

Symmetrized and non-symmetrized asymptotic mean value Laplacian in metric measure spaces

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The asymptotic mean value Laplacian—AMV Laplacian—extends the Laplace operator from \mathbb{R}^n to metric measure spaces through limits of averaging integrals. The AMV Laplacian is however not a symmetric operator in general. Therefore, we consider a symmetric version of the AMV Laplacian, and focus lies on when the symmetric and non-symmetric AMV Laplacians coincide. Besides Riemannian and 3D contact sub-Riemannian manifolds, we show that they are identical on a large class of metric measure spaces, including locally Ahlfors regular spaces with suitably vanishing distortion. In addition, we study the context of weighted domains of \mathbb{R}^n —where the two operators typically differ—and provide explicit formulae for these operators, including points where the weight vanishes.

Keywords: Laplace operator; metric measure space; mean value property; average integral

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1. Introduction

For any $u \in C^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the second-order Taylor expansion of u at x yields the identity

$$\int_{B_r(x)} u(y) - u(x) \, dy = c_n r^2 \Delta u(x) + o(r^2)$$

as $r \downarrow 0$, where $c_n := 1/(2n + 4)$. Building upon this elementary observation, we proposed in [21] a definition for the pointwise Laplacian of a locally integrable function u defined on a general metric measure space (X, d, μ) by setting

$$\Delta_\mu^d u(x) := \lim_{r \downarrow 0} \frac{1}{r^2} \int_{B_r(x)} u(y) - u(x) \, d\mu(y) \quad (1.1)$$

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for any $x \in X$ where the limit converges, see definition 3.1 for a rigorous definition. The quantity $\Delta_\mu^d u(x)$ is called the *pointwise Asymptotic Mean Value* (AMV for short) *Laplacian* of u at x . We proved several results including maximum and comparison principles for the operator Δ_μ^d .

However, a disadvantage of Δ_μ^d is that it—in general—is not a self-adjoint operator on $L^2(X, \mu)$, see [21, Theorem 5.3 and Remark 5.4]. To face this issue, we consider in this paper a *symmetrized* version of Δ_μ^d , inspired by the symmetrized r -Laplacian introduced by Adamowicz, Kijowski, and Soultanis [2], which we define as

$$\tilde{\Delta}_\mu^d u(x) := \lim_{r \downarrow 0} \frac{1}{2r^2} \int_{B_r(x)} (u(y) - u(x)) \left(1 + \frac{\mu(B_r(x))}{\mu(B_r(y))} \right) d\mu(y).$$

We call $\tilde{\Delta}_\mu^d u(x)$ the *pointwise Symmetrized Asymptotic Mean Value* (SAMV for short) *Laplacian* of u at x . The Fubini theorem together with the symmetry of the metric show that

$$\int_X (\tilde{\Delta}_\mu^d u)v \, d\mu = \int_X u(\tilde{\Delta}_\mu^d v) \, d\mu$$

for any $u, v \in L^2(X, \mu)$ for which $\tilde{\Delta}_\mu^d u$ and $\tilde{\Delta}_\mu^d v$ converge in $L^2(X, \mu)$; see remark 3.6.

The primary goal of this article is to study for which instances the operators Δ_μ^d and $\tilde{\Delta}_\mu^d$ coincide, and in which sense they do. This coincidence depends on the infinitesimal interplay between the metric and the measure. For that reason, the asymptotic behaviour as $r \downarrow 0$ of the function

$$\delta_r : (x, y) \mapsto 1 - \frac{\mu(B_r(x))}{\mu(B_r(y))}$$

which we call the r -*distortion* of the space (see definition 2.10), plays a critical role. Indeed, the assumption

$$\sup_{y \in B_r(x)} |\delta_r(x, y)| = o(r^2)$$

as $r \downarrow 0$ ensures that the pointwise equality $\Delta_\mu^d u(x) = \tilde{\Delta}_\mu^d u(x)$ holds, see proposition 4.2. On smooth Riemannian manifolds equipped with the canonical Riemannian distance and volume measure, this observation coupled with the classical volume expansion of asymptotically small balls easily yield that the AMV and SAMV Laplacians coincide almost everywhere, see proposition 4.3. A similar expansion found by Barilari, Beschatnyi, and Lerario [7] implies that the same holds true on 3-dimensional contact sub-Riemannian manifolds equipped with the Popp volume, see proposition 4.4; the case of general sub-Riemannian manifolds would be worth a deeper investigation.

In our previous article, we also defined a weak AMV Laplacian [21, Definition 5.5]. We refine this notion in the present paper, and we additionally propose a definition of weak SAMV Laplacian: see definition 3.10. In proposition 4.8, we

prove that on a locally Ahlfors regular space, if

$$\frac{1}{r} \left(\int_{B_r(\cdot)} |\delta_r(\cdot, y)| d\mu(y) \right) d\mu(\cdot) \rightharpoonup 0$$

as $r \downarrow 0$, in the sense of weak convergence of measures against compactly supported continuous functions, then the weak AMV and SAMV Laplacians of any Lipschitz function coincide if one of the two exists. This allows us to remove the curvature assumption made in [2] where the coincidence between the weak AMV and SAMV Laplacians was established for non-collapsed $RCD(K, N)$ spaces with vanishing metric-measure boundary, see remark 4.11.

The secondary goal of this article is to derive concrete formulae for the pointwise AMV and SAMV Laplacians in weighted domains of \mathbb{R}^n , that is to say metric measure spaces of the form (Ω, d, μ) where Ω is a domain of \mathbb{R}^n , d is a distance on Ω and μ is absolutely continuous with respect to the restriction of the Lebesgue measure \mathcal{L}^n to Ω , with Radon–Nikodym derivative w [17]. In this context, there are two different cases depending on whether w is strictly positive or zero at the point $x \in \Omega$.

If $w(x) > 0$ we assume that $w \in C^1(\Omega)$ and that d satisfies a mild symmetry assumption, (5.7). We show that in this case, if the matrix of limits of average second moments

$$M(x) := \lim_{r \downarrow 0} \left(\int_{B_r(x)} \frac{(y-x)_i (y-x)_j}{r^2} dy \right)_{1 \leq i, j \leq n}$$

exists, then for any $u \in C^2(\Omega)$,

$$\Delta_\mu^d u(x) = \frac{1}{2} \text{Tr}(M(x) \nabla^2 u(x)) + \frac{1}{w(x)} \langle \nabla w(x), M(x) \nabla u(x) \rangle. \tag{1.2}$$

If in addition one has

$$\frac{1}{r} \int_{B_r(x)} \left| 1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))} \right| dy \rightarrow 0 \tag{1.3}$$

as $r \downarrow 0$, then

$$\tilde{\Delta}_\mu^d u(x) = \frac{1}{2w(x)} \text{Tr}(M(x) \nabla(w \nabla u)(x)), \tag{1.4}$$

see proposition 5.3. In case d is associated with a norm, equality (1.2) has already been established in [1] in an L^p_{loc} sense (see § 2); in this case the limit matrix $M(x)$ always exists and is independent of x . For the Euclidean distance, (1.2) was proven in [21, Proposition 2.3]. It is likely that (1.4) also holds in an L^p_{loc} sense under an appropriate L^p_{loc} replacement of (1.3), but we do not investigate this question in this paper.

If $w(x) = 0$, we restrict our study to the case where d is associated with a norm, hence we can assume without loss of generality that $x = 0_n \in \Omega$. We also assume that w is locally integrable, in which case the assumption $w(x) = 0$ must

be understood in a Lebesgue point sense. In proposition 5.8, we show that under the infinitesimal evenness property

$$\mathcal{L}^n - \operatorname{ess\,sup}_{x \in B_r(0_n)} |w(x) - w(-x)| = o\left(r \int_{B_r(0_n)} w \, d\mathcal{L}^n\right) \quad \text{as } r \downarrow 0,$$

and the existence of a weak limit measure

$$\nu := \lim_{r \downarrow 0} \frac{w(r \cdot) r^n}{\mu(B_r(x))} \mathcal{L}^n \llcorner B_1(0_n),$$

setting $M_\nu := (\int_{B_1(0_n)} y_i y_j \, d\nu(y))_{1 \leq i, j \leq n}$, we get

$$\Delta_\mu^d u(0_n) = \frac{1}{2} \operatorname{Tr}(M_\nu \nabla^2 u)(0_n).$$

Under a natural comparability assumption in the spirit of [4, 5, 23] (see definition 2.1), we get an analogous result in the case of the SAMV Laplacian, see proposition 5.10.

Let us conclude this section with a last comment: proposition 5.1 shows that on weighted Euclidean domains the SAMV Laplacian coincides with the usual weighted Laplacian $\Delta u + \langle \nabla \ln w, \nabla u \rangle$ (also known as drifted Laplacian, or f -Laplacian, or Witten Laplacian, see e.g. [11, 22] and the references therein) while the AMV does not. This suggests that Δ_μ^d is the natural AMV Laplacian to consider in the future, in accordance with [2] where local minimizers of the Korevaar–Schoen energy with target \mathbb{R} [19] are identified as those Sobolev maps having zero weak SAMV Laplacian.

2. Preliminaries

Throughout the paper, we write AMV as a shorthand for ‘Asymptotic Mean Value’ and SAMV for ‘Symmetrized Asymptotic Mean Value’. We keep a positive integer n fixed and we denote by 0_n the origin of the vector space \mathbb{R}^n , by d_e the Euclidean distance on \mathbb{R}^n , by ω_n the Lebesgue measure of the unit Euclidean ball and by σ_{n-1} the total surface measure of the unit Euclidean sphere.

We say that a triple (X, d, μ) is a metric measure space if (X, d) is a metric space and μ is a fully supported Borel measure on (X, d) such that $0 < \mu(B_r(x)) < +\infty$ for any $x \in X$ and $r > 0$. In case X is an open subset of a space equipped with a norm $\|\cdot\|$ and d is the canonical distance associated with $\|\cdot\|$, we write $(X, \|\cdot\|, \mu)$.

Let (X, d, μ) be a metric measure space. We use classical notation to denote function spaces on X , like $C(X)$ for the set of continuous functions, $\operatorname{Lip}(X)$ for the set of Lipschitz functions, and so on. We set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We shall often identify a μ -measurable function $f : X \rightarrow \overline{\mathbb{R}}$ with its equivalence class under μ -a.e. equality. For any $p > 0$, we let $L^p(X, \mu)$ be the set of p -integrable functions, that is, μ -measurable functions $f : X \rightarrow \overline{\mathbb{R}}$ such that $\int_X |f|^p \, d\mu < +\infty$. We also let $L^p_{\text{loc}}(X, \mu)$ be the set of locally p -integrable functions, that is, μ -measurable functions $f : X \rightarrow \overline{\mathbb{R}}$ such that any $x \in X$ admits a neighbourhood V such that $\int_V |f|^p \, d\mu < +\infty$. We let $L^\infty(X, \mu)$ be the set of essentially bounded μ -measurable

functions on X , and $L_{loc}^\infty(X, \mu)$ be the set of locally essentially bounded μ -measurable functions on X , that is, the μ -measurable functions $f : X \rightarrow \overline{\mathbb{R}}$ such that μ -a.e. $x \in X$ admits a neighbourhood V such that the essential supremum of f on V is bounded from above. In case X is an open subset of \mathbb{R}^n and μ is the Lebesgue measure, we simply write $L^p(X)$, $L_{loc}^p(X)$, $L^\infty(X)$, and $L_{loc}^\infty(X)$ respectively. If A is a bounded Borel subset of X , we write $\mathbf{1}_A$ for the characteristic function of A , and for any locally integrable function $u : X \rightarrow \mathbb{R}$, we set

$$\int_A u \, d\mu := \frac{1}{\mu(A)} \int_A u \, d\mu.$$

Assume that (X, d) is locally compact. Under this assumption, we shall often make use of compact sets. We use the subscript c to denote the restriction of a function space to compactly supported functions: for instance, by $C_c(X)$ we mean the set of compactly supported continuous functions. We recall that a Radon measure ν on (X, d) is a Borel measure which is locally finite and inner regular, meaning that any $x \in X$ admits a neighbourhood V such that $\nu(V) < +\infty$, and any measurable subset $A \subset X$ satisfies $\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\}$, respectively. The Riesz–Markov–Kakutani theorem states that the space of Radon measures $\text{Rad}(X)$ is the topological dual of $C_c(X)$. Therefore, we equip $\text{Rad}(X)$ with the associated weak topology, and we denote by \rightharpoonup the convergence in this topology. Lastly, for any $f \in L_{loc}^1(X, \mu)$, we denote by $f\mu$ the Radon measure defined by

$$(f\mu)(A) := \int_A f \, d\mu$$

for any measurable $A \subset X$.

Lebesgue points. Let (X, d, μ) be a metric measure space. For any $u \in L_{loc}^1(X, \mu)$, we recall that $x \in X$ is a Lebesgue point of u if there exists some real number $u^*(x)$ such that

$$\lim_{r \downarrow 0} \int_{B_r(x)} |u(y) - u^*(x)| \, d\mu(y) = 0. \tag{2.1}$$

We write $\text{Leb}(u)$ for the set of Lebesgue points of u . It is easily checked that neither the set $\text{Leb}(u)$ nor the real number $u^*(x)$ for any $x \in \text{Leb}(u)$ depend on the choice of a representative in the equivalence class of u . Moreover, the following hold.

- (1) If (X, d) is locally compact, then for any $u \in C(X)$ a simple argument based on the Heine–Cantor theorem shows that $\text{Leb}(u) = X$ and $u^*(x) = u(x)$ for any $x \in X$.
- (2) If (X, d, μ) is infinitesimally doubling, meaning that

$$\limsup_{r \downarrow 0} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < +\infty \tag{2.2}$$

for μ -a.e. $x \in X$, then $\mu(X \setminus \text{Leb}(u)) = 0$ for any $u \in L_{loc}^1(X, \mu)$ and $u^*(x) = u(x)$ for μ -a.e. $x \in X$. This follows from the Lebesgue Differentiation Theorem for infinitesimally doubling metric measure spaces: see [15, Theorem 3.4.3 and page 77].

Comparability conditions. We now introduce a couple of definitions which originate at least from [23]. They have also appeared in a different way in [4, 5].

DEFINITION 2.1 Comparability conditions. Let (X, d, μ) be a metric measure space.

- (1) We say that μ satisfies a comparability condition on $A \subset X$ if for any $x \in A$ there exist $r_0 = r_0(x), C = C(x) > 0$ such that for any $r \in (0, r_0)$, for μ -a.e. $y \in B_r(x)$,

$$\mu(B_r(x)) \leq C\mu(B_r(y)). \tag{2.3}$$

In case $A = \{x\}$ for some $x \in X$ we often say that μ satisfies a comparability condition at x .

- (2) We say that μ satisfies a locally uniform comparability condition if any $x_0 \in X$ admits a neighbourhood V and two constants $r_0 = r_0(x_0), C = C(x_0) > 0$ such that the inequality (2.3) holds for μ -a.e. $x \in V$, for any $r \in (0, r_0)$ and for μ -a.e. $y \in B_r(x)$.
- (3) We say that μ satisfies a uniform comparability condition if there exist $r_0, C > 0$ such that for any $x \in X$ the inequality (2.3) holds for any $r \in (0, r_0)$ and μ -a.e. $y \in B_r(x)$.

Observe that: (3) \Rightarrow (2) \Rightarrow (1).

REMARK 2.2. If (X, d) is locally compact, then the neighbourhoods in the previous definition can be chosen compact. In particular, if μ satisfies a locally uniform comparability condition, it is not difficult to show that for any compact set $K \subset X$ there exists $r_K, C_K > 0$ such that

$$\mu(B_r(x)) \leq C_K\mu(B_r(y))$$

for μ -a.e. $x \in K$, any $r \in (0, r_K)$ and μ -a.e. $y \in B_r(x)$.

EXAMPLE 2.3. A metric measure space is doubling at scale $r_0 > 0$ if there exists a constant $C \geq 1$ such that

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x))$$

for any $x \in X$ and $r \in (0, r_0/2)$. Any such space trivially satisfies a uniform comparability condition (3), since for any $y \in B_r(x)$ where x, r are as above,

$$\mu(B_r(x)) \leq \mu(B_{2r}(y)) \leq C\mu(B_r(y)).$$

EXAMPLE 2.4. For any $Q > 0$, we say that a metric measure space (X, d, μ) is Ahlfors Q -regular if there exist $C \geq 1$ and $r_0 \in (0, +\infty]$ such that

$$C^{-1}r^Q \leq \mu(B_r(x)) \leq Cr^Q \tag{2.4}$$

for any $x \in X$ and $r \in (0, r_0)$. Non-trivial examples including fractal-type ones may be found in [9, 18, 20], see also [8] where they are called fractional metric

measure spaces. Ahlfors regular spaces are obviously doubling, hence they satisfy a uniform comparability condition (3). Moreover, we say that (X, d, μ) is locally Ahlfors Q -regular if any $x_0 \in X$ admits a neighbourhood V and two constants $r_0 = r_0(x_0), C = C(x_0) > 0$ such that

$$C^{-1}r^Q \leq \mu(B_r(x)) \leq Cr^Q$$

for any $x \in V$ and $r \in (0, r_0)$. In this case μ trivially satisfies a locally uniform comparability condition (2).

EXAMPLE 2.5. Let us provide an example of a space which, according to the terminology introduced in definition 2.1, satisfies (1) at any point but does not satisfy (3). For any $n \in \mathbb{N} \setminus \{0\}$, let \mathcal{C}_n be the circle of centre $x_n := (3n, 0)$ with radius $1/n$ and $y_{n,1}, \dots, y_{n,n}$ be distinct points on \mathcal{C}_n . Join each $y_{n,i}$ with x_n by the geodesic segment between them and consider the set $X \subset \mathbb{R}^2$ obtained as the union of all these segments and $[1, +\infty) \times \{0\}$. Equip X with the length distance d induced by the Euclidean one on \mathbb{R}^2 and with the Borel measure μ defined as the sum of the 1-dimensional Hausdorff measure and Dirac masses at each $x_n, y_{n,i}$. Then $\mu(B_{3/(2n)}(x_n)) \leq 1 + n + 3/2$ while $\mu(B_{3/(2n)}(y_{n,i})) = 5/2 + 1/n$ for any n and any i , hence (X, d, μ) cannot satisfy (3).

Averaging operator. We recall now the definition and some properties of the averaging operator.

DEFINITION 2.6 Averaging operator. Let (X, d, μ) be a metric measure space. For any $r > 0$, the averaging operator A_r is defined by setting, for all $f \in L^1_{loc}(X, \mu)$ and μ -a.e. $x \in X$,

$$A_r f(x) := \int_{B_r(x)} f \, d\mu.$$

REMARK 2.7. Obviously $\sup_{r>0} \|A_r\|_{L^\infty(X,\mu) \rightarrow L^\infty(X,\mu)} \leq 1$. In addition, if $f \in L^\infty_{loc}(X, \mu)$, then any $x \in X$ admits a neighbourhood V and a radius $r_0 = r_0(x)$ such that $\|A_r f\|_{L^\infty(V,\mu)} < +\infty$ for any $r \in (0, r_0)$. Indeed, if W is a neighbourhood of x such that $\|f\|_{L^\infty(W,\mu)} < +\infty$, then there exists $r_0 > 0$ such that $B_{2r_0}(x) \subset W$, and for any $r \in (0, r_0)$ and μ -a.e. $y \in B_{r_0}(x)$,

$$\|A_r f(y)\| \leq \int_{B_r(y)} |f(z)| \, d\mu(z) \leq \|f\|_{L^\infty(B_r(y),\mu)} \leq \|f\|_{L^\infty(W,\mu)} < +\infty,$$

hence we can take $V = B_{r_0}(x)$.

LEMMA 2.8. Let (X, d, μ) be a metric measure space and $p \in [1, +\infty)$.

- (1) Assume that μ satisfies a uniform comparability condition (3) with parameters r_0, C . Then, the operator A_r is bounded from $L^p(X, \mu)$ into itself, with norm at most $C^{1/p}$, for any $r \in (0, r_0)$.
- (2) Assume that μ satisfies a locally uniform comparability condition. Then, for any $f \in L^p_{loc}(X, \mu)$ and $x \in X$, there exist $r_0 = r_0(x, f), C = C(x, f) > 0$

and a neighbourhood V of x also depending of f , such that for any $r \in (0, r_0)$,

$$\int_V |A_r f|^p \, d\mu < C \int_V |f|^p \, d\mu < +\infty.$$

Proof. (1) If μ satisfies a uniform comparability condition with parameters r_0, C , then for any $x \in X$ and $r \in (0, r_0)$,

$$a_r(x) := \int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))} \leq C.$$

This implies boundedness of A_r from $L^1(X, \mu)$ to itself with norm at most C , see [4, Theorem 3.3] for instance. Then, the Jensen inequality implies the case $p > 1$, see e.g. [5, Theorem 2.9].

(2) Consider $x \in X$ and $f \in L^p_{\text{loc}}(X, \mu)$. By (4.2), there exist $r_0 = r_0(x, f)$, $C = C(x, f) > 0$ and a neighbourhood V of x also depending of f such that $\int_V |f|^p \, d\mu < +\infty$, $B_{2r_0}(x) \subset V$ and $\mu(B_r(y)) \leq C\mu(B_r(z))$ for μ -a.e. $y \in V$, any $r \in (0, r_0)$ and μ -a.e. $z \in B_r(y)$. The Hölder inequality yields

$$\left(\int_{B_r(z)} |f(y)| \, d\mu(y) \right)^p \leq \int_{B_r(z)} |f(y)|^p \, d\mu(y).$$

Then,

$$\begin{aligned} \int_V |A_r f|^p \, d\mu &\leq \int_V \int_{B_r(z)} |f(y)|^p \, d\mu(y) \, d\mu(z) = \int_V \int_V 1_{B_r(z)}(y) \frac{|f(y)|^p}{\mu(B_r(z))} \, d\mu(y) \, d\mu(z) \\ &\leq \int_V |f(y)|^p \int_{B_r(y)} \frac{d\mu(z)}{\mu(B_r(z))} \, d\mu(y) \\ &\leq C \int_V |f(y)|^p \, d\mu(y). \end{aligned}$$

□

REMARK 2.9 Adjoint of A_r . Assume that a uniform comparability condition holds with parameters r_0, C . Then, for any $r \in (0, r_0)$, the operator A_r^* defined by

$$A_r^* f(x) := \int_{B_r(x)} \frac{f(y) \, d\mu(y)}{\mu(B_r(y))}$$

for μ -a.e. $x \in X$, is bounded from $L^p(X, \mu)$ into itself for any $p \geq 1$. Indeed,

$$A_r^* f(x) = \int_{B_r(x)} \frac{f(y) \mu(B_r(x)) \, d\mu(y)}{\mu(B_r(y))} \leq C \int_{B_r(x)} f(y) \, d\mu(y) = C A_r f(x).$$

Moreover, the Fubini theorem and the symmetry of the distance easily imply that

$$A_r^* f(x) := \int_{B_r(x)} \frac{f(y) \, d\mu(y)}{\mu(B_r(y))}$$

for μ -a.e. $x \in X$, for all $f \in L^2(X, \mu)$. This means that A_r^* is the adjoint of A_r with respect to the L^2 scalar product. More generally, we have that for any $p, q \geq 1$ such

that $1/p + 1/q = 1$, one has

$$\int_X (A_r g)h \, d\mu = \int_X g(A_r^* h) \, d\mu$$

for all $g \in L^p(X, \mu)$ and $h \in L^q(X, \mu)$.

Distortion. We finally introduce a couple of notions which helps us formalizing the infinitesimal relation between the metric and the measure of a metric measure space. Compare with [16].

DEFINITION 2.10. Let (X, d, μ) be a metric measure space. For any $r > 0$, we define:

(1) for any $x, y \in X$,

$$\delta_r(x, y) := 1 - \frac{\mu(B_r(x))}{\mu(B_r(y))},$$

(2) for any $x \in X$,

$$z_r(x) := \int_{B_r(x)} |\delta_r(x, y)| \, d\mu(y).$$

We say that δ_r is the r -distortion function of (X, d, μ) .

REMARK 2.11. In [16], the authors considered the r -deviation functions

$$v_r(x) := 1 - \frac{\mu(B_r(x))}{\omega_n r^n}$$

which measure in a very rough way how much a metric measure space differs from the Euclidean \mathbb{R}^n . These deviation functions may behave quite differently from the distortion ones. For instance, consider a smooth Riemannian manifold equipped with its canonical Riemannian distance and volume measure. At an interior point x of such a space,

$$v_r(x) = cS(x)r^2 + O(r^4) \quad \text{as } r \downarrow 0,$$

where $S(x)$ is the scalar curvature at x and c is a dimensional constant (see (4.8)). However, we show in the proof of proposition 4.3 that if $y = y(r) \in B_r(x)$ for any $r > 0$, then

$$\delta_r(x, y) = c(S(y) - S(x))r^2 + O(r^4) \quad \text{as } r \downarrow 0,$$

where $O(r^4)$ does not depend on y . In particular, if the scalar curvature is constant but non-zero, then $v_r(x) = cr^2 + O(r^4)$ but $\delta_r(x, y) = O(r^4)$.

3. Definitions

In this section, we refine the definitions of AMV Laplacian proposed in [1, 21] and we put forward similar definitions for the SAMV Laplacian.

Pointwise AMV and SAMV Laplacians. In order to define the pointwise AMV Laplacian of a locally integrable function u over a metric measure space (X, d, μ) , in [21, Definition 1.1] we made the convention of choosing the representative \tilde{u} of u defined by

$$\tilde{u}(x) := \lim_{r \downarrow 0} \int_{B_r(x)} u(y) \, d\mu(y)$$

for any $x \in X$ where this limit exist and then to set

$$\Delta_\mu^d u(x) := \frac{1}{r^2} \int_{B_r(x)} u(y) - \tilde{u}(x) \, d\mu(y). \tag{3.1}$$

However, this definition has a couple of disadvantages which we point out in remark 3.2. Therefore, we propose a new definition, formalized with the aid of Lebesgue points, along with a similar definition for the SAMV Laplacian.

DEFINITION 3.1 Pointwise AMV/SAMV Laplacian. *Let (X, d, μ) be a metric measure space and $u \in L^1_{loc}(X, \mu)$ be given.*

(1) *For any $x \in \text{Leb}(u)$ and $r > 0$, we set*

$$\Delta_{\mu,r}^d u(x) := \frac{1}{r^2} \int_{B_r(x)} u(y) - u^*(x) \, d\mu(y).$$

If there exists a real number $\Delta_\mu^d u(x)$ such that

$$\lim_{r \downarrow 0} \Delta_{\mu,r}^d u(x) = \Delta_\mu^d u(x), \tag{3.2}$$

then we say that u admits a pointwise AMV Laplacian at x , and we call $\Delta_\mu^d u(x)$ the pointwise AMV Laplacian of u at x .

(2) *For any $x \in \text{Leb}(u)$ and $r > 0$, we set*

$$\tilde{\Delta}_{\mu,r}^d u(x) := \frac{1}{2r^2} \int_{B_r(x)} (u(y) - u^*(x)) \left(1 + \frac{\mu(B_r(x))}{\mu(B_r(y))} \right) d\mu(y).$$

If there exists a real number $\tilde{\Delta}_\mu^d u(x)$ such that

$$\lim_{r \downarrow 0} \tilde{\Delta}_{\mu,r}^d u(x) = \tilde{\Delta}_\mu^d u(x), \tag{3.3}$$

then we say that u admits a pointwise SAMV Laplacian at x , and we call $\tilde{\Delta}_\mu^d u(x)$ the pointwise SAMV Laplacian of u at x .

REMARK 3.2. Let us explain why we favour (3.2) over our previous definition (3.1). A first reason is that unlike (3.1) this new definition does not require to choose any representative in the equivalent class of u . A second reason is the following: If $x \in \text{Leb}(u)$ then $\tilde{u}(x)$ exists and is equal to $u^*(x)$. However, it may happen that $\tilde{u}(x)$ exists while $x \notin \text{Leb}(u)$. For instance, let $u \in L^1(\mathbb{R})$ be the equivalent class of the sign function $\mathbb{R} \setminus \{0\} \ni x \mapsto x/|x|$. Equip \mathbb{R} with the natural distance associated with

the absolute value. Then, $\tilde{u}(0)$ exists and is equal to 0 while $0 \notin \text{Leb}(u)$. Therefore, according to (3.1) the function u is pointwise AMV harmonic on \mathbb{R} , while (3.2) yields that $\Delta_\mu^d u(0)$ is not defined and u is pointwise AMV harmonic on $\mathbb{R} \setminus \{0\}$ only: this is in better agreement with the distributional Laplacian of u being equal to a multiple of the derivative of the Dirac distribution in 0, as mentioned in the last sentence of [21].

Strong AMV/SAMV Laplacian. In [1] the authors proposed a definition of strongly AMV-harmonic functions by appealing on compact sets. In order to work in a more general setting, we define here a strong AMV Laplacian and a strong SAMV Laplacian without involving compact sets. Let (X, d, μ) be a metric measure space. Consider $u \in L^\infty_{loc}(X, \mu)$. According to remark 2.7, any $x \in X$ admits a neighbourhood V and a radius $r_0 > 0$ such that

$$\|\Delta_{\mu,r}^d u\|_{L^\infty(B_{r_0}(x), \mu)} \leq \frac{2\|u\|_{L^\infty(V, \mu)}}{r^2}$$

for any $r \in (0, r_0)$. This observation guarantees the well-posedness of the following definition.

DEFINITION 3.3 Strong AMV Laplacian. *Let (X, d, μ) be a metric measure space. We say that $u \in L^\infty_{loc}(X, \mu)$ admits a strong AMV-Laplacian if there exists a function $v \in L^\infty_{loc}(X, \mu)$ such that any $x \in X$ admits a neighbourhood V such that*

$$\lim_{r \downarrow 0} \|\Delta_{\mu,r}^d u - v\|_{L^\infty(V, \mu)} = 0,$$

in which case we say that $\Delta_\mu^d u := v$ is the strong AMV-Laplacian of u .

Let us provide a analogous definition in the symmetrized case. To this aim, we assume that μ satisfies a local comparability condition. Under this condition, for any $u \in L^\infty_{loc}(X, \mu)$ and $x \in X$, there exist $C, r_0 > 0$ and a neighbourhood V of x such that

$$\|\tilde{\Delta}_{\mu,r}^d u\|_{L^\infty(B_{r_0}(x))} \leq \frac{2C\|u\|_{L^\infty(V, \mu)}}{r^2}$$

for any $r \in (0, r_0)$.

DEFINITION 3.4 Strong SAMV Laplacian. *Let (X, d, μ) be a metric measure space. Assume that μ satisfies a local comparability condition. We say that $u \in L^\infty_{loc}(X, \mu)$ admits a strong SAMV-Laplacian if there exists a function $v \in L^\infty_{loc}(X, \mu)$ such that any $x \in X$ admits a neighbourhood V such that*

$$\lim_{r \downarrow 0} \|\tilde{\Delta}_{\mu,r}^d u - v\|_{L^\infty(V, \mu)} = 0,$$

in which case we say that $\tilde{\Delta}_\mu^d u := v$ is the strong SAMV-Laplacian of u .

L^p/L^p_{loc} -AMV/SAMV Laplacian. Let us consider $p \in [1, +\infty)$. Recall that lemma 2.8 ensures that under a uniform comparability condition, the operator

A_r is bounded from L^p to itself for any sufficiently small r . As a consequence, also $\Delta_{\mu,r}^d$ is bounded from L^p to itself. Then, the following definition is well-posed.

DEFINITION 3.5 L^p AMV/SAMV Laplacian. *Let (X, d, μ) be a metric measure space such that μ satisfies a uniform comparability condition. Consider $u \in L^1_{loc}(X, \mu)$ and $p \in [1, +\infty)$.*

- (1) *We say that u admits an L^p -AMV Laplacian if there exists a function $v \in L^p(X, \mu)$ such that*

$$\lim_{r \downarrow 0} \|\Delta_{\mu,r}^d u - v\|_{L^p(X, \mu)} = 0,$$

in which case we say that $\Delta_{\mu}^d u := v$ is the L^p -AMV Laplacian of u .

- (2) *We say that u admits an L^p -SAMV Laplacian if there exists a function $v \in L^p(X, \mu)$ such that*

$$\lim_{r \downarrow 0} \|\tilde{\Delta}_{\mu,r}^d u - v\|_{L^p(X, \mu)} = 0,$$

in which case we say that $\tilde{\Delta}_{\mu}^d u := v$ is the L^p -SAMV Laplacian of u .

REMARK 3.6. Let $u, v \in L^2(X, \mu)$ be both admitting an L^2 -SAMV Laplacian. Since

$$\tilde{\Delta}_{\mu,r}^d u(x) = \int_X k_r(x, y)(u(y) - u(x)) \, d\mu(y)$$

for any $r > 0$ and μ -a.e. $x \in X$, where

$$k_r(x, y) := \frac{1}{2r^2} \mathbf{1}_{[0,r)}(d(x, y)) \left(\frac{1}{\mu(B_r(x))} + \frac{1}{\mu(B_r(y))} \right),$$

the Fubini theorem implies

$$\int_X (\tilde{\Delta}_{\mu,r}^d u)v \, d\mu = \int_X u(\tilde{\Delta}_{\mu,r}^d v) \, d\mu,$$

hence from letting $r \downarrow 0$ we obtain

$$\int_X (\tilde{\Delta}_{\mu}^d u)v \, d\mu = \int_X u(\tilde{\Delta}_{\mu}^d v) \, d\mu. \tag{3.4}$$

Let us now provide a definition of L^p_{loc} -AMV Laplacian and L^p_{loc} -SAMV Laplacian. In this case we assume a locally uniform comparability condition.

DEFINITION 3.7 L^p_{loc} AMV/SAMV Laplacian. *Let (X, d, μ) be a metric measure space such that μ satisfies a locally uniform comparability condition. Consider $u \in L^1_{loc}(X, \mu)$ and $p \in [1, +\infty)$.*

- (1) *We say that u admits an L^p_{loc} -AMV Laplacian if there exists a function $v \in L^p_{loc}(X, \mu)$ such that for any $x \in X$ there exists a neighbourhood V of x such*

that

$$\lim_{r \downarrow 0} \|\Delta_{\mu,r}^d u - v\|_{L^p(V,\mu)} = 0,$$

in which case we say that $\Delta_{\mu}^d u := v$ is the L^p_{loc} -AMV Laplacian of u .

- (2) We say that u admits an L^p_{loc} -SAMV Laplacian if there exists a function $v \in L^p_{loc}(X, \mu)$ such that for any $x \in X$ there exists a neighbourhood V of x such that

$$\lim_{r \downarrow 0} \|\tilde{\Delta}_{\mu,r}^d u - v\|_{L^p(V,\mu)} = 0,$$

in which case we say that $\tilde{\Delta}_{\mu}^d u := v$ is the L^p_{loc} -SAMV Laplacian of u .

REMARK 3.8. By lemma 2.8, the locally uniform comparability condition guarantees that for any $x \in X$ there exist a neighbourhood V of x and a constant r_0 such that $\Delta_{\mu,r}^d u \in L^p(V, \mu)$ for any $r \in (0, r_0)$.

REMARK 3.9. The Hölder inequality implies that if u admits an L^p_{loc} -AMV Laplacian, then u admits an L^q_{loc} -AMV Laplacian for any $q \geq p$ which coincides μ -a.e. with the L^p_{loc} -AMV Laplacian. This remark holds in the SAMV case too.

Weak AMV/SAMV Laplacian. Let us now define a notion of weak AMV Laplacian and weak SAMV Laplacian. We focus on locally compact metric measure spaces (X, d, μ) such that μ satisfies a locally uniform comparability condition. Under these assumptions, remark 2.2 implies that if K is the support of a function $\varphi \in C_c(X)$, then there exists $r_K > 0$ such that for any $r \in (0, r_K)$,

$$\left| \int_K \Delta_{\mu,r}^d \varphi \, d\mu \right| + \left| \int_K \tilde{\Delta}_{\mu,r}^d \varphi \, d\mu \right| < +\infty.$$

Then, the following definition is well-posed.

DEFINITION 3.10 Weak AMV/SAMV Laplacian. Let (X, d, μ) be a locally compact metric measure space such that μ satisfies a locally uniform comparability condition. Let $u \in L^1_{loc}(X, \mu)$ be given.

- (1) We say that u admits a weak Asymptotic Mean Value Laplacian (weak AMV Laplacian for short) if there exists a Radon measure $\Delta_{\mu}^d \mathbf{u}$ such that

$$(\Delta_{\mu,r}^d u) \mu \rightharpoonup \Delta_{\mu}^d \mathbf{u}$$

as $r \downarrow 0$, in which case we say that $\Delta_{\mu}^d \mathbf{u}$ is the weak AMV Laplacian of u .

- (2) We say that u admits a weak Symmetrized Asymptotic Mean Value Laplacian (weak SAMV Laplacian for short) if there exists a Radon measure $\tilde{\Delta}_{\mu}^d \mathbf{u}$ such that

$$(\tilde{\Delta}_{\mu,r}^d u) \mu \rightharpoonup \tilde{\Delta}_{\mu}^d \mathbf{u}$$

as $r \downarrow 0$, in which case we say that $\tilde{\Delta}_{\mu}^d \mathbf{u}$ is the weak SAMV Laplacian of u .

REMARK 3.11. A corresponding definition for the distributional AMV/SAMV Laplacian can be made for smooth manifolds by considering weak convergence in the dual of $C_c^\infty(X)$ instead of the dual of $C_c(X)$.

We point out that if u admits a weak SAMV Laplacian and $\varphi \in C_c(X)$ admits an L^2 -SAMV Laplacian, it follows from (3.4) that

$$\int_X \varphi \, d\tilde{\Delta}_\mu^d \mathbf{u} = \int_X (\tilde{\Delta}_{\mu,r}^d \varphi) \mathbf{u} \, d\mu.$$

The connection between the weak and L^p_{loc} Laplacians is the following.

PROPOSITION 3.12. *Let (X, d, μ) be a locally compact metric measure space such that μ satisfies a locally uniform comparability condition. Assume that $u \in L^1_{loc}(X, \mu)$ admits an L^p_{loc} -AMV Laplacian $\Delta_\mu^d u$ for some $p \in [1, \infty]$. Then, u admits a weak AMV Laplacian which is absolutely continuous with respect to μ with Radon–Nikodym derivative equal to $\Delta_\mu^d u$. The same holds with AMV replaced by SAMV.*

Proof. Consider $\varphi \in C_c(X)$ with support K . Let q be the conjugate exponent of p , i.e. $1/p + 1/q = 1$. Then, for any sufficiently small $r > 0$,

$$\begin{aligned} \left| \int_X \varphi (\Delta_{\mu,r}^d u - \Delta_\mu^d u) \, d\mu \right| &\leq \int_K |\varphi| \left| \Delta_{\mu,r}^d u - \Delta_\mu^d u \right| \, d\mu \\ &\leq \|\varphi\|_{L^q(K,\mu)} \left\| \Delta_{\mu,r}^d u - \Delta_\mu^d u \right\|_{L^p(K,\mu)}. \end{aligned}$$

The last term tends to zero as $r \downarrow 0$, hence $\lim_{r \downarrow 0} \int_X \varphi \Delta_{\mu,r}^d u \, d\mu = \int_X \varphi \Delta_\mu^d u \, d\mu$. The SAMV case is proved the same way. \square

Some functions admit a distributional or weak AMV Laplacian but no L^p_{loc} -AMV Laplacian. Here is an example.

EXAMPLE 3.13. Consider $(\mathbb{R}, d_e, \mathcal{L}^1)$ and $u(x) = x/|x|$, defined to be zero for $x = 0$. Then, an easy computation shows that

$$\Delta_{\mathcal{L}^1,r}^{d_e} u(x) = \begin{cases} \frac{x - r \operatorname{sgn} x}{r^3}, & r > |x|, \\ 0, & r \leq |x|. \end{cases}$$

Let V be any open neighbourhood around the origin on which v is p -integrable. Then, there exists an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq V$, and we can assume without loss

of generality that $3r < \epsilon$. Then, for $p \in (1, +\infty)$,

$$\begin{aligned} \|\Delta_{\mathcal{L}^1, r}^{d_\epsilon} u - v\|_{L^p(V)} &\geq \|\Delta_{\mathcal{L}^1, r}^{d_\epsilon} u - v\|_{L^p([- \epsilon, \epsilon])} \\ &\geq \|\Delta_{\mathcal{L}^1, r}^{d_\epsilon} u\|_{L^p([- \epsilon, \epsilon])} - \|v\|_{L^p([- \epsilon, \epsilon])} \\ &\geq \frac{1}{r^3} \left(\int_{2r}^\epsilon (x - r)^p dx \right)^{1/p} - \|v\|_{L^p([- \epsilon, \epsilon])} \\ &\geq \frac{1}{r^3} ((\epsilon - 2r)(2r - r)^p)^{1/p} - \|v\|_{L^p([- \epsilon, \epsilon])} \\ &\geq \frac{r^{1/p} \cdot r}{r^3} - \|v\|_{L^p([- \epsilon, \epsilon])} \rightarrow +\infty, \quad r \downarrow 0. \end{aligned}$$

This shows that u does not admit an L^p_{loc} -AMV Laplacian for any $p \in (1, +\infty)$. This result trivially extends to $p = \infty$. Instead, u has a distributional AMV Laplacian which can be shown to be the distributional derivative of the Dirac delta at the origin divided by three [21, p. 21].

4. Equality between the AMV and SAMV Laplacians

In this section, we study some contexts where the equality $\Delta_\mu^d = \tilde{\Delta}_\mu^d$ holds in a suitable sense.

Topological groups. We recall that a topological group is a set G equipped with a group law \cdot and a topology \mathcal{T} with respect to which the maps $(g, h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$ are continuous. A left-invariant distance d on a topological group (G, \cdot, \mathcal{T}) is a distance on G which induces the same topology as \mathcal{T} and such that

$$d(g \cdot x, g \cdot y) = d(x, y)$$

for any $g, x, y \in G$. When a topological group (G, \cdot, \mathcal{T}) is locally compact, it admits a unique—up to a positive multiplicative constant—non-zero Borel measure μ , called Haar measure, which is finite on compact sets, quasi-regular, and left-invariant in the sense that

$$\mu(g \cdot A) = \mu(A)$$

for any $g \in G$ and $A \in \mathcal{T}$.

PROPOSITION 4.1. *Let $(G, \cdot, \mathcal{T}, \mu)$ be a locally compact topological group equipped with a Haar measure μ and metrized by a left-invariant distance d . Then, the following hold:*

- (1) *For any $u \in L^1_{loc}(G, \mu)$ and any $x \in \text{Leb}(u)$, one has that $\Delta_{\mu, r}^d u(x)$ converges if and only if $\tilde{\Delta}_{\mu, r}^d u(x)$ does, in which case*

$$\Delta_\mu^d u(x) = \tilde{\Delta}_\mu^d u(x). \tag{4.1}$$

- (2) *For any $p \in [1, +\infty)$, a function $u \in L^p(G, \mu)$ admits a L^p_{loc} -AMV Laplacian if and only if it admits a L^p_{loc} -SAMV Laplacian, in which case*

$$\Delta_\mu^d u = \tilde{\Delta}_\mu^d u \quad \mu\text{-a.e.} \tag{4.2}$$

Proof. For any $x \in G$, $r > 0$ and $y \in B_r(x)$, since \mathbf{d} is left-invariant,

$$y \cdot x^{-1} \cdot B_r(x) = B_r(y),$$

and since μ is left-invariant, this implies that

$$\mu(B_r(x)) = \mu(B_r(y)). \tag{4.3}$$

Then, for any $u \in L^1(G, \mu)$ and any $x \in \text{Leb}(u)$, one has $\Delta_{\mu,r}^{\mathbf{d}}u(x) = \tilde{\Delta}_{\mu,r}^{\mathbf{d}}u(x)$ for any $r > 0$, from which (4.1) follows as $r \downarrow 0$. Moreover, (4.3) ensures that μ trivially satisfies a uniform comparability condition. \square

Riemannian and sub-Riemannian manifolds. We provide in the next proposition a general criterion to get that $\Delta_{\mu}^{\mathbf{d}}$ and $\tilde{\Delta}_{\mu}^{\mathbf{d}}$ coincide. We phrase this result in terms of the r -distortion function $\delta_r(x, y)$ introduced in definition 2.10.

PROPOSITION 4.2. *Let (X, \mathbf{d}, μ) be a metric measure space. For any $x \in X$ and $r > 0$, set*

$$\varepsilon_x(r) := \sup_{y \in B_r(x)} |\delta_r(x, y)|. \tag{4.4}$$

Let $u \in L^1_{loc}(X, \mu)$ be given. Then, for any $x \in \text{Leb}(u)$ such that

$$\varepsilon_x(r) = O(r^2), \quad r \downarrow 0, \tag{4.5}$$

it holds that

$$\lim_{r \downarrow 0} |\Delta_{\mu,r}^{\mathbf{d}}u(x) - \tilde{\Delta}_{\mu,r}^{\mathbf{d}}u(x)| = 0. \tag{4.6}$$

In particular, if $x \in \text{Leb}(u)$ satisfies (4.5), then $\Delta_{\mu,r}^{\mathbf{d}}u(x)$ converges if and only if $\tilde{\Delta}_{\mu,r}^{\mathbf{d}}u(x)$ does, in which case $\Delta_{\mu}^{\mathbf{d}}u(x) = \tilde{\Delta}_{\mu}^{\mathbf{d}}u(x)$.

Proof. For any $x \in \text{Leb}(u)$ and $r > 0$,

$$\Delta_{\mu,r}^{\mathbf{d}}u(x) - \tilde{\Delta}_{\mu,r}^{\mathbf{d}}u(x) = \frac{1}{2r^2} \int_{B_r(x)} \delta_r(x, y)(u(y) - u^*(x)) \, d\mu(y). \tag{4.7}$$

Then,

$$|\Delta_{\mu,r}^{\mathbf{d}}u(x) - \tilde{\Delta}_{\mu,r}^{\mathbf{d}}u(x)| \leq \frac{\varepsilon_x(r)}{2r^2} \int_{B_r(x)} |u(y) - u^*(x)| \, d\mu(y).$$

The conclusion follows since $\int_{B_r(x)} |u(y) - u^*(x)| \, d\mu(y) \rightarrow 0$ as $r \downarrow 0$. \square

Thanks to proposition 4.2 we can immediately show that $\Delta_{\mu}^{\mathbf{d}}$ and $\tilde{\Delta}_{\mu}^{\mathbf{d}}$ coincide on regular enough Riemannian manifolds. Indeed, let M be a smooth n -dimensional manifold equipped with a C^2 Riemannian metric. Let μ be the associated Riemannian volume measure. As is well-known (see e.g. [13]), the scalar curvature of

(M, g) is a continuous function $S_g : M \rightarrow \mathbb{R}$ such that

$$\frac{\mu(B_r(x))}{\omega_n r^n} = 1 - c_n S_g(x) r^2 + O(r^4) \quad \text{as } r \downarrow 0 \tag{4.8}$$

for any $x \in M$, where $c_n := (n + 2)^{-1} 6^{-1}$ and $O(r^4)$ depends only on curvature terms of g at x — in particular, if K is a compact subset of M , then $O(r^4)$ can be made independent of $x \in K$.

PROPOSITION 4.3. *Let M be a smooth manifold equipped with a C^2 Riemannian metric g . Let μ be the associated Riemannian volume measure. Let $u \in L^1_{loc}(M, \mu)$ be given. Then, for μ -a.e. $x \in M$,*

$$\lim_{r \downarrow 0} |\Delta_{\mu,r}^d u(x) - \tilde{\Delta}_{\mu,r}^d u(x)| = 0.$$

In particular, for μ -a.e. $x \in M$, one has that $\Delta_{\mu,r}^d u(x)$ converges if and only if $\tilde{\Delta}_{\mu,r}^d u(x)$ does, in which case $\Delta_{\mu}^d u(x) = \tilde{\Delta}_{\mu}^d u(x)$.

Proof. Let n be the dimension of M . For any $x \in M$ and $y \in B_r(x)$, thanks to (4.8) we find that

$$\begin{aligned} \frac{\mu(B_r(y))}{\mu(B_r(x))} &= \frac{1 - c_n S_g(y) r^2 + O(r^4)}{1 - c_n S_g(x) r^2 + O(r^4)} \\ &= (1 - c_n S_g(y) r^2 + O(r^4))(1 + c_n S_g(x) r^2 + O(r^4)) \\ &= 1 - c_n (S_g(y) - S_g(x)) r^2 + O(r^4) \end{aligned}$$

as $r \downarrow 0$, where we choose $O(r^4)$ uniform over points in the closed ball $\bar{B}_1(x)$. Since S_g is continuous it is uniformly continuous on $\bar{B}_1(x)$ hence it admits a non-decreasing modulus of continuity ω . Therefore,

$$\frac{\varepsilon_x(r)}{r^2} \leq c_n \left(\sup_{y \in B_r(x)} |S_g(y) - S_g(x)| \right) + O(r^2) \leq c_n \omega(r) + O(r^2)$$

as $r \downarrow 0$. This shows that (4.5) holds at any $x \in M$.

Let us show now that (M, g) satisfies the infinitesimally doubling condition (2.2). Thanks to (4.8), for any $x \in M$,

$$\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} = 2^n \frac{\mu(B_{2r}(x))}{\omega_n (2r)^n} \frac{\omega_n r^n}{\mu(B_r(x))} = 2^n \frac{1 - c_n S_g(z) (2r)^2 + O(r^4)}{1 - c_n S_g(z) r^2 + O(r^4)},$$

as $r \downarrow 0$, where we choose $O(r^4)$ uniform over points in the closed ball $\bar{B}_1(x)$, hence

$$\lim_{r \downarrow 0} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} = 2^n.$$

From (2.2) we get that μ -a.e. point $x \in M$ is a Lebesgue point of u , hence the result follows from proposition 4.2. □

Let us pass now to the context of sub-Riemannian manifolds. We recall that a sub-Riemannian structure on a manifold M is a pair (D, g) where D is a bracket generating subbundle of the tangent bundle TM , i.e. Lie brackets of vector fields tangent to D span the full tangent bundle, and g is a smooth metric defined on D . Such a structure yields a well-defined distance d called the sub-Riemannian (or Carnot–Carathéodory) distance. More precisely, $d(p, q)$ is the infimum of the length of Lipschitz curves tangent to D (also called horizontal curves) joining two points p and q . Here the length of the curve is computed with respect to the metric g . We refer to [3] for a complete introduction to sub-Riemannian geometry.

In this context, the analysis made in [7] yields the following.

PROPOSITION 4.4. *Let (M, D, g) be a 3-dimensional, contact sub-Riemannian manifold equipped with the Popp volume μ . Let $u \in L^1_{loc}(M, \mu)$ be given. Then, for μ -a.e. $x \in M$, it holds that*

$$\lim_{r \downarrow 0} |\Delta_{\mu,r}^d u(x) - \tilde{\Delta}_{\mu,r}^d u(x)| = 0.$$

In particular, for μ -a.e. $x \in M$, one has that $\Delta_{\mu,r}^d u(x)$ converges if and only if $\tilde{\Delta}_{\mu,r}^d u(x)$ does, in which case $\Delta_{\mu}^d u(x) = \tilde{\Delta}_{\mu}^d u(x)$.

Proof. From [7, Theorem 1] we know the following for any $x \in M$,

$$\frac{\mu(B_r(x))}{c_o r^4} = 1 - c_1 \kappa(x) r^2 + O(r^3) \quad \text{as } r \downarrow 0, \tag{4.9}$$

where c_o and c_1 are positive constants and κ depends smoothly of x . Then, the proof of proposition 4.3 carries over and yields the result. \square

Spaces with suitably vanishing distortion. We investigate now the case of spaces where the distortion functions δ_r satisfy more subtle assumptions. We begin with the following elementary lemma. Recall that $z_r(x)$ is defined in definition 2.10 as the average of $|\delta_r(x, y)|$.

LEMMA 4.5. *Let (X, d, μ) be a locally compact metric measure space. Then, for any $u \in \text{Lip}(X)$, $\varphi \in C_c(X)$, and $r > 0$,*

$$\left| \int_X \varphi(\Delta_{\mu,r}^d u - \tilde{\Delta}_{\mu,r}^d u) \, d\mu \right| \leq \frac{\text{Lip}(u)}{2} \int_X |\varphi| \frac{z_r}{r} \, d\mu.$$

Proof. The result follows from a direct computation:

$$\begin{aligned} \left| \int_X \varphi(\Delta_{\mu,r}^d u - \tilde{\Delta}_{\mu,r}^d u) \, d\mu \right| &\leq \frac{1}{2} \int_X |\varphi(x)| \int_{B_r(x)} |\delta_r(x, y)| \frac{|u(y) - u(x)|}{r^2} \, d\mu(y) \, d\mu(x) \\ &\leq \frac{\text{Lip}(u)}{2} \int_X |\varphi(x)| \int_{B_r(x)} |\delta_r(x, y)| \frac{d(x, y)}{r^2} \, d\mu(y) \, d\mu(x) \\ &\leq \frac{\text{Lip}(u)}{2} \int_X |\varphi(x)| \int_{B_r(x)} \frac{|\delta_r(x, y)|}{r} \, d\mu(y) \, d\mu(x) \\ &= \frac{\text{Lip}(u)}{2} \int_X |\varphi(x)| \frac{z_r(x)}{r} \, d\mu(x). \end{aligned}$$

□

As an immediate corollary, we obtain the following result by letting $r \downarrow 0$.

COROLLARY 4.6. *Let (X, d, μ) be a locally compact metric measure space such that $z_r \in L^1_{loc}(X, \mu)$ for all $r > 0$ small enough. Assume that*

$$\frac{z_r}{r} \mu \rightharpoonup 0 \quad \text{as } r \downarrow 0. \tag{4.10}$$

Then, $u \in \text{Lip}(X)$ admits a weak AMV Laplacian if and only if u admits a weak SAMV Laplacian, in which case $\Delta^d_\mu \mathbf{u} = \tilde{\Delta}^d_\mu \mathbf{u}$.

REMARK 4.7. If μ satisfies a locally uniform comparability condition, then $z_r \in L^1_{loc}(X, \mu)$. Indeed, this assumption and the local compactness of (X, d) imply that there exist $r_0, C > 0$ such that for any $x \in X, r \in (0, r_0)$ and any compact neighbourhood K of x ,

$$\int_K z_r \, d\mu \leq (1 + C)\mu(\cup_{\bar{x} \in K} B_r(\bar{x})) < +\infty.$$

We are going to apply corollary 4.6 in the context of locally Ahlfors regular spaces (see example 2.4 for the definition). Let Q be a fixed positive number. We set

$$\omega_Q := \frac{\pi^{Q/2}}{\Gamma(Q/2 + 1)}$$

where Γ is the classical Gamma function; if Q is an integer n , then ω_Q coincides with the Lebesgue measure of the unit Euclidean ball in \mathbb{R}^n . For a locally Ahlfors Q -regular metric measure space (X, d, μ) , we define

$$\theta_r(x) := \frac{\mu(B_r(x))}{\omega_Q r^Q}$$

for any $x \in X$ and $r > 0$, and

$$\mu_r = \frac{1 - \theta_r}{r} \mu.$$

Each μ_r is a signed Radon measure. We let

$$|\mu_r| = \frac{|1 - \theta_r|}{r} \mu$$

be the associated total variation measure. Here is the main result of this paragraph.

PROPOSITION 4.8. *Let (X, d, μ) be a locally compact, locally Ahlfors Q -regular metric measure space satisfying*

$$|\mu_r| \rightharpoonup 0 \quad \text{as } r \downarrow 0. \tag{4.11}$$

Then, $u \in \text{Lip}(X)$ admits a weak AMV Laplacian if and only if u admits a weak SAMV Laplacian, in which case $\Delta^d_\mu \mathbf{u} = \tilde{\Delta}^d_\mu \mathbf{u}$.

To prove proposition 4.8, we need the following lemma.

LEMMA 4.9. *Let (X, d, μ) be a locally compact metric measure space satisfying a locally uniform comparability condition. For any compact set $K \subset X$, there exist $r_K, C_K > 0$ depending only on K , and another compact set $K' \subset X$ containing K such that for any $f \in L^\infty_{loc}(X, \mu)$,*

$$\sup_{r \in (0, r_K)} \|A_r^* f - f\|_{L^\infty(K, \mu)} \leq C_K \|f\|_{L^\infty(K', \mu)}. \tag{4.12}$$

Proof. Let $K \subset X$ be a compact set. The local compactness of (X, d) ensures that there exist $r_K > 0$ and another compact set $K' \subset X$ containing K such that

$$\bigcup_{x \in K} B_{r_K}(x) \subset K'.$$

Consider $f \in L^\infty_{loc}(X, \mu)$. For μ -a.e. $x \in K$ and any $r \in (0, r_K)$,

$$\begin{aligned} |A_r^* f(x) - f(x)| &= \left| \int_{B_r(x)} \left(\frac{f(y)}{\mu(B_r(y))} - \frac{f(x)}{\mu(B_r(x))} \right) d\mu(y) \right| \\ &\leq \left| \int_{B_r(x)} \frac{f(y) - f(x)}{\mu(B_r(y))} d\mu(y) \right| \\ &\quad + \left| \int_{B_r(x)} \left(\frac{f(x)}{\mu(B_r(y))} - \frac{f(x)}{\mu(B_r(x))} \right) d\mu(y) \right| \\ &\leq \int_{B_r(x)} \frac{|f(y) - f(x)|}{\mu(B_r(y))} d\mu(y) + |f(x)| \left| \int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))} - 1 \right| \\ &\leq 2\|f\|_{L^\infty(K', \mu)} \int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))} \\ &\quad + \|f\|_{L^\infty(K', \mu)} \left(\int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))} + 1 \right) \\ &= \|f\|_{L^\infty(K', \mu)} \left(3 \int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))} + 1 \right). \end{aligned}$$

By remark 2.2, the locally uniform comparability condition implies that

$$\int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))} \leq C'_K$$

for some $C'_K > 0$ depending only on K . We obtain the desired result by setting $C_K := 3C'_K + 1$. □

We are now in a position to prove proposition 4.8.

Proof of proposition 4.8. By example 2.4 our space fulfils a locally uniform comparability condition, and by remark 4.7 we infer that $z_r \in L^1_{\text{loc}}(X, \mu)$. Hence corollary 4.6 shows that we only need to prove that

$$\frac{z_r}{r} \mu \rightharpoonup 0$$

as $r \downarrow 0$. Take $\varphi \in C_c(X)$. Up to decomposing φ as $\varphi^+ - \varphi^-$, we do not lose any generality in assuming that φ is non-negative, what we do from now on. Let K be the support of φ . Let r_K, C_K and K' be given by lemma 4.9. Consider $r \in (0, r_K)$. Observe that if $x \in K$ and $y \in B_r(x)$, then

$$\begin{aligned} |\delta_r(x, y)| &= \frac{|\mu(B_r(y)) - \mu(B_r(x))|}{\mu(B_r(y))} = \frac{|\theta_r(y) - \theta_r(x)|}{\theta_r(y)} \\ &\leq \frac{1}{\theta_r(y)} (|\theta_r(y) - 1| + |1 - \theta_r(x)|) \\ &\leq \omega_Q^{-1} C_{K'} \left(|\theta_r(y) - 1| + |1 - \theta_r(x)| \right) \end{aligned}$$

where we have multiplied and divided by $\omega_Q r^Q$ to get the second equality and we have used the local Ahlfors regularity property on K' to get the last one. Then,

$$\begin{aligned} \int_X \varphi \frac{z_r}{r} d\mu &= \int_X \varphi(x) \int_{B_r(x)} \frac{|\delta_r(x, y)|}{r} d\mu(y) d\mu(x) \\ &\leq \omega_Q^{-1} C_{K'} \left(\int_X \varphi(x) \int_{B_r(x)} \frac{|\theta_r(y) - 1|}{r} d\mu(y) d\mu(x) \right. \\ &\quad \left. + \int_X \varphi(x) \frac{|\theta_r(x) - 1|}{r} d\mu(x) \right) \end{aligned}$$

By (4.11), the second term on the right-hand side converges to 0 when $r \downarrow 0$. Let us show that also the first one converges to 0. We have

$$\begin{aligned} \int_X \varphi(x) \int_{B_r(x)} \frac{|\theta_r(y) - 1|}{r} d\mu(y) d\mu(x) &= \int_X \varphi(x) A_r \left[\frac{|\theta_r(\cdot) - 1|}{r} \right] (x) d\mu(x) \\ &= \int_X A_r^* \varphi(x) \frac{|\theta_r(x) - 1|}{r} d\mu(x). \end{aligned}$$

Since $x \notin K'$ yields $B_r(x) \cap \text{supp } \varphi = \emptyset$ which in turn implies that $A_r^* \varphi(x) = 0$, we may replace the previous integral over X with an integral over K' . Then,

$$\begin{aligned} \int_X \varphi(x) \int_{B_r(x)} \frac{|\theta_r(y) - 1|}{r} d\mu(y) d\mu(x) \\ \leq \int_{K'} |A_r^* \varphi(x) - \varphi(x)| \frac{|\theta_r(x) - 1|}{r} d\mu(x) + \int_{K'} \varphi \frac{|\theta_r - 1|}{r} d\mu \end{aligned}$$

$$\begin{aligned} &\leq C\|\varphi\|_{L^\infty(X,\mu)} \int_{K'} \frac{|\theta_r - 1|}{r} \, d\mu \\ &\leq C\|\varphi\|_{L^\infty(X,\mu)} \int_X \psi \frac{|\theta_r - 1|}{r} \, d\mu \end{aligned}$$

where we have used (4.12) to get the second inequality and we have set $\psi(\cdot) := (1 - d(\cdot, K')/\rho)^+$ for $\rho > 0$ small enough to ensure that $\psi \in C_c(X)$. By (4.11), we get

$$\lim_{r \downarrow 0} \int_X \psi \frac{|\theta_r - 1|}{r} \, d\mu = 0. \quad \square$$

REMARK 4.10. In [16], the authors introduced the following definition: a metric measure space (X, d, μ) has *vanishing metric measure boundary* if $\mu_r \rightarrow 0$ as $r \downarrow 0$. It would be interesting to study whether a result like proposition 4.8 may be obtained with (4.11) replaced by this weaker assumption.

REMARK 4.11. For $K \in \mathbb{R}$ and $N \in [1, +\infty)$, an $\text{RCD}(K, N)$ space is a proper (hence locally compact) metric measure space satisfying a synthetic notion of Ricci curvature bounded below by K and dimension bounded above by N ; we refer to [6, 14], for instance, for a nice account on these spaces. According to [12], an $\text{RCD}(K, N)$ space (X, d, μ) is called non-collapsed if N is an integer and $\mu = \mathcal{H}^N$. Non-collapsed $\text{RCD}(K, N)$ spaces are locally Ahlfors N -regular; this is a consequence of [12, Theorem 1.3]. By [10, Theorem 1.2] (see also [2, Theorem 3.7]), any non-collapsed $\text{RCD}(K, N)$ space with vanishing metric-measure boundary satisfies (4.11). Building upon this, Adamowicz, Kijowski, and Soultanis proved in [2, Corollary 3.9] that the weak AMV and SAMV Laplacians coincide on such a space. Our result provides the same conclusion in a setting where no curvature-dimension condition is assumed. In this regard, it would be worth investigating the validity of (4.11) in contexts which are not RCD, like sub-Riemannian or Finsler spaces.

5. Weighted Lebesgue measures

In this section, we study the context of weighted Lebesgue measures where the two notions of pointwise Laplacian *do not* coincide. We first focus on points where the weight is positive, before tackling points where the weight vanishes.

5.1. Positive weights

We begin this subsection with a straightforward result.

PROPOSITION 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set equipped with the Euclidean distance d_e , and $\mu := w \mathcal{L}^n \llcorner \Omega$ where $w \in C^1(\Omega)$. Consider (Ω, d_e, μ) . Then, for any $x \in \Omega$ such that $w(x) > 0$, any $u \in C^2(\Omega)$ admits a pointwise AMV Laplacian and a*

pointwise SAMV Laplacian at x , and

$$\Delta_\mu^{d_e} u(x) = \frac{1}{2(n+2)} (\Delta u(x) + 2\langle \nabla \ln w, \nabla u \rangle(x)), \tag{5.1}$$

$$\tilde{\Delta}_\mu^{d_e} u(x) = \frac{1}{2(n+2)} (\Delta u(x) + \langle \nabla \ln w, \nabla u \rangle(x)). \tag{5.2}$$

Proof. The first equality was proved in [21, Proposition 2.3]. Let us prove the second equality following the same lines. Let $r_0 > 0$ be such that $B_{r_0}(x) \subset \Omega$ and $w(y) > w(x)/2$ for any $y \in B_{r_0}(x)$. Then, for any $r \in (0, r_0)$,

$$\tilde{\Delta}_{\mu,r}^{d_e} u(x) = \frac{1}{2} (\Delta_{\mu,r}^{d_e} u(x) + I(r)). \tag{5.3}$$

where

$$I(r) := \frac{1}{r^2} \int_{B_r(x)} (u(y) - u(x)) \frac{w(y)}{\mu(B_r(y))} dy.$$

We claim that

$$I(r) \rightarrow \frac{\Delta u(x)}{2(n+2)} \quad \text{as } r \downarrow 0. \tag{5.4}$$

Once this claim is proved, (5.2) follows from letting $r \downarrow 0$ in (5.3) and using (5.1). We now prove (5.4). Since w is C^1 , a first-order Taylor expansion shows that for any $0 < r < r_0/2$, $y \in B_r(x)$, and $z \in B_r(y)$,

$$w(z) = w(y) + \langle R_y(z), y - z \rangle$$

for some $R_y(z) \in \mathbb{R}^n$ such that

$$|R_y(z)| \leq C := \sup_{\xi \in \overline{B_{r_0}(x)}} |\nabla w(\xi)|.$$

Then,

$$\mu(B_r(y)) = \int_{B_r(y)} w(z) dz = \mathcal{L}^n(B_r(y)) \left(w(y) + \int_{B_r(y)} \langle R_y(z), y - z \rangle dz \right)$$

so that

$$\frac{w(y)}{\mu(B_r(y))} = \frac{1}{\mathcal{L}^n(B_r(y)) \left(1 + \frac{1}{w(y)} \int_{B_r(y)} \langle R_y(z), y - z \rangle dz \right)}. \tag{5.5}$$

Now

$$\left| \frac{1}{w(y)} \int_{B_r(y)} \langle R_y(z), y - z \rangle dz \right| \leq \frac{2C}{w(x)} r = O(r) \tag{5.6}$$

and $\mathcal{L}^n(B_r(y)) = \mathcal{L}^n(B_r(x))$, hence we get

$$I(r) = \frac{1}{r^2(1 + O(r))} \int_{B_r(x)} u(y) - u(x) dy \rightarrow \frac{\Delta u(x)}{2(n+2)} \quad \text{as } r \downarrow 0.$$

□

REMARK 5.2. Proposition 5.1 shows that if $w > 0$ everywhere in Ω , then $\tilde{\Delta}_\mu^{d_e}$ coincides with a dimensional constant times the classical weighted Laplacian of $(\mathbb{R}^n, d_e, w\mathcal{L}^n)$ (also known as drifted Laplacian, or f -Laplacian, or Witten Laplacian, see e.g. [11, 22] and the references therein), namely

$$\Delta u + \langle \nabla \ln w, \nabla u \rangle.$$

Moreover, working in exponential coordinates, it is not difficult to show that (5.1) and (5.2) extend to the setting of weighted Riemannian manifolds $(M, g, w\mu)$ where μ is the canonical Riemannian measure and $w : M \rightarrow (0, +\infty)$ is a smooth map.

We now study the case where the Euclidean distance on Ω is replaced with a more general distance d . To this aim, for any $x \in \Omega$ and $r > 0$, we set

$$M^r(x) := \left(M_{ij}^r(x) := \frac{1}{r^2} \int_{B_r(x)} (y-x)_i (y-x)_j \, dy \right)_{1 \leq i < j \leq n}.$$

In case d is associated with a norm, the change of variable $\xi = (y-x)/r$ shows that the matrices $M^r(x)$ are all equal to the second-moment matrix

$$M(0) := \left(M_{ij}(0) := \int_{B_1(0)} \xi_i \xi_j \, d\xi \right)_{1 \leq i < j \leq n}.$$

We will also work under the assumption that balls for d are symmetric with respect to the vector space structure of \mathbb{R}^n , in the sense that for any $x \in \Omega$ and $r > 0$ such that $B_r(x) \subset \Omega$, for any $v \in \mathbb{R}^n$,

$$x + v \in B_r(x) \iff x - v \in B_r(x). \tag{5.7}$$

This is trivially satisfied when d is associated with a norm. But it may fail in general. For instance, set

$$d(x, y) := \begin{cases} |x - y|, & x, y \leq 0, \\ \frac{1}{2}|x - y|, & x, y > 0, \\ \frac{1}{2}y - x, & x \leq 0, y > 0, \\ \frac{1}{2}x - y, & y \leq 0, x > 0. \end{cases}$$

Then, d is a metric on \mathbb{R} such that $\frac{1}{2}d_e \leq d \leq 2d_e$ and $B_r(0) = (-r, 2r)$ for any $r > 0$.

A large class of distances d which satisfy (5.7) without being associated with a norm is given by the following: For $\alpha = (\alpha_1, \dots, \alpha_n) \in \{1, 2\}^n$, set

$$\Phi_\alpha(x) := (x_1^{\alpha_1}, \dots, x_n^{\alpha_n})$$

for any $x = (x_1, \dots, x_n) \in (0, +\infty)^n$. Set

$$d_\alpha(x, y) := \|\Phi_\alpha(x) - \Phi_\alpha(y)\|$$

for any $x, y \in (0, +\infty)^n$, where $\|\cdot\|$ is any l^p -norm on \mathbb{R}^n . Then, d_α is a distance on $(0, +\infty)^n$ which always satisfies (5.7); it is associated with a norm if and only if $\alpha_1 = \dots = \alpha_n = 1$, because otherwise homogeneity fails.

Our next result makes use of $z_r^{\mathcal{L}^n}$, i.e. the average of the absolute value of the distortion function, defined for any $r > 0$ by

$$z_r^{\mathcal{L}^n}(x) := \int_{B_r(x)} \left| 1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))} \right| dy. \tag{5.8}$$

for any $x \in \mathbb{R}^n$.

PROPOSITION 5.3. *Let $\Omega \subset \mathbb{R}^n$ be open with respect to a distance d satisfying (5.7). For $w \in C^1(\Omega)$ set $\mu := w\mathcal{L}^n \llcorner \Omega$. Consider (Ω, d, μ) . Let $x \in \Omega$ be such that $w(x) > 0$.*

- (1) *Assume that the limit $M(x) := \lim_{r \downarrow 0} M^r(x)$ exists. Then, any $u \in C^2(\Omega)$ admits a pointwise AMV Laplacian at x , and*

$$\Delta_\mu^d u(x) = \frac{1}{2} \text{Tr}(M(x) \nabla^2 u(x)) + \frac{1}{w(x)} \langle \nabla w(x), M(x) \nabla u(x) \rangle. \tag{5.9}$$

- (2) *If in addition*

$$z_r^{\mathcal{L}^n}(x) = o(r) \tag{5.10}$$

as $r \downarrow 0$, then u admits a pointwise SAMV Laplacian at x , and

$$\tilde{\Delta}_\mu^d u(x) = \frac{1}{2w(x)} \text{Tr}(M(x) \nabla(w \nabla u)(x)). \tag{5.11}$$

Proof. Step 1. We first prove (5.9). To this aim, we claim that for any $r > 0$,

$$\begin{aligned} \Delta_{\mu,r}^d u(x) &= \frac{\mathcal{L}^n(B_r(x))w(x)}{\mu(B_r(x))} \left(\frac{1}{2} \text{Tr}(\nabla^2 u(x) M^r(x)) \right. \\ &\quad \left. + \frac{1}{w(x)} \langle M^r(x) \nabla u(x), \nabla w(x) \rangle \right) + O(r). \end{aligned} \tag{5.12}$$

Once this is proved, (5.9) follows immediately since $\mu(B_r(x))/\mathcal{L}^n(B_r(x)) \rightarrow w(x)$ as $r \downarrow 0$. Let us prove (5.12). By a second-order Taylor expansion of u and a first-order Taylor expansion of w , for any $y \in B_r(x)$ we infer that

$$\begin{aligned} (u(y) - u(x))w(y) &= \sum_{i=1}^n [wu_i](x)(y - x)_i \\ &\quad + \sum_{i,j=1}^n \left[\frac{wu_{ij}}{2} + w_i u_j \right] (x) (y - x)_i (y - x)_j + O(r^3) \end{aligned}$$

where $O(r^3)$ is independent of y and where we have denoted by u_i the partial derivatives of u , by u_{ij} the second-order partial derivatives of u , and similarly for w . Integrate with respect to $y \in B_r(x)$ and then divide by $r^2\mu(B_r(x))$. Then, the left-hand side becomes $\Delta_{\mu,r}^d u(x)$. Due to antisymmetry, assumption (5.7) implies that the first term on the right-hand side vanishes when integrating w.r.t. y over $B_r(x)$. Then, we get

$$\begin{aligned} \Delta_{\mu,r}^d u(x) &= \sum_{i,j=1}^n \left[\frac{wu_{ij}}{2} + w_i u_j \right] (x) \frac{1}{r^2\mu(B_r(x))} \\ &\quad \times \int_{B_r(x)} (y - x)_i (y - x)_j \, dy + \frac{\mathcal{L}^n(B_r(x))}{\mu(B_r(x))} O(r) \\ &= \frac{\mathcal{L}^n(B_r(x))}{\mu(B_r(x))} \sum_{i,j=1}^n \left[\frac{wu_{ij}}{2} + w_i u_j \right] (x) M_{ij}^r(x) + O(r) \\ &= \frac{\mathcal{L}^n(B_r(x))}{\mu(B_r(x))} \left(\frac{w(x)}{2} \text{Tr}(\nabla^2 u(x) M^r(x)) \right. \\ &\quad \left. + \langle M^r(x) \nabla u(x), \nabla w(x) \rangle \right) + O(r). \end{aligned}$$

Step 2. Now we prove (5.11). Observe that

$$\tilde{\Delta}_{\mu,r}^d u(x) = \frac{1}{2} (\Delta_{\mu,r}^d u(x) + I(r)) \tag{5.13}$$

where

$$I(r) := \frac{1}{r^2} \int_{B_r(x)} (u(y) - u(x)) \frac{w(y)}{\mu(B_r(y))} \, dy.$$

We claim that

$$I(r) \rightarrow \Delta_{\mathcal{L}^n}^d u(x) = \frac{1}{2} \text{Tr}(M(x) \nabla^2 u(x)). \tag{5.14}$$

Once this claim is proved, letting $r \downarrow 0$ in (5.13) yields (5.11), since a direct computation gives

$$\frac{1}{2w(x)} \text{Tr}(M(x) \nabla(w \nabla u)(x)) = \frac{1}{2} \text{Tr}(M(x) \nabla^2 u(x)) + \frac{1}{2w(x)} \langle M(x) \nabla u(x), \nabla w(x) \rangle. \tag{5.15}$$

Thus, we prove (5.14). Thanks to (5.5) and (5.6), we get

$$\begin{aligned}
 I(r) &= \frac{1}{r^2(1+O(r))} \int_{B_r(x)} (u(y) - u(x)) \frac{1}{\mathcal{L}^n(B_r(y))} dy \\
 &= \frac{1}{r^2(1+O(r))} \left[\int_{B_r(x)} (u(y) - u(x)) dy \right. \\
 &\quad \left. - \int_{B_r(x)} (u(y) - u(x)) \left(1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))}\right) dy \right] \\
 &= \frac{1}{1+O(r)} \left[\Delta_{\mathcal{L}^n, r}^d u(x) - \frac{1}{r^2} \int_{B_r(x)} (u(y) - u(x)) \left(1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))}\right) dy \right].
 \end{aligned}
 \tag{5.16}$$

Now using that u is Lipschitz, we get

$$\begin{aligned}
 &\frac{1}{r^2} \left| \int_{B_r(x)} (u(y) - u(x)) \left(1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))}\right) dy \right| \\
 &\leq \frac{\text{Lip}(u)}{r^2} \int_{B_r(x)} d(x, y) \left| 1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))} \right| dy \\
 &\leq \frac{\text{Lip}(u)}{r} \int_{B_r(x)} \left| 1 - \frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_r(y))} \right| dy \\
 &= \text{Lip}(u) \frac{z_r^{\mathcal{L}^n}(x)}{r} \rightarrow 0 \quad \text{as } r \downarrow 0
 \end{aligned}$$

by assumption (5.10). Then, (5.14) follows from letting $r \downarrow 0$ in (5.16). □

REMARK 5.4. It follows from the previous proof that

$$\tilde{\Delta}_\mu^d u(x) = \frac{1}{2} (\Delta_\mu^d u(x) + \Delta_{\mathcal{L}^n}^d u(x)).
 \tag{5.17}$$

REMARK 5.5. Let $\|\cdot\|$ denote the infinity matrix norm. By the Jensen inequality,

$$\|M^r(x)\| \leq \int_{B_r(x)} \left\| \left(\frac{(y-x)_i (y-x)_j}{r^2} \right)_{1 \leq i, j \leq n} \right\| dy \leq 1$$

for any $r > 0$, so the sequence $\{M^r(x)\}_{r>0}$ is bounded and therefore it admits a set of accumulation points $\{M_\alpha\}$ as $r \downarrow 0$. This set is a singleton if and only if the limit $M(x) := \lim_{r \downarrow 0} M^r(x)$ exists. If this is not the case, then $\Delta_\mu^d u(x)$ and $\tilde{\Delta}_\mu^d u(x)$ may be understood as multivalued:

$$\begin{aligned}
 \Delta_\mu^d u(x) &= \left\{ \frac{1}{2} \text{Tr}(M_\alpha(x) \nabla^2 u(x)) + \frac{1}{w(x)} \langle \nabla w(x), M_\alpha(x) \nabla u(x) \rangle \right\}, \\
 \tilde{\Delta}_\mu^d u(x) &= \left\{ \frac{1}{2w(x)} \text{Tr}(M_\alpha(x) \nabla(w \nabla u)(x)) \right\}.
 \end{aligned}$$

5.2. Vanishing weights

We are now interested in the pointwise AMV and SAMV Laplacians at a point where the weight vanishes. With no loss of generality, we assume that this point is the origin 0_n . We let $\|\cdot\|$ be a norm on \mathbb{R}^n .

DEFINITION 5.6. *Let w be a Lebesgue integrable function defined on a neighbourhood Ω of 0_n . We say that w is a vanishing weight at 0_n if $w(x) > 0$ for \mathcal{L}^n -a.e. $x \in \Omega$ and $0_n \in \text{Leb}(w)$ with $w^*(0_n) = 0$.*

For any Radon measure ν on \mathbb{R}^n , we define the second-moment matrix of ν as

$$M_\nu := \left(\int_{B_1(0_n)} y_i y_j \, d\nu(y) \right)_{1 \leq i, j \leq n}. \tag{5.18}$$

5.2.1. *Infinitesimally even weights*

DEFINITION 5.7. *We say that a weight w is infinitesimally even at 0_n if it is \mathcal{L}^n -essentially bounded in a neighbourhood of 0_n and such that*

$$\mathcal{L}^n - \text{ess sup}_{x \in B_r(0_n)} |w(x) - w(-x)| = o \left(r \int_{B_r(0_n)} w \, d\mathcal{L}^n \right) \quad \text{as } r \downarrow 0.$$

Our first result is the following.

PROPOSITION 5.8. *Consider $(\Omega, \|\cdot\|, w\mathcal{L}^n)$ where w is a vanishing weight at 0_n of domain Ω which is infinitesimally even at 0_n . Set $\mu := w\mathcal{L}^n$. Assume that the blow-up probability measures*

$$\nu_r := \left(\frac{w(r \cdot) r^n}{\mu(B_r(0_n))} \right) \mathcal{L}^n \llcorner B_1(0_n)$$

weakly converge to some Radon measure ν supported in $B_1(0_n)$.¹ Then, the point-wise AMV Laplacian at 0_n of any function $u : \Omega \rightarrow \mathbb{R}$ two times differentiable at 0_n exists and satisfies

$$\Delta_\mu^{\|\cdot\|} u(0_n) = \frac{1}{2} \text{Tr}(M_\nu \nabla^2 u)(0_n). \tag{5.19}$$

¹Observe that

$$\frac{1}{r^2} \int_{B_r(0_n)} x_i x_j \, d\mu(x) = \int_{B_1(0_n)} y_i y_j \, d\nu_r(y)$$

Proof. From the Taylor theorem, there exist a neighbourhood $\Omega' \subset \Omega$ of 0_n and functions $h_{ij} : \Omega' \rightarrow \mathbb{R}$ such that $\lim_{x \rightarrow 0_n} h_{ij}(x) = 0$ and

$$u(x) - u(0_n) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n)x_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(0_n)x_i x_j + \sum_{i,j=1}^n h_{ij}(x)x_i x_j$$

for any $x \in \Omega'$. Then, for any $r > 0$ such that $B_r(0_n) \subset \Omega'$,

$$\begin{aligned} \Delta_{\mu,r}^{\|\cdot\|} u(0_n) &= \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n) \frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(0_n) \frac{1}{r^2} \int_{B_r(0_n)} x_i x_j \, d\mu(x) \\ &\quad + \sum_{i,j=1}^n \frac{1}{r^2} \int_{B_r(0_n)} h_{ij}(x)x_i x_j \, d\mu(x). \end{aligned} \tag{5.20}$$

Step 1. We fix $i, j \in \{1, \dots, n\}$ and show that

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{B_r(0_n)} h_{ij}(x)x_i x_j \, d\mu(x) = 0. \tag{5.21}$$

Fix $\varepsilon > 0$. Then, there exists $\delta \in (0, r_0)$ such that $|h_{ij}(x)| < \varepsilon$ for any $x \in B_\delta(0_n)$. For any $r \in (0, \delta)$,

$$\left| \frac{1}{r^2} \int_{B_r(0_n)} h_{ij}(x)x_i x_j \, d\mu(x) \right| \leq \frac{1}{r^2} \int_{B_r(0_n)} |h_{ij}(x)||x_i||x_j| \, d\mu(x) < \varepsilon,$$

where we have used that $|x_i| \leq r$ and $|x_j| \leq r$ to get the last inequality. This yields (5.21).

Step 2. We fix $i \in \{1, \dots, n\}$ and show that

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) = 0. \tag{5.22}$$

Observe that

$$\int_{B_r(0_n)} x_i \, d\mu(x) = \frac{1}{\int_{B_r(0_n)} w(x) \, dx} \int_{B_r(0_n)} x_i w(x) \, dx \tag{5.23}$$

and

$$\begin{aligned} \int_{B_r(0_n)} x_i w(x) \, dx &= \frac{1}{2} \int_{B_r(0_n)} x_i w(x) \, dx + \frac{1}{2} \int_{B_r(0_n)} x_i w(x) \, dx \\ &= \frac{1}{2} \int_{B_r(0_n)} x_i w(x) \, dx + \frac{1}{2} \int_{B_r(0_n)} -x'_i w(-x') \, dx' \\ &= \frac{1}{2} \int_{B_r(0_n)} x_i [w(x) - w(-x)] \, dx, \end{aligned} \tag{5.24}$$

where we have used the change of variable $x' = -x$ in the second integral to pass from the first line to the second. Set $S_w(r) := \mathcal{L}^n - \text{ess sup}_{x \in B_r(0_n)} |w(x) - w(-x)|$. Then,

$$\begin{aligned} \left| \frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) \right| &= \frac{1}{2r^2 \left| \int_{B_r(0_n)} w(x) \, dx \right|} \left| \int_{B_r(0_n)} x_i [w(x) - w(-x)] \, dx \right| \\ &\leq \frac{1}{2r^2 \left| \int_{B_r(0_n)} w(x) \, dx \right|} \left(\int_{B_r(0_n)} |x_i| |w(x) - w(-x)| \, dx \right) \\ &\leq \frac{S_w(r)}{2r \left| \int_{B_r(0_n)} w(y) \, dy \right|} \int_{B_r(0_n)} 1 \, dx \\ &= \frac{S_w(r)}{2r \left| \int_{B_r(0_n)} w(y) \, dy \right|} \end{aligned}$$

where we have used $|x_i| \leq r$ and $|w(x) - w(-x)| \leq S_w(r)$ to get the second inequality. Then, (5.22) follows from the infinitesimal evenness assumption on w .

Step 3. We fix $i, j \in \{1, \dots, n\}$ and show that

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{B_r(0_n)} x_i x_j \, d\mu(x) = \int_{B_1(0_n)} y_i y_j \, d\nu(y). \tag{5.25}$$

Performing the change of variable $y = x/r$, we have

$$\begin{aligned} \frac{1}{r^2} \int_{B_r(0_n)} x_i x_j \, d\mu(x) &= \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_1(0_n)} (ry_i)(ry_j) w(ry) r^n \, dy \\ &= \int_{B_1(0_n)} y_i y_j \, d\nu_r(y) \rightarrow \int_{B_1(0_n)} y_i y_j \, d\nu(y) \quad \text{as } r \downarrow 0. \end{aligned}$$

□

REMARK 5.9. By the Prokhorov theorem, the probability measures $\{\nu_r\}$ always admit accumulation points in the weak topology of measures as $r \downarrow 0$. The assumption made in the previous proposition demands that there is a unique limit. If this is not satisfied, then $\Delta_{\mu}^{\|\cdot\|} u(0_n)$ may be understood as multivalued, namely we get

$$\Delta_{\mu}^{\|\cdot\|} u(0_n) = \left\{ \frac{1}{2} \text{Tr}(M_{\bar{\nu}_\alpha} \nabla^2 u)(0_n) \right\}$$

where $\{\bar{\nu}_\alpha\}$ are the limit points of $\{\nu_r\}$ as $r \downarrow 0$.

A version of proposition 5.8 also holds in the case of the SAMV Laplacian, but under slightly different assumptions.

PROPOSITION 5.10. Consider $(\Omega, \|\cdot\|, w, \mathcal{L}^n)$ where w is a vanishing weight at 0_n of domain Ω which is infinitesimally even at 0_n . Assume that $\mu := w, \mathcal{L}^n$ satisfies

a comparability condition at 0_n , that the weight

$$\tilde{w}(\cdot) := \frac{w(\cdot)}{2} \left(1 + \frac{\mu(B_r(0_n))}{\mu(B_r(\cdot))} \right) \tag{5.26}$$

is infinitesimally even at 0_n , and that there exists a Radon measure $\tilde{\nu}$ on $B_1(0_n)$ such that

$$\left(\frac{\tilde{w}(r \cdot)r^n}{\mu(B_r(0_n))} \right) \mathcal{L}^n \llcorner B_1(0_n) =: \tilde{\nu}_r \rightarrow \tilde{\nu} \quad \text{as } r \downarrow 0. \tag{5.27}$$

Then, the pointwise SAMV Laplacian at 0_n of any function $u : \Omega \rightarrow \mathbb{R}$ two times differentiable at 0_n exists and satisfies

$$\tilde{\Delta}_\mu \|\cdot\| u(0_n) = \frac{1}{2} \text{Tr}(M_{\tilde{\nu}} \nabla^2 u)(0_n). \tag{5.28}$$

Proof. Let $u : \Omega \rightarrow \mathbb{R}$ be two times differentiable at 0_n . Using the Taylor theorem like in the proof of proposition 5.8, we get that for any small enough r , it holds that

$$\begin{aligned} \tilde{\Delta}_{\mu,r} \|\cdot\| u(0_n) &= \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n) \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} x_i \tilde{w}(x) \, dx \\ &\quad + \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(0_n) \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} x_i x_j \tilde{w}(x) \, dx \\ &\quad + \sum_{i,j=1}^n \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} h_{ij}(x) x_i x_j \tilde{w}(x) \, dx. \end{aligned} \tag{5.29}$$

Fix $i, j \in \{1, \dots, n\}$. For any $\varepsilon > 0$ there exists $\delta \in (0, r_0)$ such that $|h_{ij}(x)| < \varepsilon$ for any $x \in B_\delta(0_n)$. For any $r \in (0, \delta)$,

$$\begin{aligned} &\left| \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} h_{ij}(x) x_i x_j \tilde{w}(x) \, dx \right| \\ &\leq \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} |h_{ij}(x)| |x_i| |x_j| \tilde{w}(x) \, dx \\ &\leq \frac{\varepsilon}{\mu(B_r(0_n))} \int_{B_r(0_n)} \tilde{w}(x) \, dx \leq (1 + C)\varepsilon/2, \end{aligned}$$

where we have used the local comparability condition at 0_n to get the last inequality. This yields

$$\lim_{r \downarrow 0} \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} h_{ij}(x) x_i x_j \tilde{w}(x) \, dx = 0. \tag{5.30}$$

Moreover, we have that

$$\lim_{r \downarrow 0} \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} x_i \tilde{w}(x) \, dx = 0. \tag{5.31}$$

Indeed, acting like in (5.24) we get

$$\int_{B_r(0_n)} x_i \tilde{w}(x) \, dx = \frac{1}{2} \int_{B_r(0_n)} x_i [\tilde{w}(x) - \tilde{w}(-x)] \, dx.$$

Set $S_{\tilde{w}}(r) := \mathcal{L}^n - \text{ess sup}_{x \in B_r(0_n)} |\tilde{w}(x) - \tilde{w}(-x)|$. Like in the previous proof, we obtain

$$\left| \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} x_i \tilde{w}(x) \, dx \right| \leq \frac{S_{\tilde{w}}(r)}{2r | \int_{B_r(0_n)} w(x) \, dx |} \leq \frac{(1 + C) S_{\tilde{w}}(r)}{4r | \int_{B_r(0_n)} \tilde{w}(x) \, dx |}$$

where we have used the local comparability condition at 0_n to get the last inequality. Then, (5.31) follows from the infinitesimal evenness assumption on \tilde{w} .

Lastly, acting as in the previous proof and using the definition of $\tilde{\nu}_r$, we obtain that

$$\lim_{r \downarrow 0} \frac{1}{r^2 \mu(B_r(0_n))} \int_{B_r(0_n)} x_i x_j \tilde{w}(x) \, dx = \int_{B_1(0_n)} y_i y_j \, d\tilde{\nu}(y) \tag{5.32}$$

for any $i, j \in \{1, \dots, n\}$.

The conclusion follows by letting $r \downarrow 0$ in (5.29) and using (5.30), (5.31) and (5.32). □

REMARK 5.11. A simple computation shows that

$$\tilde{\nu}_r = \frac{1}{2} \left(\nu_r + \frac{r^n}{\mu(B_r(\cdot))} \mu \llcorner B_1(0_n) \right). \tag{5.33}$$

In this way, it clearly appears that there exists $r_0 > 0$ such that the measures $\{\tilde{\nu}_r\}_{0 < r < r_0}$ are uniformly bounded, thanks to the local comparability condition at 0_n of μ . Therefore, like in the non-symmetrized case (see remark 5.9), the Prokhorov theorem implies that the measures $\{\tilde{\nu}_r\}_{r > 0}$ admit limit points in the weak topology of measures as $r \downarrow 0$. If these limit points are not unique then $\tilde{\Delta}_\mu^{\|\cdot\|} u(0)$ may be understood as multivalued, namely

$$\tilde{\Delta}_\mu^{\|\cdot\|} u(0_n) = \left\{ \frac{1}{2} \text{Tr}(M_{\hat{\nu}_\alpha} \nabla^2 u)(0_n) \right\}$$

where $\{\hat{\nu}_\alpha\}$ are the limit points of $\{\tilde{\nu}_r\}$ as $r \downarrow 0$.

REMARK 5.12. If w is even, then so is \tilde{w} . Indeed, in this case, obvious changes of variable show that

$$\begin{aligned} \mu(B_r(-x)) &= \int_{B_r(0_n)} w(-x + y) \, dy = \int_{B_r(0_n)} w(-x - y) \, dy \\ &= \int_{B_r(0_n)} w(x + y) \, dy = \mu(B_r(x)) \end{aligned}$$

from which evenness of \tilde{w} follows from the definition in (5.26).

5.2.2. *Weights* $w(\cdot) = |\cdot|^\alpha$ Proposition 5.8 may be applied to the case of weights $w(\cdot) = |\cdot|^\alpha$ where $|\cdot|$ is the Euclidean norm and $\alpha > 0$.

COROLLARY 5.13. *Take $\alpha > 0$ and set $w(\cdot) := |\cdot|^\alpha$ where $|\cdot|$ denotes the Euclidean norm. Consider (\mathbb{R}^n, d_e, μ) where $\mu := w\mathcal{L}^n$. Then, for any $u : \mathbb{R}^n \rightarrow \mathbb{R}$ two times differentiable at 0_n the pointwise AMV and SAMV Laplacians of u in 0_n exist and are given by*

$$\Delta_\mu^{d_e} u(0_n) = \frac{n + \alpha}{2n(n + \alpha + 2)} \Delta u(0_n), \tag{5.34}$$

$$\tilde{\Delta}_\mu^{d_e} u(0_n) = \frac{n^2 + (2 + \alpha)n + \alpha}{2n(n + \alpha + 2)(n + 2)} \Delta u(0_n), \tag{5.35}$$

where $\Delta := \sum_i \partial_{ii}$ is the classical Laplacian.

Proof. The weight w is continuous, hence 0_n is a Lebesgue point with $w(0_n) = 0$. Moreover, the infinitesimal evenness of w is trivially satisfied because w is even. Finally the α -homogeneity of w implies that the measures $\{\nu_r\}_{r>0}$ are constantly equal to the measure

$$\nu = \frac{n + \alpha}{\sigma_{n-1}} |\cdot|^\alpha \mathcal{L}^n \llcorner B_1(0_n),$$

hence proposition 5.8 applies and yields

$$\Delta_\mu^{d_e} u(0_n) = \frac{1}{2} \text{Tr}(M_\nu \nabla^2 u)(0_n) \tag{5.36}$$

where we recall that M_ν is defined in (5.18). Let us compute M_ν . If $n = 1$,

$$\int_{-1}^1 y^2 |y|^\alpha d\nu(y) = \frac{1 + \alpha}{\sigma_0} \int_{-1}^1 y^2 |y|^\alpha d\nu(y) = \frac{1 + \alpha}{2 + \alpha + 1}.$$

Assume now $n > 1$. Consider $i, j \in \{1, \dots, n\}$ such that $i \neq j$. Then, the map $y \mapsto y_i y_j |y|^\alpha$ is odd, so

$$\int_{B_1(0_n)} y_i y_j |y|^\alpha dy = 0.$$

Moreover, the invariance of the Lebesgue measure under exchanging coordinates θ_i and θ_n yields

$$\begin{aligned} \int_{B_1(0_n)} y_i^2 |y|^\alpha dy &= \int_{B_1(0_n)} y_n^2 |y|^\alpha dy = \int_0^1 \int_{\mathbb{S}^{n-1}} r^{\alpha+n+1} \theta_n^2 d\sigma(\theta) dr \\ &= \frac{1}{n + \alpha + 2} \int_{\mathbb{S}^{n-1}} \theta_n^2 d\sigma(\theta) \end{aligned}$$

where σ is the usual surface measure on \mathbb{S}^{n-1} . Since

$$\sum_{1 \leq \ell \leq n} \int_{\mathbb{S}^{n-1}} \theta_\ell^2 d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} |\theta|^2 d\sigma(\theta) = \sigma_{n-1}$$

we get

$$\int_{\mathbb{S}^{n-1}} \theta_n^2 \, d\sigma(\theta) = \frac{\sigma_{n-1}}{n}.$$

In the end, for any $n \geq 1$,

$$M_\nu = \frac{n + \alpha}{n(n + \alpha + 2)} I_n$$

and the result follows.

For the SAMV Laplacian, consider \tilde{w} as defined in (5.26). We want to apply proposition 5.10. The weight \tilde{w} is even by remark 5.12. Thanks to (5.33), in order to prove (5.27), one is left with computing the limit as $r \downarrow 0$ of

$$\frac{r^n}{\mu(B_r(x))}$$

for any $x \neq 0$. This follows from the Lebesgue differentiation theorem:

$$\frac{r^n}{\mu(B_r(x))} = \left(\omega_n \int_{B_r(x)} w(y) \, dy \right)^{-1} \rightarrow \frac{1}{|x|^\alpha \omega_n}.$$

Consequently,

$$\tilde{\nu} = \frac{1}{2} \left(\nu + \frac{\mathcal{L}^n}{\omega_n} \right).$$

Finally, computing the second order moments w.r.t. to $\tilde{\nu}$, simplifying and putting back into (5.28) gives (5.35). □

5.2.3. *Separable weights* After proposition 5.8 it is natural to consider the class of separable weights, defined as follows.

DEFINITION 5.14. *We say that a vanishing weight w of domain Ω is separable in a neighbourhood of 0_n if there exist $r > 0$ such that $B_r(0_n) \subset \Omega$ and $f \in L^1([0, r], \mathcal{L}^1)$, $g \in L^1(\mathbb{S}^{n-1}, \sigma)$, such that $w(x) = f(|x|)g(x/|x|)$ for \mathcal{L}^n -a.e. $x \in \Omega \setminus \{0_n\}$. We say that f (resp. g) is the radial (resp. angular) part of w .*

PROPOSITION 5.15. *Consider $(\Omega, d_e, w\mathcal{L}^n)$ where w is a vanishing weight of domain Ω which is separable in a neighbourhood of 0_n . Let f and g be the radial and angular parts of w , respectively. Set $\mu = w\mathcal{L}^n$ and*

$$\nu := (c_f g) \sigma$$

where σ is the normalized surface measure on the sphere \mathbb{S}^{n-1} and

$$c_f := \int_0^1 f(\rho) \rho^{n+1} \, d\rho \in (0, +\infty).$$

Then, the following hold.

- (1) For any function u two-times differentiable at 0_n , the AMV Laplacian $\Delta_\mu^{de} u(0_n)$ exists and satisfies

$$\Delta_\mu^{de} u(0_n) = \frac{1}{2} \text{Tr}(M_\nu \nabla^2 u)(0_n) \tag{5.37}$$

if and only if

$$\left\langle \nabla u(0_n), \int_{\mathbb{S}^{n-1}} \theta g(\theta) \, d\sigma(\theta) \right\rangle = 0. \tag{5.38}$$

- (2) For any function u two-times differentiable at 0_n such that $\nabla u(0_n) = 0$, the AMV Laplacian $\Delta_\mu^{de} u(0_n)$ exists and satisfies (5.37).
 (3) If $\int_{\mathbb{S}^{n-1}} \theta g(\theta) \, d\sigma(\theta) = 0_n$, then for any function u two-times differentiable at 0_n , the AMV Laplacian $\Delta_\mu^{de} u(0_n)$ exists and satisfies (5.37).

Proof. We only prove the first assertion since the others are direct consequences. Acting as in the proof of proposition 5.8, we get that $\Delta_\mu^{de} u(0_n)$ exists and satisfies (5.37) if and only if

$$\lim_{r \downarrow 0} \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n) \frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) = 0.$$

Thanks to (5.23), we know that for any $i \in \{1, \dots, n\}$,

$$\frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) = \frac{1}{r \int_{B_1(0_n)} w(ry) \, dy} \left(\int_{B_1(0_n)} y_i w(ry) \, dy \right)$$

from which the separation assumption yields

$$\frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) = \frac{\int_0^1 f(r\rho) \rho^{n+1} \, d\rho \int_{\mathbb{S}^{n-1}} \theta_i [g(\theta) - g(-\theta)] \, d\sigma(\theta)}{r \int_0^1 f(r\rho) \rho \, d\rho \int_{\mathbb{S}^{n-1}} g(\theta) \, d\sigma(\theta)}.$$

Now

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \theta_i [g(\theta) - g(-\theta)] \, d\sigma(\theta) &= \int_{\mathbb{S}^{n-1}} \theta_i g(\theta) \, d\sigma(\theta) - \int_{\mathbb{S}^{n-1}} \theta_i g(-\theta) \, d\sigma(\theta) \\ &= 2 \int_{\mathbb{S}^{n-1}} \theta_i g(\theta) \, d\sigma(\theta) \end{aligned}$$

where we have used the change of variable $\theta' = -\theta$ in the second integral to get the last term. Thus,

$$\begin{aligned} \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n) \frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) &= \frac{\int_0^1 f(r\rho) \rho^{n+1} \, d\rho}{r \int_0^1 f(r\rho) \rho \, d\rho} \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n) \frac{2 \int_{\mathbb{S}^{n-1}} \theta_i g(\theta) \, d\sigma(\theta)}{\int_{\mathbb{S}^{n-1}} g(\theta) \, d\sigma(\theta)} \\ &= \frac{\int_0^1 f(r\rho) \rho^{n+1} \, d\rho}{r \int_0^1 f(r\rho) \rho \, d\rho} \left\langle \nabla u(0_n), 2 \int_{\mathbb{S}^{n-1}} \theta g(\theta) \, d\sigma(\theta) \right\rangle. \end{aligned} \tag{5.39}$$

This immediately shows that if (5.38) holds, then $\Delta_\mu^{d_e} u(0_n)$ exists and is given by (5.19).

Assume now that (5.38) does not hold. Then,

$$c := \left| \left\langle \nabla u(0_n), 2 \int_{\mathbb{S}^{n-1}} \theta g(\theta) \, d\sigma(\theta) \right\rangle \right| > 0.$$

Let $m \in (0, 1)$ be the median of the measure $f(r\rho) \, d\rho$ on $[0, 1]$, so that

$$\int_0^m f(r\rho) \, d\rho = \int_m^1 f(r\rho) \, d\rho = \frac{1}{2} \int_0^1 f(r\rho) \, d\rho.$$

Since f is non-negative and $\int_m^1 f(r\rho)\rho^{n+1} \, d\rho \geq m^{n+1} \int_m^1 f(r\rho) \, d\rho$, we get

$$\frac{\int_0^1 f(r\rho)\rho^{n+1} \, d\rho}{r \int_0^1 f(r\rho)\rho \, d\rho} \geq \frac{m^{n+1} \int_m^1 f(r\rho) \, d\rho}{r \int_0^1 f(r\rho)\rho \, d\rho} = \frac{m^{n+1} \int_m^1 f(r\rho) \, d\rho}{2r \int_m^1 f(r\rho)\rho \, d\rho} \geq \frac{m^{n+1}}{2r},$$

where we have used $\int_m^1 f(r\rho)\rho \, d\rho \leq \int_m^1 f(r\rho) \, d\rho$ to get the last inequality. Then, from (5.39) we get

$$\left| \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0_n) \frac{1}{r^2} \int_{B_r(0_n)} x_i \, d\mu(x) \right| \geq \frac{m^{n+1}}{2r} c \rightarrow +\infty$$

as $r \downarrow 0$, so $\Delta_\mu^{d_e} u(0_n)$ does not converge. □

When $n = 2$, the Fourier theory easily yields a concrete equivalent form of the sufficient condition given in (3).

COROLLARY 5.16. *Assume $n = 2$ and consider $(\Omega, d_e, w\mathcal{L}^n)$ and u as in the previous proposition. If there exist $c \geq 1$ and $\varphi(\cdot) \in \text{Span}(\{\cos(m\cdot), \sin(m\cdot)\}_{m \geq 2})$ such that*

$$g(e^{i\cdot}) = c + \varphi(\cdot),$$

then for any function u two times differentiable at 0_2 , the AMV Laplacian $\Delta_\mu^{d_e} u(0_2)$ exists and is given by (5.37).

Proof. Let $h \in L^1(\mathbb{R})$ be 2π -periodic and such that $g(e^{it}) = h(t)$ for any $t \in \mathbb{R}$. Since

$$\int_{\mathbb{S}^1} \theta g(\theta) \, d\sigma(\theta) = \int_0^{2\pi} e^{it} g(e^{it}) \, dt = \int_0^{2\pi} e^{it} h(t) \, dt$$

then

$$\int_{\mathbb{S}^1} \theta g(\theta) \, d\sigma(\theta) = 0_2 \iff (*) : \begin{cases} \int_0^{2\pi} \cos(t)h(t) \, dt = 0, \\ \int_0^{2\pi} \sin(t)h(t) \, dt = 0. \end{cases}$$

Since g is non-negative, so is h , hence $(*)$ is equivalent to

$$h = c + \varphi$$

where $c \geq 1$ and $\varphi \in \text{Span}(\{\cos(m\cdot), \sin(m\cdot)\}_{m \geq 2})$, thanks to the Fourier theory. \square

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