

ON POSITIVE VECTOR SUPERFUNCTIONS OF ELLIPTIC SYSTEMS

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1. Let Ω be a bounded domain in R^n with coordinates $x = (x_1, \dots, x_n)$ such that Ω contains the origin. We consider the elliptic operators

$$(1.1) \quad L_k = \sum_{i,j=1}^n a_{ij}^{(k)}(x) \partial_{x_i} \partial_{x_j}, \quad k = 1, 2, \dots, m,$$

where $a_{ij}^{(k)}$ are real valued and in $C^2(\Omega)$. And we assume that there are positive constants c_0, c_1 with $nc_1 - 2c_0 < (n-1) \cdot c_0$ such that for any $x \in \Omega$ and any $\xi \in R^n$

$$(1.2) \quad \sum_{i,j=1}^n a_{ij}^{(k)}(x) \xi_i \xi_j \geq c_0 |\xi|^2, \quad 0 < a_{ii}^{(k)}(x) \leq c_1.$$

Obviously $c_0 \leq c_1$.

In this article we consider the solutions $\{u_k\}$ of the elliptic differential inequalities in Ω

$$(1.3) \quad L_k u_k \leq \sum_{i=1}^m \sum_{j=1}^n b_{ij}^{(k)}(x) \partial_{x_i} u_j + \sum_{\ell=1}^m c_\ell^{(k)}(x) u_\ell, \quad k = 1, 2, \dots, m.$$

Here all coefficients are real valued and

$$b_{ij}^{(k)} \in C^1(\Omega), \quad c_\ell^{(k)} \in L_{loc}^\infty(\Omega).$$

The assumption on $c_\ell^{(k)}$ will be weakened later (see Remark 1 in the final section).

Let F be a compact subset of Ω such that the n -dimensional measure of F is not zero and

$$(1.4) \quad \int_{\Omega-F} [\text{dis}(x, F)]^{n-2-\alpha'} dx < \infty^1$$

for some α' satisfying

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1) We denote by $\text{dis}(x, F)$ the distance between x and F .

$$(1.5) \quad nc_1 - 2c_0 < \alpha'c_0 < (n-1)c_0.$$

From now on we write $\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $x = (x_i, \hat{x}_i)$. If we put

$$G_{\hat{x}_i} = F^c \cap \{(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \mid y \in R^1\},$$

it can be written by

$$(1.6) \quad G_{\hat{x}_i} = (-\infty, a_0) \cup (b_0, \infty) \cup \left(\bigcup_{\nu=1}^{\infty} (a_\nu, b_\nu) \right)$$

for some a_ν, b_ν ($\nu = 0, 1, 2, \dots$), since $G_{\hat{x}_i}$ is an open set in R^1 . We assume that there are positive constants C and σ with $\sigma < 1$ independent of i and \hat{x}_i such that

$$(1.7) \quad \sum_{\nu=1}^{\infty} (b_\nu - a_\nu)^\sigma \leq C.$$

We define

$$|u|_{s,A} = \sup_{\substack{\ell \leq s \\ x \in A}} |\partial_{x_1}^{\ell_1} \cdots \partial_{x_n}^{\ell_n} u(x)| \quad (\ell = \ell_1 + \cdots + \ell_n)$$

for non-negative integers s and subsets A of Ω . As well-known, this norm is extended also for real positive s .

Our objective is to prove

THEOREM. *Let $\{u_k\}$ be solutions of (1.3) in Ω and be in $C^2(\Omega)$. Let $u_k \geq 0$ in Ω . Assume that F satisfies the above conditions. Then for any p with $1 \leq p < \infty$ there are positive constants $\gamma (< 1)$, C, d and an open subset G of Ω such that the following holds:*

If $|u_k|_{2,\Omega} \leq K$ and $|u_k|_{1,F} \leq \varepsilon$ for ε, K with $0 \leq \varepsilon \leq dK$, we have

$$(1.8) \quad \|u_k\|_{L^p(G)} \leq C\varepsilon^\gamma K^{1-\gamma},$$

where γ, C, d and G depend only on F, p and the coefficients of (1.3).

The proof is given in the final section. In the next section we give an example of compact subsets which satisfies (1.4) and (1.7), but contains neither interior points nor continuous arcs. And for this example we see that the assumption of $|u_k|_{1,F} \leq \varepsilon$ can be replaced by the weaker one of $|u_k|_{0,F} \leq \varepsilon$ in our Theorem (see Proposition 2).

The system (1.3) was considered first by K. Akô [1] for more extensive non-linear elliptic operators. He derived the minimum principle for non-

negative solutions of (1.3) which are named “vector superfunctions” in [1]. When particularly $\varepsilon = 0$ in our Theorem, it means that u_k vanish identically. A. Ancona [2] has discussed the distribution of zero points for non-negative solutions u of the inequality $\Delta u \leq \alpha \cdot u$ in the weak sense, where the assumption on α is weaker than ours, that is, α is locally integrable in Ω .

By V. A. Kondrat’ev and S. D. Èidel’man [6], the Harnack’s inequality holds in the wide sense for positive solutions of (1.3). This implies that

$$\sum_{k=1}^m \int_{\Omega'} u_k dx \leq C \sum_{k=1}^m \int_{\Omega''} u_k dx$$

for any two fixed subdomains Ω' and Ω'' with $\Omega'' \subset \Omega' \subset \Omega$. Hence our Theorem is effective, when F has no interior point. In [6] more general non-stationary systems were treated. For elliptic systems (1.3) F. Mandras [7] also has shown an inequality of Harnack’s type in the ordinary sense, where it needs some hypothesis on the sign of $c_i^{(k)}(x)$.

Recently, the problem of elliptic variational inequalities with obstacles has been studied in great detail for vector-valued functions by several authors (cf. e.g. [4]), which is closely related to inequalities (1.3). In this connection, we give a remark in the final section (see Remark 4).

In category of complex valued functions, a theorem analogous to ours is obtained for elliptic operators with simple characteristics and two variables (see [3]), where the method is restricted to the case of $n = 2$ and the motivation is derived from the ill-posed problems.

We have also

COROLLARY. *Let $\{u_k\}$ and F be such as in our Theorem. Then for any p with $1 \leq p < \infty$ and for any subdomain Ω' with $\Omega' \subset \Omega$, there are positive constants C, d and $\gamma (< 1)$ depending only on F, p, Ω' and the coefficients of (1.3) such that the following is valid:*

If $|u_k|_{2, \Omega} \leq K$ and $|u_k|_{1, F} \leq \varepsilon$ for ε, K with $0 \leq \varepsilon \leq dK$, the inequality

$$\|u_k\|_{L^p(\Omega')} \leq C\varepsilon^\gamma K^{1-\gamma}$$

holds.

In fact, combining (1.8) and the inequality of [6], we have

$$\begin{aligned} \|u_k\|_{L^p(\Omega')} &\leq K^{(p-1)/p} \left(\int_{\Omega'} u_k dx \right)^{1/p} \\ &\leq CK^{(p-1)/p} \sum_{j=1}^m \left(\int_G u_j dx \right)^{1/p} \end{aligned}$$

$$\begin{aligned} &\leq CK^{(p-1)/p}(C_\varepsilon^\gamma K^{1-\gamma})^{1/p} \\ &\leq C\varepsilon^{\gamma/p} K^{(p-\gamma)/p} . \end{aligned}$$

Replacing γ/p newly with γ , we obtain our Corollary.

2. In this section we denote the one-dimensional variable by x or y . Let $0 < \delta < 1$ and let $Q = \{(x, y); |x| < a, |y| < b\}$, where $b < a < 1$. We calculate first the integral

$$\iint_Q [\text{dis}((x, y), Q^c)]^{-\delta} dx dy .$$

If we put $Q' = \{(x, y); 0 < bx/a < y < b\}$, we then have

$$\begin{aligned} &\iint_{Q'} [\text{dis}((x, y), Q^c)]^{-\delta} dx dy \\ &= \iint_{Q'} (b - y)^{-\delta} dx dy \\ &= (1 - \delta)^{-1} (2 - \delta)^{-1} ab^{1-\delta} . \end{aligned}$$

If we put

$$\begin{aligned} Q'' &= \{(x, y); 0 < y < bx/a < b\} , \\ Q''_1 &= Q'' \cap \{0 < x < a - b\} , \\ Q''_2 &= Q'' \cap \{y > x + b - a\} \cap \{x > a - b\} , \end{aligned}$$

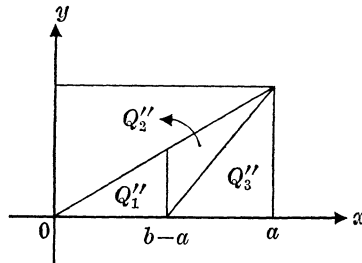
and

$$Q''_3 = Q'' \cap \{y < x + b - a\} ,$$

it follows that

$$\text{dis}((x, y), Q^c) = \begin{cases} b - y & \text{in } Q''_1 \cup Q''_2 \\ a - x & \text{in } Q''_3 \end{cases}$$

(see Figure 1).



(Figure 1)

Hence we get

$$\begin{aligned} \iint_{Q'_1} [\text{dis}((x, y), Q^c)]^{-\delta} dx dy &= (1 - \delta)^{-1}[(a - b)b^{1-\delta} + (2 - \delta)^{-1}(b^{2-\delta} - a^{2-\delta})(b/a)^{1-\delta}] , \\ \iint_{Q'_2} [\text{dis}((x, y), Q^c)]^{-\delta} dx dy &= (1 - \delta)^{-1}(2 - \delta)^{-1}(a^{1-\delta} - b^{1-\delta})a^{\delta-1}b^{2-\delta} \end{aligned}$$

and

$$\iint_{Q'_3} [\text{dis}((x, y), Q^c)]^{-\delta} dx dy = (1 - \delta)^{-1}(2 - \delta)^{-1}b^{2-\delta} .$$

Combining these equalities we obtain

$$(2.1) \quad \iint_Q [\text{dis}((x, y), Q^c)]^{-\delta} dx dy \leq C(ab)^{1-\delta} .$$

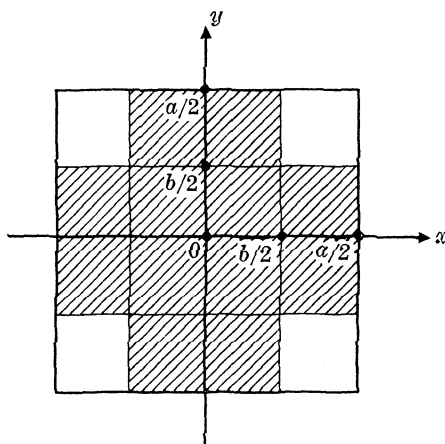
Secondly we put

$$A = \{|x| < a/2, |y| < a/2\} \cap [\{|x| < b/2\} \cup \{|y| < b/2\}]$$

and

$$\begin{aligned} A_1 &= \{|x| < b/2, |y| < b/2\} , \\ A_2 &= \{|x| < b/2, b/2 < y < a/2\} , \\ A_3 &= \{|x| < b/2, -a/2 < y < -b/2\} , \\ A_4 &= \{-a/2 < x < -b/2, |y| < b/2\} , \\ A_5 &= \{b/2 < x < a/2, |y| < b/2\} \end{aligned}$$

(see Figure 2).



(Figure 2)

Then $A = \bigcup_{i=1}^5 A_i$ and $\text{dis}((x, y), A^c) \geq \text{dis}((x, y), A_i^c)$. Thus it follows that

$$\begin{aligned} & \iint_A [\text{dis}((x, y), A^c)]^{-\delta} dx dy \\ & \leq \sum_{i=1}^5 \iint_{A_i} [\text{dis}((x, y), A_i^c)]^{-\delta} dx dy . \end{aligned}$$

By (2.1) this implies that

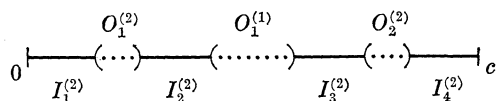
$$(2.2) \quad \begin{aligned} & \iint_A [\text{dis}((x, y), A^c)]^{-\delta} dx dy \\ & \leq C[b^{2(1-\delta)} + 4(b(a - b))^{1-\delta}] . \end{aligned}$$

Now we give an example of compact subsets F with the assumptions in the previous section.

We consider in a closed interval $[0, c]$ ($0 < c < 1$). For the time being we take a sequence in such a way that

$$(2.3) \quad 0 < a_n < 2^{1-n} \left(c - \sum_{j=1}^{\infty} 2^{j-1} a_j \right) , \quad n = 1, 2, \dots .$$

We eliminate from $[0, c]$ an open interval $O_1^{(1)}$ with its length a_1 and with its center $c/2$. The remained closed intervals are denoted respectively by $I_1^{(1)}$ and $I_2^{(1)}$ in turn from the left. Next we eliminate from each $I_j^{(1)}$ an open interval $O_j^{(2)}$ with its length a_2 whose center is identical with that of $I_j^{(1)}$. Then there are remained four closed intervals which are denoted by $I_1^{(2)}, I_2^{(2)}, I_3^{(2)}$ and $I_4^{(2)}$ in turn from the left, respectively (see Figure 3). We repeat inductively this process, that is, $O_j^{(n+1)}$ is eliminated from the center of $I_j^{(n)}$ ($j = 1, 2, \dots, 2^n$). The lastly remained closed set is written by E .



(Figure 3)

Obviously, $|E| = c - \sum_{j=1}^{\infty} 2^{j-1} a_j > 0^{(2)}$. Since

$$|O_j^{(n)}| = a_n , \quad |I_j^{(n)}| = 2^{-n} \left(c - \sum_{j=1}^n 2^{j-1} a_j \right) ,$$

we see by (2.3)

2) The notation $|\cdot|$ means the one-dimensional measure.

$$(2.4) \quad |I_j^{(n)} \cup O_{j'}^{(n)}|/|I_j^{(n)}| \leq M,$$

where M is independent of n, j and j' .

For the above set E we have

PROPOSITION 1. *Suppose that $u \in C^2([0, c])$ and $|u''| \leq K$ in $[0, c]$. If $|u| \leq \varepsilon$ ($\leq K$) on E , it follows that*

$$(2.5) \quad |u'| \leq C_0 \sqrt{\varepsilon K}$$

on E , where C_0 is independent of ε and K .

Proof. First we note that this proposition is correct, if $|u| \leq \varepsilon$ and $u' \neq 0$ in $[0, c]$. In fact, we have for some ξ

$$|u'(\xi)| = |(u(c) - u(0))/(c - 0)| \leq 2\varepsilon/c$$

and in virtue of Cauchy's mean value theorem we get

$$|(u'(x)^2 - u'(\xi)^2)/(u(x) - u(\xi))| = 2|u'(\xi')u''(\xi')/u'(\xi')| \leq 2K$$

for any $x \in (0, c)$. Thus (2.5) is obtained.

Now, by the uniform continuity of u there is a number n such that $|u| \leq 2\varepsilon$ on $\bigcup_{j=1}^{2n} I_j^{(n)}$, since $\bigcup_{j=1}^{2n} I_j^{(n)} \searrow E$ as $n \rightarrow \infty$. It is sufficient to prove (2.5) on each $I_k^{(n)}$ for such n and for any fixed k .

Naturally it may be assumed that $\dot{I}_k^{(n)} \cap \{u'(x) \neq 0\} \neq \phi$, where $\dot{I}_k^{(n)}$ is the interior of $I_k^{(n)}$. This set can be written as a union of open connected components, that is, $\dot{I}_k^{(n)} \cap \{u'(x) \neq 0\} = \bigcup_{j=1}^{\infty} (\gamma_j, \delta_j)$. It suffices to show (2.5) for each (γ_j, δ_j) . If $u'(\gamma_j) = 0$, it follows from the Cauchy's mean value theorem that

$$(u'(x)^2 - u'(\gamma_j)^2)/(u(x) - u(\gamma_j)) = 2u''(\xi)$$

for any $x \in (\gamma_j, \delta_j)$ and for some $\xi \in (\gamma_j, x)$. Thus (2.5) holds in (γ_j, δ_j) . It is similar, if $u'(\delta_j) = 0$.

When $u'(\gamma_j) \neq 0$ and $u'(\delta_j) \neq 0$, it is clear that $\dot{I}_k^{(n)} = (\gamma_j, \delta_j)$ and $u' \neq 0$ on $I_k^{(n)}$. Hence $u' > 0$ or $u' < 0$ there. We may assume that $u' > 0$ on $I_k^{(n)}$ without loss of generality. It is immediately seen that some $O_{k'}^{(n)}$ is adjacent either on the left or right side of $I_k^{(n)}$. For example, let $O_{k'}^{(n)}$ be on the right side of $I_k^{(n)}$. Let $I_k^{(n)} = [\gamma, \delta]$, and let us put

$$\eta_0 = \sup \{\eta | u' > 0 \text{ on } [\gamma, \eta], \gamma < \eta \leq c\}.$$

Then $\delta < \eta_0 \leq c$. It suffices to assume that $u'(\eta_0) = 0$. In fact, unless it is so, we may treat η_0' in place of η_0 such as

$$\eta'_0 = \inf \{ \eta \mid u' > 0 \text{ on } [\eta, \delta], 0 \leq \eta < \delta \} .$$

When $\eta_0 \in \overline{O_{k'}^{(n)}}$, we see that

$$u'(x_0)^2 = \frac{u(\delta) - u(\gamma)}{(\delta - \gamma)^2} (u(\delta) - u(\gamma))$$

for some $x_0 \in (\gamma, \delta)$, and

$$u(\delta) - u(\gamma) = (\delta - \gamma)u'(\gamma) + \frac{1}{2}(\delta - \gamma)^2 u''(x_1)$$

for some $x_1 \in (\gamma, \delta)$. Since $\gamma, \delta \in E$, this implies

$$u'(x_0)^2 \leq 4\varepsilon(|u'(\gamma)|/(\delta - \gamma) + \frac{1}{2}K) .$$

On the other hand it follows from (2.4) that

$$\begin{aligned} |u'(\gamma)|/(\delta - \gamma) &\leq M|u'(\eta_0) - u'(\gamma)|/(\eta_0 - \gamma) \\ &\leq MK . \end{aligned}$$

Hence we obtain $u'(x_0)^2 \leq 2(2M + 1)\varepsilon K$. Applying again Cauchy's mean value theorem, we have

$$|(u'(x)^2 - u'(x_0)^2)/(u(x) - u(x_0))| \leq 2K ,$$

for any $x \in [\gamma, \delta]$, that is, $u'(x)^2 \leq u'(x_0)^2 + 2K(|u(x)| + |u(x_0)|)$. Therefore we conclude that on $I_k^{(n)}$

$$(2.6) \quad u'(x)^2 \leq 2(2M + 5)\varepsilon K .$$

This means that (2.5) holds in $I_k^{(n)}$.

Next let $\eta_0 \in \overline{O_{k'}^{(n)}}$. Then we write by μ the right terminus of $O_{k'}^{(n)}$. Naturally $\mu \in E$ and $u' > 0$ in $[\gamma, \mu]$. Hence we see

$$-\varepsilon \leq u(\gamma) \leq u(x) \leq u(\mu) \leq \varepsilon$$

for $x \in [\gamma, \mu]$, that is, $|u(x)| \leq \varepsilon$ in $I_k^{(n)} \cup O_{k'}^{(n)}$. By the way of making E there are $I_{k''}^{(n-1)}$ such that $I_{k''}^{(n-1)} \supset I_k^{(n)}$ and $I_{k''}^{(n-1)} = I_k^{(n)} \cup O_{k'}^{(n)} \cup I_{k+1}^{(n)}$. Since $|u| \leq 2\varepsilon$ on $I_{k+1}^{(n)}$, the same inequality holds also on $I_{k''}^{(n-1)}$.

If $\eta_0 \in I_{k+1}^{(n)}$, the Cauchy's mean value theorem implies immediately that $u'(x)^2 \leq 8\varepsilon K$ on $I_k^{(n)}$, that is, (2.5) holds in $I_k^{(n)}$.

If $\eta_0 \in I_{k''}^{(n)}$, $u' > 0$ on $I_{k''}^{(n-1)}$. Hence replacing $I_k^{(n)}$ by $I_{k''}^{(n-1)}$, we can repeat the above argument. Doing inductively over and over again, we finally arrive at the case that $u' > 0$ in $[0, c]$. This is nothing but the beginning of the proof. Thus we complete the proof.

We set $F = E \times E (\subset \Omega)$ for the above set E . From Proposition 1 we have immediately

PROPOSITION 2. *Let $u \in C^2(\Omega)$, and let $|u|_{2,\Omega} \leq K$. If $|u|_{0,F} \leq \varepsilon (\leq K)$ on F , then $|u|_{1,F} \leq C_0\sqrt{\varepsilon K}$.*

Let α' be the constant in (1.5) for $n = 2$ and σ be any fixed positive number less than 1. In addition to (2.3) we impose on E the following conditions:

$$(2.7) \quad \sum_{j=1}^{\infty} 2^{j-1} a_j^\alpha < \infty$$

and

$$(2.8) \quad \sum_{j=1}^{\infty} 4^j a_j^{2(1-\alpha')}, \quad \sum_{j=1}^{\infty} 4^j (2^{1-j} a_j)^{1-\alpha'} < \infty .$$

Then we see that F satisfies (1.4) and (1.7).

In fact, (1.7) is trivial from (2.7). We write $J = [0, c] \times [0, c]$. By virtue of (2.2) we easily see

$$\begin{aligned} & \iint_{J-F} [\text{dis}((x, y), F)]^{-\alpha'} dx dy \\ & \leq C \left[\sum_{j=1}^{\infty} 4^{j-1} a_j^{2(1-\alpha')} + \sum_{j=1}^{\infty} 4^j \left(2^{1-j} \left(c - \sum_{k=1}^j 2^{k-1} a_k \right) a_j \right)^{1-\alpha'} \right] . \end{aligned}$$

The sums on the right-hand side is finite from (2.8), which implies immediately (1.4). It is clear that there is actually a sequence which satisfies (2.3), (2.7) and (2.8) simultaneously.

If F is particularly the above set in our Theorem, the assumption $|u_k|_{1,F} \leq \varepsilon$ is weakened by $|u_k|_{0,F} \leq \varepsilon$. And (1.8) becomes

$$\begin{aligned} \|u_k\|_{L^p(G)} & \leq C(C_0\sqrt{\varepsilon K})^\gamma K^{1-\gamma} \\ & \leq C'\varepsilon^{\gamma/2} K^{1-\gamma/2} , \end{aligned}$$

if $0 \leq \varepsilon \leq d^2 K / C_0^2$.

3. From now on let α be a fixed number such that

$$(3.1) \quad nc_1 - 2c_0 < \alpha c_0 , \quad \alpha < \alpha' .$$

We rewrite (1.1) simply by

$$L = \sum_{i,j} a_{ij} \partial_{x_i} \partial_{x_j}$$

where a_{ij} satisfy the assumptions in (1.2).

Let r be the Euclidean metric of x . By (1.2) we have

$$\begin{aligned} Lr^{-\alpha} &= \alpha(2 + \alpha)r^{-4-\alpha} \sum_{i,j} a_{ij}x_i x_j - \alpha r^{-2-\alpha} \sum_i a_{ii} \\ &\geq \alpha(\alpha c_0 - (nc_1 - 2c_0))r^{-2-\alpha}. \end{aligned}$$

Hence it follows from (3.1) that

$$(3.2) \quad Lr^{-\alpha} \geq c_2 r^{-2-\alpha}$$

in Ω for some positive constant c_2 .

We construct the function

$$\psi_F(x) = \int_F |x - y|^{-\alpha} dy$$

for the set F in the first section.

First we see that ψ_F is in $C^1(R^n)$. In fact, let $f_\delta(r)$ ($\delta > 0$) be a C^1 -function such that

$$|\partial_r f_\delta(r)| \leq 1 \quad \text{in } r \geq 0$$

and

$$\begin{aligned} f_\delta(r) &= r && \text{for } r \geq \delta, \\ \delta/2 &\leq f_\delta(r) \leq \delta && \text{for } r \leq \delta. \end{aligned}$$

If we set

$$\psi_F^{(\delta)}(x) = \int_F (f_\delta(|x - y|))^{-\alpha} dy,$$

we obtain

$$\begin{aligned} &\left| \partial_{x_i} \psi_F^{(\delta)}(x) - \int_F \partial_{x_i} (|x - y|^{-\alpha}) dy \right| \\ &\leq \alpha \left(\int_{F \cap D_\delta} |\partial_r f_\delta| f^{-1-\alpha} dy + \int_{F \cap D_\delta} |x - y|^{-1-\alpha} dy \right), \end{aligned}$$

where $D_\delta = \{y; |x - y| \leq \delta\}$. By (1.5) and (3.1) the right-hand side tends to zero uniformly as $\delta \rightarrow 0$. Hence $\psi_F \in C^1(R^n)$.

Next we have

$$(3.3) \quad |\partial^2 \psi_F(x)| \leq C [\text{dis}(x, F)]^{n-2-\alpha'}.$$

Indeed, since $\text{dis}(x, F) \leq |x - y|$ for $y \in F$, it follows that in F^c

$$\begin{aligned} |\partial^2 \psi_F(x)| &\leq C \int_F |x - y|^{-2-\alpha} dy \\ &\leq C [\text{dis}(x, F)]^{n-2-\alpha'} \int_F |x - y|^{\alpha'-\alpha-n} dy. \end{aligned}$$

The last integral is uniformly bounded by virtue of $\alpha' > \alpha$. Hence (3.3) is valid.

LEMMA 1³⁾. For any given $\rho > 0$, there are positive numbers c_3, c_4 ($c_3 < c_4$), a closed subset F' of F , and subdomains, $\Omega_1, \Omega_2, \Omega'$ of Ω with $F' \subset \Omega_1 \subset \Omega_2 \subset \Omega' \subset \Omega$ such that

- (i) F' has the same assumptions as F .
- (ii) $\psi_{F'}(x), |\partial\psi_{F'}(x)| \leq \rho L\psi_{F'}(x)$ in $\Omega' - F'$.
- (iii) $\psi_{F'}(x) \geq c_4$ in Ω_1 and $\psi_{F'}(x) \leq c_3$ in $\Omega' - \Omega_2$.

Proof. We may assume that the origin is the Lebesgue density point of F without loss of generality.

By (3.2) there are a positive number δ and a compact subdomain Ω' of Ω containing the origin such that if $|y| \leq \delta$ and $x \in \Omega'$, then

$$(3.4) \quad |x - y|^{-\alpha}, \quad |\partial_x(|x - y|^{-\alpha})| \leq \rho L_x(|x - y|^{-\alpha}).$$

Secondly we choose subdomains, Ω_1, Ω_2 of Ω and a_1, a_2 so that $0 \in \Omega_1 \subset \Omega_2 \subset \Omega'$, $0 < a_1 < a_2$ and

$$|x - y| \begin{cases} \leq a_1 & (x \in \Omega_1, |y| \leq \delta), \\ \geq a_2 & (x \in \Omega_2, |y| \leq \delta), \end{cases}$$

where δ is retaken, if necessary.

Since the origin is the density point of F , there is a closed cube J such that $J \subset \{x | r < \delta\} \cap \Omega_1$ and the n -dimensional measure of $F \cap J$ is not zero. If we put $F' = F \cap J$, naturally F' is closed and satisfies (1.7). In addition, we see

$$\begin{aligned} & \int_{\Omega - F'} [\text{dis}(x, F')]^{n-2-\alpha'} dx \\ &= \int_{\Omega - F} [\text{dis}(x, F')]^{n-2-\alpha'} dx + \int_{F - F'} [\text{dis}(x, F')]^{n-2-\alpha'} dx. \end{aligned}$$

The first integral of the right-hand side is estimated from the above by

$$\int_{\Omega - F} [\text{dis}(x, F)]^{n-2-\alpha'} dx$$

and the second integral is estimated by

$$\int_{F - J} [\text{dis}(x, J)]^{n-2-\alpha'} dx.$$

Hence F' also satisfies (1.4).

3) This appears also in [3] for $n = 2$. We repeat its proof in order to make sure.

The statement (i) has just been shown. In view of (3.4), (ii) is valid. Putting $c_3 = a_2^{-\alpha} \text{mes } F'$ and $c_4 = a_1^{-\alpha} \text{mes } F'$, we have (iii). The proof is complete.

4. Let σ be as in (1.7) and c_5 be a fixed positive number such that $\psi_{F'} < c_5$ in Ω' . Then we have

LEMMA 2. Let $v \in C^2(\Omega')$ with compact support in Ω' and $v \geq 0$ in Ω' . If $|v|_{2, \Omega'} \leq K$ and $|v|_{1, F'} \leq \varepsilon (\leq K)$, it follows that for any $\lambda \geq \lambda_0 (>0)$

$$\int_{F'c} vL^*(\exp(\lambda\psi_{F'}))dx \leq C \left(\varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_5} + \int_{F'c} Lv \cdot \exp(\lambda\psi_{F'}) dx \right)^4,$$

where C, λ_0 are independent of ε and K .

Proof. For brevity we denote by ψ and φ the functions $\psi_{F'}$ and $\exp(\lambda\psi_{F'})$ respectively.

We replace F by F' in the definition of $G_{\hat{x}_i}$. Let the new $G_{\hat{x}_i}$ be rewritten in the same form as (1.6). We note that the points (a_ν, \hat{x}_i) and (b_ν, \hat{x}_i) are all in F' . For $g \in C^2(\Omega')$ we see that

$$(4.1) \quad \int_{a_\nu}^{b_\nu} vg \partial_{x_i} \partial_{x_j} \varphi dx_i = [(vg \partial_{x_j} \varphi)(y, \hat{x}_i)]_{y=a_\nu}^{y=b_\nu} - \int_{a_\nu}^{b_\nu} \partial_{x_i}(vg) \cdot \partial_{x_j} \varphi dx_i$$

and

$$\begin{aligned} [(vg \partial_{x_j} \varphi)(y, \hat{x}_i)]_{y=a_\nu}^{y=b_\nu} &= (g \partial_{x_j} \varphi)(b_\nu, \hat{x}_i)(v(b_\nu, \hat{x}_i) - v(a_\nu, \hat{x}_i)) \\ &\quad + v(a_\nu, \hat{x}_i)(\partial_{x_j} \varphi)(b_\nu, \hat{x}_i)(g(b_\nu, \hat{x}_i) - g(a_\nu, \hat{x}_i)) \\ &\quad + (gv)(a_\nu, \hat{x}_i)((\partial_{x_j} \varphi)(b_\nu, \hat{x}_i) - (\partial_{x_j} \varphi)(a_\nu, \hat{x}_i)). \end{aligned}$$

We use the trivial inequality

$$|X - Y| \leq C(X^{1-\sigma} + Y^{1-\sigma})|X - Y|^\sigma \quad (X, Y \geq 0).$$

It follows from the mean value theorem that

$$\begin{aligned} |v(b_\nu, \hat{x}_i) - v(a_\nu, \hat{x}_i)| &\leq C(v(a_\nu, \hat{x}_i)^{1-\sigma} + v(b_\nu, \hat{x}_i)^{1-\sigma})|v(b_\nu, \hat{x}_i) - v(a_\nu, \hat{x}_i)|^\sigma \\ &\leq C\varepsilon^{1-\sigma} K^\sigma (b_\nu - a_\nu)^\sigma. \end{aligned}$$

Hence, noting that $|\partial_{x_j} \varphi| \leq C\lambda e^{\lambda c_5}$, we obtain

4) L^* is the formal adjoint operator of L .

$$\begin{aligned} & |[(vg\partial_{x_j}\varphi)(y, \hat{x}_i)]_{y=a_\nu}^{y=b_\nu}| \\ & \leq C \left(\lambda e^{\lambda c_5} \varepsilon^{1-\sigma} K^\sigma (b_\nu - a_\nu)^\sigma + \varepsilon \lambda e^{\lambda c_5} (b_\nu - a_\nu) \right. \\ & \quad \left. + \varepsilon \int_{a_\nu}^{b_\nu} |(\partial_{x_i} \partial_{x_j} \varphi)(x_i, \hat{x}_i)| dx_i \right). \end{aligned}$$

Since $|\partial^2 \varphi| \leq \lambda(|\partial^2 \psi| + \lambda|F\psi|^p)\varphi$, this implies that

$$\begin{aligned} & |[(vg\partial_{x_j}\varphi)(y, \hat{x}_i)]_{y=a_\nu}^{y=b_\nu}| \\ & \leq C e^{2\lambda c_5} \left(\varepsilon^{1-\sigma} K^\sigma (b_\nu - a_\nu)^\sigma + \varepsilon \int_{a_\nu}^{b_\nu} |(\partial^2 \psi)(x_i, \hat{x}_i)| dx_i \right). \end{aligned}$$

We integrate the both sides of (4.1) with x_j and repeat the same calculus as there. Then by virtue of (1.4), (1.7) and (3.3) it follows immediately that

$$\left| \int_{F'\varepsilon} v g \partial_{x_i} \partial_{x_j} \varphi dx - \int_{F'\varepsilon} \varphi \partial_{x_j} \partial_{x_i} (gv) dx \right| \leq C \varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_5}.$$

We verify more easily

$$(4.2) \quad \left| \int_{F'\varepsilon} v g \partial_{x_i} \varphi dx - \int_{F'\varepsilon} \varphi \partial_{x_i} (vg) dx \right| \leq C \varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_5}.$$

Therefore we obtain

$$\left| \int_{F'\varepsilon} v L^* \varphi dx - \int_{F'\varepsilon} \varphi L v dx \right| \leq C \varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_5}.$$

This completes the proof.

5. We prove our Theorem.

Let $\theta(x)$ be a function in $C^2(\Omega')$ with compact support in Ω' such that $\theta \geq 0$ in Ω' and $\theta = 1$ in Ω_2 . We set $v_k = \theta u_k$. Then (1.3) is rewritten by

$$(5.1) \quad L_k v_k \leq \sum_{i,\ell} b_{i\ell}^{(k)} \partial_{x_i} v_\ell + \sum_{\ell} c_\ell^{(k)} v_\ell + R_k,$$

where

$$R_k = u_k L_k \theta + 2 \sum_{i,j} a_{ij}^{(k)} \partial_{x_i} \theta \cdot \partial_{x_j} u_k - \sum_{i,\ell} b_{i\ell}^{(k)} \partial_{x_i} \theta \cdot u_\ell.$$

By virtue of Lemma 2 it follows that for $\lambda \geq \lambda_0$

$$\sum_{k=1}^m \int_{F'\varepsilon} v_k L_k^* \varphi dx \leq C \left(\varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_5} + \sum_{k=1}^m \int_{F'\varepsilon} \varphi L_k v_k dx \right).$$

Naturally L_k^* is written:

$$L_k^* = L_k + \sum_i b_i^{(k)}(x) \partial_{x_i} + c^{(k)}(x).$$

Thus we see

$$L_k^* \varphi = [\lambda^2 \sum_{i,j} a_{ij}^{(k)} \partial_{x_i} \psi \cdot \partial_{x_j} \psi + \lambda (\sum_{i,j} a_{ij}^{(k)} \partial_{x_i} \partial_{x_j} \psi + \sum_i b_i^{(k)} \partial_{x_i} \psi) + c^{(k)}] \varphi.$$

By (3.2) and ellipticity of L_k there are a positive constant c_8 and λ_1 ($\geq \lambda_0$) such that for $\lambda \geq \lambda_1$

$$L_k^* \varphi \geq c_8 \lambda \varphi \left(\int_{F'} |x - y|^{-2-\alpha} dy + \lambda |\nabla \psi|^2 \right)$$

in $\Omega' - F'$.

On the other hand it follows from (4.2) that

$$\left| \int_{F'^c} \varphi b_{i\ell}^{(k)} \partial_{x_i} v_\ell dx \right| \leq C \left(\varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_8} + \int_{F'^c} (1 + \lambda |\partial \psi|) v_\ell \varphi dx \right).$$

Combining the above inequalities, we obtain for $\lambda \geq \lambda_1$,

$$\int_{F'^c} \varphi (\sum_k v_k) dx \leq C \left(\varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_8} + \int_{F'^c} \varphi (\sum_k |R_k|) dx \right),$$

where it is seen that the integral on the left-hand side can be replaced by

$$\int \varphi (\sum_k v_k) dx \quad \text{since} \quad \int_{F'} \varphi (\sum_k v_k) dx \leq C \varepsilon e^{\lambda c_8}.$$

We note that $R_k = 0$ in Ω_2 . Reducing the integral domain of the left-hand side, we have by Lemma 1

$$(5.2) \quad e^{\lambda c_4} \int_{\Omega_1} (\sum_k u_k) dx \leq C (\varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_8} + K e^{\lambda c_8}) \quad (\lambda \geq \lambda_1).$$

That is,

$$\int_{\Omega_1} (\sum_k u_k) dx \leq C (\varepsilon^{1-\sigma} K^\sigma e^{\lambda(2c_8 - c_4)} + K e^{\lambda(c_8 - c_4)}).$$

Now we use the idea of F. John [5] (see p. 559) with respect to the ill-posed Cauchy problem. Setting $h = (1 - \sigma)(2c_8 - c_3)^{-1}$ and $\lambda = \log(K/\varepsilon)^h$, we see

$$\int_{\Omega_1} (\sum_k u_k) dx \leq C \varepsilon^\beta K^{1-\beta},$$

where $\beta = h(c_4 - c_3)$. Hence we obtain

$$\begin{aligned} \left(\int_{\Omega_1} u_k^p dx\right)^{1/p} &\leq \left(K^{p-1} \int_{\Omega_1} u_k dx\right)^{1/p} \\ &\leq C\varepsilon^{\beta/p} K^{1-(\beta/p)}. \end{aligned}$$

If we choose $G = \Omega_1$, $\gamma = \beta/p$ and $d = \exp(-\lambda_1/h)$, the proof of our Theorem is complete.

Remark 1. From the above proof we immediately see that the assumption on $c_\ell^{(k)}$ is weakened as follows:

$$c_\ell^{(k)}(x) = O\left(\int_{F'} |x - y|^{-2-\alpha} dy\right) \quad (\text{dis}(x, F) \rightarrow 0).$$

Remark 2. Let δ be any fixed positive number less than 1. In our Theorem we easily verify that the assumption $|u|_{2,\Omega} \leq K$ is weakened to $|u|_{1+\delta,\Omega} \leq K$, if the real number δ in (1.7) is restricted to $0 < \sigma < \delta$.

Remark 3. Instead of (1.3) we consider the solutions $\{u_k\}$ of the inhomogeneous inequalities

$$L_k u_k \leq \sum_{\ell=1}^m \sum_{i=1}^n b_{i\ell}^{(k)}(x) \partial_{x_i} u_\ell + \sum_{\ell=1}^m c_\ell^{(k)}(x) u_\ell + f_k, \quad k = 1, 2, \dots, m,$$

where $f_k \in C^0(\Omega)$. Then (5.2) is replaced with

$$e^{\lambda c_4} \int_{\Omega_1} \left(\sum_k u_k\right) dx \leq C \left(\varepsilon^{1-\sigma} K^\sigma e^{2\lambda c_5} + K e^{\lambda c_3} + e^{\lambda c_5} \int_{\Omega'} \left(\sum_k |f_k|\right) dx\right),$$

which implies that for $\lambda \geq \lambda_1$ we have

$$\int_{\Omega_1} \left(\sum_k u_k\right) dx \leq C \left(\varepsilon^\beta K^{1-\beta} + (K/\varepsilon)^{\beta(c_5-c_4)} \int_{\Omega'} \left(\sum_k |f_k|\right) dx\right).$$

Remark 4. Let $\{u_k\}$ be in $C^{1+\delta}(\Omega)$ ($0 < \delta < 1$) and solutions of (1.3) in the weak sense, that is,

$$(5.3) \quad \int u_k L_k^* \phi dx \leq \sum_{i,\ell} b_{i\ell}^{(k)} \partial_{x_i} u_\ell \cdot \phi dx + \sum_\ell \int c_\ell^{(k)} u_\ell \phi dx$$

for any $\phi \in C_0^2(\Omega)$ with $\phi \geq 0$ in Ω . We assume (1.7) for σ with $0 < \sigma < \delta$ and that L_k has constant coefficients independent of k , which means naturally that $L^* = L (=L_k)$. Moreover let $c_\ell^{(k)}$ be in $C(\Omega)$.

We construct the well-known regular approximation of u_k such that

$$u_k^{(\rho)}(x) = \int u_k(y) j_\rho(x - y) dy$$

for sufficiently small $\rho > 0$. By (5.3) we have

$$\begin{aligned}
Lu_k^{(\rho)}(x) &= \int u_k(y)L^*j_\rho(x-y)dy \\
&\leq \sum_{i,\ell} \int b_{i\ell}^{(k)}(y)\partial_{y_i}u_\ell(y)\cdot j_\rho(x-y)dy + \sum_\ell \int c_\ell^{(k)}(y)u_\ell(y)j_\rho(x-y)dy \\
&= \sum_{i,\ell} b_{i\ell}^{(k)}\partial_{x_i}u_k^{(\rho)} + \sum_\ell c_\ell^{(k)}u_\ell^{(\rho)} + f_k^{(\rho)}.
\end{aligned}$$

Here $f_k^{(\rho)}$ is of the form:

$$\begin{aligned}
f_k^{(\rho)}(x) &= \sum_{i,\ell} \int (b_{i\ell}^{(k)}(y) - b_{i\ell}^{(k)}(x))\partial_{y_i}u_\ell(y)\cdot j_\rho(x-y)dy \\
&\quad + \sum_\ell \int (c_\ell^{(k)}(y) - c_\ell^{(k)}(x))u_\ell(y)j_\rho(x-y)dy.
\end{aligned}$$

Let Ω'' be a subdomain such that $\Omega' \subset \Omega'' \subset \Omega$. Since the support of $j_\rho(x)$ is in $\{|x| < \rho\}$, it follows that $|f_k^{(\rho)}|_{0,\Omega'} \leq C(\rho)(\sum_\ell |u_\ell|_{1,\Omega''})$, where $C(\rho) \rightarrow 0$ as $\rho \rightarrow 0$.

Let ε and K be positive numbers in our Theorem. Suppose that $|u_k|_{1,\Omega'} \leq \varepsilon$ and $|u_k|_{1+\delta,\Omega} \leq K$. Noting that $|u_k^{(\rho)} - u_k|_{1,\Omega''} \rightarrow 0$ ($\rho \rightarrow 0$), we have in view of Remarks 2 and 3

$$\int_{\Omega_1} (\sum_k u_k)dx \leq C\varepsilon^{\beta'}K^{1-\beta'} \quad (0 < \beta' < 1).$$

Hence the conclusion in our Theorem holds also for this case.

It is known that vector-valued solutions of variational inequalities satisfy (5.3) for some closed convex sets of vector functions. Recently, S. Hildebrandt and K. O. Widman [4] have studied the Hölder continuity for first derivatives of these solutions, where quasi-linear elliptic systems have been treated.

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