy.

Some considerations not yet taken into account should be mentioned here which improve the situation considerably. The wheel of conjunctions has 3 spokes. If we now include oppositions, which from the point of view of the mirror condition are just as good at reversing perturbations, it has six spokes. Consecutive spokes are therefore separated by 120° and therefore the worst possible case is not $\alpha_o \sim 90$ but $\alpha_o \sim 60^{\circ}$. Then the maximum time before or after t_o at which a conjunction <u>or opposition</u> occurred or will occur within $7^{\circ}.5$ of the apse line conjunction direction at t_o is ~ 470 years. In this time the apse lines would have separated by about $3^{\circ}.4$. We are now therefore considering true anomalies of order 7° or smaller and for the orbits of Jupiter and Saturn the departure from a perfect mirror condition is certainly within $0^{\circ}.2$ of 90°

In the Sun-Jupiter-Saturn system, therefore, whenever the apse lines coincide, a very close approximation to a mirror configuration will take or has taken place within a short time interval of the apse line coincidence event. This event occurs every $\Pi_{23}/2 \approx 25000$ years since it is only necessary that the apse lines coincide and not that there be a conjunction of perihelia.

A general three-body system of almost circular orbits, almost commensurable in mean motions, and with small perturbations is therefore in a stable, or trapped mode. Before perturbations can build up disastrously, an apse line coincidence event will take place ensuring that a close approximation to a mirror condition event occurs or has occurred. This latter event almost completely ensures that perturbations produced in the system will be reversed.

For three-body systems of higher eccentricity and large perturbations, the chance of finding an efficient reversal of perturbation changes is decreased by the necessity(because of the higher eccentricities) of finding a conjunction of much smaller true anomalies in a much smaller time interval (because of the increased secular speeds of the apse lines).

5. FOUR- OR MORE- BODY HIERARCHICAL SYSTEMS

Such systems can be looked upon as a 'nested' mirror-seeking set. It may be remarked that in the development of the disturbing function, first order perturbations are additive. Each sub-set of three bodies in the hierarchical system will have its orbital characteristics perturbed by the other members of the system. Nevertheless, the two apses of the sub-set must still coincide at regular intervals ensuring that the close approximation to a mirror configuration that must occur will still effectively reverse the first-order three-body perturnations. If, moreover, at much longer intervals of time, there happens to occur, with four bodies of the system, a conjunction of the three apse lines, and shortly thereafter, or before, a conjunction of all four bodies, a more complete reversal of perturbations will take place. For this higher order event, however, to occur within a reasonably short time it would appear that some commensurable locking mechanism would have to be provided such as the Laplace relation for the inner three Galilean satellites of Jupiter, Io, Europa and Ganymede. Their mean longitudes ℓ and mean motions n are related such that, in order from the planet,

$$n_1 - 3n_2 + 2n_3 = 0$$
$$\ell_1 - 3\ell_2 + 2\ell_3 = 180^\circ,$$

thus ensuring that frequent mirror reversals of mutual perturbations take place. In addition,

$$n_1 - 2n_1 \sim 0$$
$$n_2 - 2n_3 \sim 0.$$

It seems more likely, therefore, that the frequent occurrence of the three-body sub-set near mirror configurations is the main mechanism by which perturbations are squashed before they destroy the stability of the system. In doing so, they provide the system with a durability long enough to ensure that 4 and higher mirror events can occur in which more complete cancelling of perturbations occurs.

6. EFFECT OF ORBITAL INCLINATIONS

We now consider the effect of the mutual inclination of the orbits in a general hierarchical three-body system. For the sake of simplicity we consider the orbits to be circular, of radius a_2 and a_3 , $a_2 < a_3$. Let the mutual inclination be i. Then for a mirror configuration to occur, it is obvious that the conjunction or opposition of the bodies has to take place at the common node or 90° from the common node. At any other position,

https://doi.org/10.1017/S0252921100097177 Published online by Cambridge University Press

each of the velocities is not at right angles to the radius vector of the other orbit. It should be remembered that now, it is only at the common node that a conjunction or opposition results in the radius vectors being collinear. In this context a conjunction is defined as a configuration where the longitudes ℓ of the bodies are equal, where the longitude of each body is defined to be measured from the common node along the body's orbital plane to the body's radius vector. An opposition occurs when $\ell_2 = \ell_3 + \pi$. Thus perfect mirror configurations occur at

(i) $\ell_2 = \ell_3 = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$

(ii) $\ell_2 = \ell_3 + \pi,\ \ell_3 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

(iii) $\ell_3 = \ell_2 + \pi,\ \ell_2 = 0,\ \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

It is easy to show that for any other longitude of conjunction ℓ , the angle ϕ between the radius vector of one body and the velocity vector of the other is given by

$$\cos \phi = \tfrac{1}{2} (\cos i - 1) \sin 2\ell \qquad (15)$$

For a given value i of the mutual inclination, ϕ departs from 90° most when

$$\ell = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}\ .$$

Then for any of these values it is obvious that maximum departure of ϕ from 90° occurs when $i = 90^\circ$, the angle ϕ being 60° or 120°.

If we again consider the Sun-Jupiter-Saturn case, we find that $i \sim 1^\circ$. For $\ell = \frac{\pi}{4}$, $\phi = 90\overset{\circ}{.}0044$, a value so close to 90° that it is obvious that in this system the effect of true anomalies and eccentricities on departure of conjunctions or oppositions from perfect mirror configurations is more important than any inclination effect. Even in the case of asteroids, or of the Sun-Jupiter-Pluto case, where inclinations may be of order 20°, for $\ell = \frac{\pi}{4}$ and $i = 20^\circ$, $\phi = 91\overset{\circ}{.}73$. Eccentricities of order 0.2 are common in such systems. For such a value, the departure of the angle between radius and velocity vectors from 90° can be as high as $11\overset{\circ}{.}5$ for a true anomaly of $78\overset{\circ}{.}5$ and we have seen that to cut this departure to 1°, the true anomaly has to be less than 6°.

7. CONCLUSIONS

It is deduced that as far as a hierarchical n-body dynamical system of the kind found in the Solar System is concerned, it maximises its survival chances if its orbits are so spaced and shaped that inclinations and eccentricities are minimised and perturbations are such that the rates of rotation of apses are kept low. In such a system, frequent and efficient cancellation of perturbations by the occurrence of close approximations to mirror configurations is realised.

The problem of the origin of the low inclinations and eccentricities remains.

In the early days of the Solar System, the formation of the planets by accretion from the disc of dust and gas must have given rise to bodies in orbits with a large distribution in eccentricities and inclinations. Collisions, near-collisions and expulsions must have been common. It may be concluded that survival would have favoured the more massive bodies in orbits of smaller eccentricities and inclinations so that for that reason alone the system would have evolved towards the almost circular, almost coplanar system we observe today, a system of survivors.

In addition, however, in the system's early days, the dissipative and smoothing power of the remaining dust and gas of the disc the protoplanets ploughed their way through must have tended to reduce eccentricities and possibly inclinations. This power of course diminished sharply with the growth of the protoplanets as they accreted the remaining dust and gas. This process must therefore have aided the occurrence of even closer approaches to the ideal but unattainable perfect mirror condition that reverses completely the system's perturbations.

The trapping of the hierarchical systems in the Solar System could then be said to be an example of the old explanation put forward to explain the stage conjurer's tricks - "It's all done by mirrors"!

REFERENCES

Easton, R. 1971. J. Diff. Eq. 10, 371.
Easton, R. 1975. J. Diff. Eq. 19, 258.
Marchal, C. 1971. Astron. Astrophys., 10, 278.
Marchal, C. and Saari, D. 1975. Cel. Mech., 12, 115.
Roy, A.E. and Ovenden, M.W. 1955. Mon.Not.Soc.Astron.Soc.115, 3.
Saari, D.G. 1974. SIAM J. Appl. Math., 26, 806.
Smale, S. 1970. Inventiones Math. 11, 45.
Zare, K. 1976. Cel. Mech., 14, 73.
Zare, K. 1977. Cel. Mech., 16, 35.