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ON TOTALLY UMBILICAL QR-SUBMANIFOLDS OF QUATERNION KAEHLERIAN MANIFOLDS

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We introduce the notion of generalised 3-Sasakian structure on a manifold and show that a totally umbilical, but not totally geodesic, proper QR-submanifold of a quaternion Kaehlerian manifold is an extrinsic sphere and inherits such a structure.

1. INTRODUCTION

As it is well known (see [2]), the tangent bundle TM of a CR-submanifold M of a Kaehlerian manifold $\widetilde{M}$ has the decomposition $TM = D \oplus D^{\perp}$, where D and $D^{\perp}$ are invariant and anti-invariant distributions on M with respect to the complex structure $\widetilde{J}$ of $\widetilde{M}$. Equivalently, M is a CR-submanifold of $\widetilde{M}$ if and only if its normal bundle $TM^{\perp}$ has the decomposition $TM^{\perp} = \nu \oplus \nu^{\perp}$, where ν and $\nu^{\perp}$ are invariant and anti-invariant vector subbundles of $TM^{\perp}$ with respect to $\widetilde{J}$.

The above equivalence fails in the case of submanifolds of a quaternion Kaehlerian manifold. Thus we have two concepts: the quaternion CR-submanifold introduced by Barros, Chen and Urbano [1] where M has $TM = D \oplus D^{\perp}$, and the QR-submanifold introduced by Bejancu [3] where M has $TM^{\perp} = \nu \oplus \nu^{\perp}$, both decompositions being considered with respect to the quaternion structure of the ambient manifold. Taking into account the research done till now, we may conclude that quaternion CR-submanifolds and QR-submanifolds have very little in common, and that there is much room for new results on their geometry.

According to a result of Bejancu (see [3, Theorem 3.3]), any totally umbilical proper QR-submanifold M of a quaternion Kaehlerian manifold $\widetilde{M}$ with dim $\nu_x^{\perp} > 1$ for any $x \in M$ is totally geodesic. The main purpose of the present paper is to study the remaining cases. In Section 2 we recall the concepts of QR-submanifold and totally umbilical submanifold and introduce the new concept of generalised 3-Sasakian structure on a manifold. The main results are stated in Section 3. First, in the case $\dim \nu_x^{\perp} = 0$ for any $x \in M$, we prove that M is a totally geodesic quaternionic submanifold of $\widetilde{M}$. Then we prove Theorems 3.1 and 3.2 which state that when $\dim \nu_x^{\perp} = 1$ for any $x \in M$,

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and M is not totally geodesic, then it is an extrinsic sphere that inherits a generalised 3-Sasakian structure.

2. Preliminaries

2.1 QR-Submanifolds

Let $\widetilde{M}$ be a $4m$-dimensional quaternion Kaehlerian manifold with metric tensor $\widetilde{g}$. Then there exists a 3-dimensional vector bundle V of tensors of type (1, 1) on $\widetilde{M}$ with local basis of almost Hermitian structures $\{\widetilde{J}_a\}$, $a \in \{1,2,3\}$, such that

(i) $\widetilde{J}_1 \circ \widetilde{J}_2 = -\widetilde{J}_2 \circ \widetilde{J}_1 = \widetilde{J}_3$, and

(ii) If U is a coordinate neighbourhood on $\widetilde{M}$ and S and X are sections of vector bundles $V_{|U}$ and $T\widetilde{M}_{|U}$ respectively, then $\widetilde{\nabla}_X S$ is also a section of $V_{|U}$, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{g}$.

It follows (see Ishihara [7]) that (ii) is equivalent to the condition

(ii)′ There exist local 1-forms α_{ab} on U such that $\alpha_{ab} + \alpha_{ba} = 0$ and

$$\widetilde{\nabla}_X \widetilde{J}_a = \alpha_{ab}(X)\widetilde{J}_b + \alpha_{ac}(X)\widetilde{J}_c, \tag{2.1}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$.

If U and U' are two coordinate neighbourhoods such that $U \cap U' \neq \emptyset$, then on $U \cap U'$ we have

$$\widetilde{J}'_a = \sum_{b=1}^{3} A_{ab}\widetilde{J}_b, \tag{2.2}$$

where $[A_{ab}]$ is an element of $SO(3)$ and $\{\widetilde{J}'_a\}$ is a local basis for V on U'.

Next, we consider a real p-dimensional submanifold M of $\widetilde{M}$ and denote by $TM^{\perp}$ its normal bundle. Then we say that M is a *QR-submanifold (quaternionic-real submanifold)* (see Bejancu [3]) if there exists a vector subbundle ν of $TM^{\perp}$ such that

$$\widetilde{J}_a(\nu) = \nu \quad \text{and} \quad \widetilde{J}_a(\nu^{\perp}) \subset TM, \quad \forall a \in \{1,2,3\},$$

where $\nu^{\perp}$ is the complementary orthogonal vector bundle to ν in $TM^{\perp}$. If in particular, $\nu = TM^{\perp}$ or $\nu = \{0\}$ we say that M becomes a quaternionic submanifold (see Chen [6]) or an anti-quaternionic submanifold (see Pak [10]). In the case M is a real hypersurface of $\widetilde{M}$ we have $\widetilde{g}(\widetilde{J}_a N, N) = 0$ for any $a \in \{1,2,3\}$ and normal vector field N. Hence $\widetilde{J}_a(TM^{\perp}) \subset TM$, that is, M is an example of QR-submanifold with $\nu = \{0\}$.

Suppose M is a QR-submanifold of $\widetilde{M}$ which is not a quaternionic submanifold. Then for each $x \in M$, we denote $\widetilde{J}_a(\nu_x^{\perp})$ by D_{ax}, $a \in \{1,2,3\}$. It is easy to see that D_{1x}, D_{2x}, and D_{3x} are mutually orthogonal subspaces of T_xM and have the same dimension s as

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the dimension of $\nu_x^\perp$. We note that separately, the subspaces D_{ax}, $a \in \{1,2,3\}$, do not define, in general, distributions on M. However, due to (2.2) the mapping

$$D^\perp : x \longrightarrow D_x^\perp = D_{1x} \oplus D_{2x} \oplus D_{3x},$$

is a $3s$-dimensional distribution on M. Also, we have

$$\widetilde{J}_a(D_{ax}) = \nu_x^\perp \quad \text{and} \quad \widetilde{J}_a(D_{bx}) = D_{cx}, \tag{2.3}$$

for each $x \in M$, $a \in \{1,2,3\}$, where (a,b,c) is a cyclic permutation of $(1,2,3)$. Finally, we denote by D the complementary orthogonal distribution to $D^\perp$ in TM. It follows that D is invariant with respect to the action of $\{\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3\}$, that is, $\widetilde{J}_a(D) = D$, for any $a \in \{1,2,3\}$. Thus we are entitled to call it the *quaternionic distribution* on M. Hence the tangent bundle of a QR-submanifold has the decomposition

$$TM = D \oplus D^\perp, \tag{2.4}$$

where D and $D^\perp$ are the above distributions. If $D \neq \{0\}$ and $D^\perp \neq \{0\}$, we say that M is a *proper QR-submanifold* of $\widetilde{M}$.

We remark that the tangent bundle of a quaternion CR-submanifold has also a decomposition as in (2.4). But in that case $D^\perp$ is anti-invariant with respect to $\widetilde{J}_a$, that is $\widetilde{J}_a(D^\perp) \subset TM^\perp$ for any $a \in \{1,2,3\}$. Due to (2.3) we see that $D^\perp$ from the decomposition of the tangent bundle of a QR-submanifold is never an anti-invariant distribution. Actually, this is the main difference between the above two classes of submanifolds.

2.2 Totally Umbilical Submanifolds

Let M be a p-dimensional submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle E over M.

Denote by B the second fundamental form of M and by H the mean curvature vector of M, that is,

$$H = \frac{1}{p}\sum_{i=1}^{p} B(E_i, E_i),$$

where $\{E_i\}$ is an orthonormal basis of $\Gamma(TM)$. Then M is said to be totally umbilical (see Chen [5, p.50]) if the second fundamental form of M is expressed as

$$B(X,Y) = g(X,Y)H, \tag{2.5}$$

where g is the induced Riemannian metric on M. In this case, the Gauss and Weingarten formulas become

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \ \forall X, Y \in \Gamma(TM), \tag{2.6}$$

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and

$$\widetilde{\nabla}_X N = -\widetilde{g}(H, N)X + \nabla^{\perp}_X N, \quad \forall X \in \Gamma(TM), \; N \in \Gamma(TM^{\perp}), \tag{2.7}$$

respectively, where ∇ and $\widetilde{\nabla}$ are the Levi-Civita connections on M and $\widetilde{M}$ respectively, and $\nabla^{\perp}$ is the normal connection of M. Also, we note that the Codazzi equation becomes

$$\widetilde{g}\left(\widetilde{R}(X, Y)Z, N\right) = g(Y, Z)\widetilde{g}(\nabla^{\perp}_X H, N) - g(X, Z)\widetilde{g}(\nabla^{\perp}_Y H, N), \tag{2.8}$$

for any $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, where $\widetilde{R}$ is the curvature tensor field of $\widetilde{\nabla}$.

It is well known that any sphere of a Euclidean space is totally umblical and has positive constant curvature. Finally, we recall that M is an *extrinsic sphere* of $\widetilde{M}$ if it is totally umbilical and has parallel mean curvature vector $H \neq 0$, that is,

$$\nabla^{\perp}_X H = 0, \quad \forall X \in \Gamma(TM).$$

2.3 Generalised 3-Sasakian Manifolds

In the present subsection we shall define a new structure on manifolds of dimension $4k+3$, which is a generalisation of what is known as a 3-Sasakian structure on a manifold (see Kuo [8] and Udriste [11]). Consider a $(4k+3)$-dimensional Riemannian manifold (P, g) endowed with a 3-dimensional vector bundle E of tensors of type (1, 1) and a 3-dimensional distribution F. Suppose that there exist a local basis $\{\varphi_a\}$ of E and an orthonormal local basis $\{\xi_a\}$ of F satisfying the conditions:

$$\begin{gathered}(\varphi_a)^2 = -I + \eta_a \otimes \xi_a \; ; \; \varphi_a(\xi_a) = 0; \; \varphi_a(\xi_b) = -\varphi_b(\xi_a) = \xi_c; \\ \eta_a \circ \varphi_a = 0; \quad \eta_a \circ \varphi_b = -\eta_b \circ \varphi_a = \eta_c; \\ \varphi_a \circ \varphi_b - \xi_a \otimes \eta_b = -\varphi_b \circ \varphi_a + \xi_b \otimes \eta_a = \varphi_c,\end{gathered} \tag{2.9}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$ and η_a are local 1-forms given by

$$\eta_a(X) = g(X, \xi_a), \quad \forall X \in \Gamma(TP). \tag{2.10}$$

Moreover, we suppose

$$g(\varphi_a X, \varphi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y), \tag{2.11}$$

for any $a \in \{1, 2, 3\}$ and $X, Y \in \Gamma(TP)$. Further, the covariant derivative of φ_a with respect to the Levi-Civita connection ∇ on P is assumed to be expressed as follows:

$$(\nabla_X \varphi_a)Y = g(X, Y)\xi_a - \eta_a(Y)X + \alpha_{ab}(X)\varphi_b(Y) + \alpha_{ac}(X)\varphi_c(Y), \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$ and any cyclic permutation (a, b, c) of $(1, 2, 3)$, where α_{ab} are local 1-forms on P and $\alpha_{ab} + \alpha_{ba} = 0$. Further, we consider two coordinate neighbourhoods U and U' on P such that $U \cap U' \neq \emptyset$ and consider the local bases $\{\varphi_a\}$ and $\{\varphi'_a\}$ respectively.

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Finally, we suppose that $\{\varphi'_a\}$ and $\{\varphi_a\}$ are related on $U \cap U'$ as follows

$$\varphi'_a = \sum_{b=1}^{3} A_{ab}\varphi_b, \tag{2.13}$$

where $[A_{ab}]$ is an element of $SO(3)$.

The Riemannian manifold (P, g) endowed with 3-dimensional vector bundles E and F with local bases $\{\varphi_a\}$ and $\{\xi_a\}$ respectively, satisfying (2.9)-(2.13), is called a *generalised 3-Sasakian manifold*. Also we say that $(\varphi_a, \xi_a, \eta_a, g)$ is a *generalised 3-Sasakian structure*. One of our main results will show that a particular class of QR-submanifolds inherits a generalised 3-Sasakian structure from the quaternion Kaehlerian structure of the ambient manifold.

3. Main Results

Let M be a real p-dimensional submanifold of a $4m$-dimensional quaternion Kaehlerian manifold $\widetilde{M}$. It was proved by Bejancu (see [**3**, Theorem 3.3]) that if M is a totally umbilical proper QR-submanifold with $s = \dim \nu_x^\perp > 1$ for any $x \in M$, then M must be totally geodesic. Thus it remains to study the cases $s = 0$ and $s = 1$. To this end we first prove the following general lemma.

Lemma 3.1. *Let M be a totally umbilical QR-submanifold of $\widetilde{M}$ with $D \neq \{0\}$. Then the mean curvature vector H of M is a global section of $\nu^\perp$.*

Proof: Consider a unit vector field $X \in \Gamma(D)$. Then using (2.1) and (2.6) and taking into account that both D and ν are invariant with respect to $\widetilde{J}_a$ we deduce that

$$\begin{aligned}\widetilde{g}\left(\widetilde{J}_1\widetilde{\nabla}_X X, \widetilde{J}_1 N\right) &= \widetilde{g}\left(\widetilde{\nabla}_X \widetilde{J}_1 X - \alpha_{12}(X)\widetilde{J}_2 X - \alpha_{13}(X)\widetilde{J}_3 X, \widetilde{J}_1 N\right) \\ &= \widetilde{g}\left(\nabla_X \widetilde{J}_1 X + g\left(X, \widetilde{J}_1 X\right) H, \widetilde{J}_1 N\right) = 0, \quad \forall N \in \Gamma(\nu).\end{aligned}$$

On the other hand, $\widetilde{J}_1$ is a linear isometry and, using again (2.6), we infer that

$$\widetilde{g}\left(\widetilde{J}_1\widetilde{\nabla}_X X, \widetilde{J}_1 N\right) = \widetilde{g}\left(\widetilde{\nabla}_X X, N\right) = \widetilde{g}(H, N).$$

Thus for any $N \in \Gamma(\nu)$ we have $\widetilde{g}(H, N) = 0$, that is, $H \in \Gamma(\nu^\perp)$. ∎

In case $s = 0$, that is, $\dim \nu_x^\perp = 0$ for any $x \in M$, by Lemma 3.1 we deduce that H vanishes identically on M. Hence M is a totally geodesic quaternionic submanifold.

In the remaining part of the paper we suppose M is a totally umbilical, but not totally geodesic, proper QR-submanifold such that $s = 1$. Since M is not totally geodesic then there exists a coordinate neighbourhood U^* on M such that $h = \|H\|$ is nowhere vanishing on U^*. Thus we may consider on U^* the unit vector field

$$\xi = \frac{1}{h}H. \tag{3.1}$$

Further, we define on U^* the unit vector fields

$$\xi_a = \widetilde{J}_a \xi, \tag{3.2}$$

and the 1-forms

$$\eta_a(X) = g(X, \xi_a), \quad \forall X \in \Gamma(TM_{|U^*}). \tag{3.3}$$

Denote by Q the projection morphism of TM on D with respect to the decomposition (2.4). Then for any $X \in \Gamma(TM_{|U^*})$ we derive that

$$X = QX + \sum_{c=1}^{3} \eta_c(X)\xi_c. \tag{3.4}$$

Applying $\widetilde{J}_a$ to (3.4) and using (3.2) we obtain

$$\widetilde{J}_a X = \varphi_a X - \eta_a(X)\xi, \tag{3.5}$$

where we set

$$\varphi_a X = \widetilde{J}_a QX + \eta_b(X)\xi_c - \eta_c(X)\xi_b, \tag{3.6}$$

(a, b, c) being a cyclic permutation of $(1, 2, 3)$.

LEMMA 3.2. *For any $X \in \Gamma(TM_{|U^*})$ we have*

$$\nabla^{\perp}_X H = X(h)\xi, \tag{3.7}$$

and

$$hX = \varphi_a(\nabla_X \xi_a) + h\eta_a(X)\xi_a - \alpha_{ab}(X)\xi_c + \alpha_{ac}(X)\xi_b, \tag{3.8}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$.

PROOF: First, replace N by H in (2.7) and obtain

$$\widetilde{\nabla}_X H = -h^2 X + \nabla^{\perp}_X H, \quad \forall X \in \Gamma(TM_{|U^*}). \tag{3.9}$$

On the other hand, we take the covariant derivative of (3.1) and by using (3.2), (2.1), (2.6), (3.5) and (3.1) we infer that

$$\begin{aligned} \widetilde{\nabla}_X H &= X(h)\xi - h\widetilde{\nabla}_X \widetilde{J}_a \xi_a \\ &= X(h)\xi - h\big(\varphi_a(\nabla_X \xi_a) + h\eta_a(X)\xi_a - \alpha_{ab}(X)\xi_c + \alpha_{ac}(X)\xi_b\big), \end{aligned} \tag{3.10}$$

since by (3.3) we have $\eta_a(\nabla_X \xi_a) = 0$. Thus (3.7) and (3.8) are obtained by comparing the normal and tangent components from (3.9) and (3.10). ∎

Further, we recall that the curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$ satisfies (see Besse [4, p.403–405])

$$\widetilde{R}(X,Y)\widetilde{J}_aZ - \widetilde{J}_a\widetilde{R}(X,Y)Z = \frac{\rho}{m+2}\left(\widetilde{g}(\widetilde{J}_cX,Y)\widetilde{J}_bZ - \widetilde{g}(X,\widetilde{J}_bY)\widetilde{J}_cZ\right), \tag{3.11}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$, where $4m\rho$ is the scalar curvature of $\widetilde{M}$ and (a,b,c) is a cyclic permutation of $(1,2,3)$. Also we have (see Marchiafava [9])

$$\widetilde{g}\left(\widetilde{R}(X,Y)Z,W\right) = \widetilde{g}\left(\widetilde{R}(\widetilde{J}_aX,\widetilde{J}_aY)\widetilde{J}_aZ,\widetilde{J}_aW\right), \tag{3.12}$$

for any $X, Y, Z, W \in \Gamma(T\widetilde{M})$ and $a \in \{1,2,3\}$.

We are now in a position to prove the main results of the paper.

THEOREM 3.1: *Let M be a totally umbilical, but not totally geodesic, proper QR-submanifold of the quaternion Kaehlerian manifold $\widetilde{M}$ such that $\dim \nu_x^{\perp} = 1$ for any $x \in M$. Then M is an extrinsic sphere of $\widetilde{M}$.*

PROOF: First, by Lemma 3.1 and (3.7) we have

$$\nabla_X^{\perp}H \in \Gamma(\nu^{\perp}), \quad \forall X \in \Gamma(TM). \tag{3.13}$$

Then we take $a = 1$, $Z = X \in \Gamma(D)$ and $Y = \xi_1$ in (3.11) and obtain

$$\widetilde{g}\left(\widetilde{R}(X,\xi_1)\widetilde{J}_1X,\xi\right) = \widetilde{g}\left(\widetilde{R}(\xi_1,X)X,\xi_1\right).$$

On the other hand, (2.8) yields

$$\widetilde{g}\left(\widetilde{R}(X,\xi_1)\widetilde{J}_1X,\xi\right) = g(\xi_1,\widetilde{J}_1X)\widetilde{g}(\nabla_X^{\perp}H,\xi) - g(X,\widetilde{J}_1X)\widetilde{g}(\nabla_{\xi_1}^{\perp}H,\xi) = 0.$$

Hence $\widetilde{g}\left(\widetilde{R}(\xi_1,X)X,\xi_1\right) = 0$ and by linearity we deduce that

$$\widetilde{g}\left(\widetilde{R}(\xi_1,X)Y,\xi_1\right) = 0, \quad \forall X, Y \in \Gamma(D). \tag{3.14}$$

In particular, we take $Y = \widetilde{J}_1X$, where X is a unit vector field that lies in the quaternionic distribution, and by using again (3.11) and (2.8), we infer that

$$\begin{aligned} 0 = \widetilde{g}\left(\widetilde{R}(\xi_1X)\widetilde{J}_1X,\xi_1\right) &= \widetilde{g}\left(\widetilde{J}_1\widetilde{R}(\xi_1,X)X,\xi_1\right) - \frac{2\rho}{m+2}g(\xi_1,\widetilde{J}_2X)g(\xi_1,\widetilde{J}_3X) \\ &= \widetilde{g}\left(\widetilde{R}(\xi_1,X)X,\xi\right) = \widetilde{g}\left(\nabla_{\xi_1}^{\perp}H,\xi\right). \end{aligned}$$

Therefore, we have

$$\nabla_{\xi_a}^{\perp}H \in \Gamma(\nu), \quad \forall a \in \{1,2,3\}. \tag{3.15}$$

Next, from (2.8) we deduce that

$$\tilde{g}\left(\tilde{R}(\tilde{J}_1X, \tilde{J}_2X)\tilde{J}_3X, \xi\right) = g(\tilde{J}_2X, \tilde{J}_3X)\tilde{g}(\nabla^{\perp}_{\tilde{J}_1X}H, \xi) - g(\tilde{J}_1X, \tilde{J}_3X)\tilde{g}\left(\nabla^{\perp}_{\tilde{J}_2X}H, \xi\right) = 0, \tag{3.16}$$

for any $X \in \Gamma(D)$. On the other hand, by using (3.12), (3.11) and (2.8) we derive

$$\begin{aligned} \tilde{g}\left(\tilde{R}(\tilde{J}_1X, \tilde{J}_2X)\tilde{J}_3X, \xi\right) &= -\tilde{g}\left(\tilde{J}_1\tilde{R}(X, \tilde{J}_3X)\tilde{J}_2X, \xi\right) \\ &= \tilde{g}\left(\tilde{R}(\tilde{J}_3X, X)\tilde{J}_3X, \xi\right) = -g(X, X)\tilde{g}\left(\nabla^{\perp}_X H, \xi\right). \end{aligned} \tag{3.17}$$

As M is supposed to be a proper QR-submanifold, there exists a non-zero vector field $X \in \Gamma(D)$ and hence (3.17) and (3.16) imply

$$\nabla^{\perp}_X H \in \Gamma(\nu), \quad \forall X \in \Gamma(D). \tag{3.18}$$

Thus from (3.15) and (3.18) we infer that $\nabla^{\perp}_X H$ lies in $\Gamma(\nu)$ for any $X \in \Gamma(TM)$. Then, taking into account (3.13) we obtain $\nabla^{\perp}_X H = 0$. As $\nabla^{\perp}$ is a Riemannian connection on $TM^{\perp}$ we deduce that h is a positive constant on U^*. By continuity of h and connectedness of M it follows that h is a positive constant on M. Hence H is nowhere zero on M and thus M is an extrinsic sphere. ∎

THEOREM 3.2. *Let M be as in Theorem 3.1. Then there exists a generalised 3-Sasakian structure on M.*

PROOF: First, from (3.2) it follows that $\{\xi_1, \xi_2, \xi_3\}$ is a local orthonormal basis for the distribution $F = D^{\perp}$ on M. Also we have $\{\eta_1, \eta_2, \eta_3\}$ given by (3.3). Next, we consider the local tensor fields $\{\varphi_a\}$, $a \in \{1, 2, 3\}$ given by (3.6). Then it is easy to check that $(\varphi_a, \xi_a, \eta_a, g)$ satisfy (2.9)-(2.11). By using (3.2), (3.3) and (2.2) for any two neighbourhoods U^* and $U^{*'}$ on M such that $U^* \cap U^{*'} \neq \emptyset$, we obtain

$$\eta'_a = \sum_{b=1}^{3} A_{ab}\eta_b, \tag{3.19}$$

on $U^* \cap U^{*'}$, where $[A_{ab}]$ is the matrix in (2.2). Then by direct calculations using (3.5), (2.2) and (3.19), we derive (2.13). Hence we have a 3-dimensional vector bundle E of tensors of type (1, 1) on M whose local basis is $\{\varphi_1, \varphi_2, \varphi_3\}$ given by (3.6). Moreover, by using (3.5), (2.6), (2.7), (3.1) and (2.11) and taking into account that g and ξ are parallel with respect to ∇ and $\nabla^{\perp}$ respectively, we deduce that

$$\begin{aligned} \tilde{\nabla}_X \tilde{J}_1 Y &= \tilde{\nabla}_X\left(\varphi_1 Y - \eta_1(Y)\xi\right) \\ &= \nabla_X\varphi_1 Y + g(X, \varphi_1 Y)H - X\left(\eta_1(Y)\right)\xi + \eta_1(Y)\tilde{g}(H, \xi)X \\ &= \left\{\nabla_X\varphi_1 Y + h\eta_1(Y)X\right\} - \left\{hg(\varphi_1 X, Y) + \eta_1(\nabla_X Y) + g(Y, \nabla_X\xi_1)\right\}\xi, \end{aligned} \tag{3.20}$$

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for any $X, Y \in \Gamma(TM)$. On the other hand, by using (2.1), (2.6), (3.5), (3.1) and (3.2) we infer that

$$\widetilde{\nabla}_X \widetilde{J}_1 Y = \{\varphi_1(\nabla_X Y) + hg(X,Y)\xi_1 + \alpha_{12}(X)\varphi_2 Y + \alpha_{13}(X)\varphi_3 Y\} - \{\eta_1(\nabla_X Y) + \alpha_{12}(X)\eta_2(Y) + \alpha_{13}(X)\eta_3(Y)\}\xi. \tag{3.21}$$

Comparing the tangent parts from (3.20) and (3.21) we obtain

$$(\nabla_X \varphi_1)Y = h\{g(X,Y)\xi_1 - \eta_1(Y)X\} + \alpha_{12}(X)\varphi_2 Y + \alpha_{13}(X)\varphi_3 Y. \tag{3.22}$$

Finally, we consider the Riemannian metric $g^* = h^2 g$ on M and choose $\varphi_a^* = \varphi_a$ and $\xi_a^* = (1|h)\xi_a$ as local bases in $\Gamma(E)$ and $\Gamma(F)$ respectively. Then it follows that $\eta_a^* = h\eta_a$. In this way, from (3.22) we obtain (2.12) for $(\varphi_a^*, \xi_a^*, \eta_a^*, g^*)$. Moreover, as $(\varphi_a, \xi_a, \eta_a, g)$ satisfy (2.9) - (2.11) and (2.13), it follows that $(\varphi_a^*, \xi_a^*, \eta_a^*, g^*)$ satisfy these relations too. Therefore, $(\varphi_a^*, \xi_a^*, \eta_a^*, g^*)$ is a generalised 3-Sasakian structure on M. ☐

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