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A one-dimensional tearing mode equation for pedestal stability studies in tokamaks

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Starting from expressions in Connor *et al.* (*Phys. Fluids*, vol. 31, 1988, p. 577), we derive a one-dimensional tearing equation similar to the approximate equation obtained by Hegna & Callen (*Phys. Plasmas*, vol. 1, 1994, p. 2308) and Nishimura *et al.* (*Phys. Plasmas*, vol. 5, 1998, p. 4292), but for more realistic toroidal equilibria. The intention is to use this approximation to explore the role of steep profiles, bootstrap currents and strong shaping in the vicinity of a separatrix, on the stability of tearing modes which are resonant in the H-mode pedestal region of finite aspect ratio, shaped cross-section tokamaks, e.g. the Joint European Torus (JET). We discuss how this one-dimensional model for tearing modes, which assumes a single poloidal harmonic for the perturbed poloidal flux, compares with a model that includes poloidal coupling Fitzpatrick *et al.* (*Nucl. Fusion*, vol. 33, 1993, p. 1533).

Key words: fusion plasma, plasma confinement, plasma instabilities

1. Introduction

Edge localised modes (ELMs) are a ubiquitous feature of H-mode tokamak plasmas with important consequences for confinement and for transient heat loads on divertor target plates. Most theoretical models appeal to ideal magnetohydrodynamic (MHD) ballooning and peeling modes (Hegna *et al.* 1996; Connor *et al.* 1998; Wilson *et al.* 1999; Snyder *et al.* 2004) as the trigger for ELMs. While this may well be the case for larger Type I ELMs, the smaller Type III may involve resistive ballooning modes (Connor 1998). Furthermore it is unclear whether ideal peeling modes are ever unstable due to the presence of a separatrix in divertor tokamaks (Huysmans 2005; Webster & Gimblett 2009) or can lead to the required destruction of magnetic surfaces seen in resistive MHD simulations, e.g. ASDEX Team (1989). However, concerning the need for resistivity, one should mention a model for ELMs in which an unstable ideal peeling mode does play a part, triggering a Taylor relaxation in the edge plasma, thus involving reconnection. The relaxation region grows in size until the ideal mode becomes stable (Gimblett, Hastie & Helander 2006). An alternative possible explanation is that ELMs might be triggered by tearing modes being driven

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unstable by the large bootstrap current density that results from the pressure gradients in the H-mode pedestal.

The theory of tearing modes utilises asymptotic matching techniques (Furth, Killeen & Rosenbluth 1963). Thus solutions of the resistive equations (or those corresponding to more complex plasma models e.g. Antonsen & Coppi (1981), Drake *et al.* (1983), Cowley, Kulsrud & Hahm (1986), Pegoraro & Schep (1986), Porcelli (1987), Fitzpatrick (1989), Connor, Hastie & Zocco (2012)) that pertain near resonant surfaces, $m = nq(\rho_s)$, are matched to solutions of ideal MHD equations that describe the regions away from the resonance to obtain a dispersion relation determining their stability. Here m and n are poloidal and toroidal mode numbers of the perturbation, $q(\rho)$ is the safety factor, ρ is a flux surface label with dimensions of length and ρ_s is the resonance position. This matching procedure involves obtaining the asymptotic forms of the ideal MHD solutions as $\rho \rightarrow \rho_s$ from both left and right, and the matching is characterised by a quantity Δ'. Stability of a mode is determined by comparing Δ' with Δ'_{crit}, a parameter that is determined from the solution of the equation describing the narrow layer around the resonance. The quantity Δ'_{crit} is usually a large positive number (Glasser, Greene & Johnson 1975; Drake *et al.* 1983; Cowley *et al.* 1986), but physics close to the resonance can make Δ'_{crit} negative: e.g. when microtearing modes are unstable, as has been reported for the plasma region around the H-mode pedestal in the Mega Ampère Spherical Tokamak (MAST) (Dickinson *et al.* 2012) and in the Joint European Torus (JET) (Hatch *et al.* 2016).

The linear theory of tearing instability in toroidal geometry (Connor *et al.* 1988) is a complex problem, raising issues associated with the coupling of different poloidal harmonics and with the decoupling of resonances at different rational surfaces due to differing diamagnetic frequencies at such surfaces. Hegna & Callen (1994) proposed a simple approximation that the perturbed poloidal flux has a single poloidal harmonic, of admittedly uncertain accuracy, to obtain a master equation for tearing instability, with similar one-dimensional (1-D) character to that holding in a straight cylinder. This equation was derived for equilibria with weakly shaped poloidal cross-section, and under the additional assumptions of large aspect ratio, low β (where β is the ratio of plasma pressure, p, to the magnetic field energy density, $\beta = 2\mu_0 p/B^2$) and with the toroidal magnetic field greatly exceeding the poloidal field:

$$\frac{I}{q}\frac{\mathrm{d}}{\mathrm{d}\psi}\frac{q}{I}\langle g^{\psi\psi}\rangle\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\psi} - \left[m^2\langle g^{\theta\theta}\rangle + \frac{m}{m-nq}I\langle\sigma\rangle' + \frac{m^2}{(m-nq)^2}\mu_0 I p'\langle J\rangle'\right]\tilde{A} = 0, \tag{1.1}$$

where $\tilde{A}$ and ψ are respectively the perturbed and equilibrium poloidal flux, the magnetic field is $\boldsymbol{B} = I\nabla\phi + \nabla\phi \times \nabla\psi$, I is the toroidal field function $I = RB_\phi$, ϕ is the toroidal coordinate, $\sigma = \mu_0 j_\parallel/B$, with the parallel current density $j_\parallel = \boldsymbol{j}\cdot\boldsymbol{B}/B$, q is the safety factor, $'$ denotes the radial gradient with respect to ψ, $\langle Y\rangle$ is the flux surface average of Y for any quantity $Y(\psi, \theta)$,

$$\langle Y\rangle = \frac{1}{2\pi}\oint Y\,\mathrm{d}\theta \tag{1.2}$$

and the metric elements are $g^{\psi\psi} = |\nabla\psi|^2$ and $g^{\theta\theta} = |\nabla\theta|^2$. The θ coordinate is a straight field line poloidal angle and $J = (\nabla\psi \times \nabla\theta\cdot\nabla\phi)^{-1} = R^2 q/I$ is the Jacobian. Nishimura, Callen & Hegna (1998) presented numerical solutions of a similar equation, for a family of equilibrium profiles resembling those studied previously by Furth, Rutherford & Selberg (1973) in a cylindrical geometry.

To assist the tearing mode stability analysis of the H-mode pedestal, in this paper we develop a 1-D ideal MHD equation for application to realistic, fully toroidal tokamak equilibria at high β, thus generalising the earlier seminal works by Hegna & Callen (1994) and Nishimura *et al.* (1998). This contrasts with alternative approximate treatments described in Fitzpatrick *et al.* (1993), where the effect of poloidal mode coupling was calculated for toroidal equilibria of large aspect ratio, low β and weak shaping, and approximate solutions with seven poloidal harmonics were used to obtain Δ'.

2. A 1-D tearing mode equation

We start from equations (A5) and (A6) of Connor *et al.* (1988), which respectively govern the radial component of the displacement ($\boldsymbol{\xi}$), and the perturbed toroidal magnetic field: these quantities manifest themselves in Connor *et al.*'s variables $y = R_0 f \boldsymbol{\xi} \cdot \nabla\rho$ and $z = R^2 \delta \boldsymbol{B} \cdot \nabla\phi / B_0$. Here the equilibrium magnetic field is written as $\boldsymbol{B} = R_0 B_0 [g\nabla\phi + f\nabla\phi \times \nabla\rho]$, where ρ is a flux surface label with dimension of length, $B_0 g(\rho) R_0/R$ is the full toroidal magnetic field, B_0 is the vacuum toroidal field at the major radius, R_0, of the magnetic axis and $q = (\rho/R_0)(g/f)$. The variable y, which is related to the perturbed poloidal flux, now denoted ψ, by $y = \psi/(m - nq)$,[1] is assumed to contain only a single poloidal harmonic, $\mathrm{e}^{\mathrm{i}m\theta}$, where θ is the poloidal angle in straight field line coordinates. These equations can be used to generate the 1-D ideal MHD equation for ψ.

Then equations (A5) and (A6) of Connor *et al.* (1988) take the form

$$\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\mathrm{e}^{\mathrm{i}m\theta} = -\frac{\partial}{\partial\theta}\left[\psi \mathrm{e}^{\mathrm{i}m\theta}\left(\mathrm{i}T + \frac{U}{m - nq}\right)\right] + \left(Sz + \frac{\partial}{\partial\theta} Q \frac{\partial z}{\partial\theta}\right) \tag{2.1}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\theta} - \mathrm{i}nq\right)\frac{\partial z}{\partial\rho} &= \psi \mathrm{e}^{\mathrm{i}m\theta}\left[\mathrm{i}W + \frac{X}{m - nq} + (m - nq)V\right] - \mathrm{i}\psi \mathrm{e}^{\mathrm{i}m\theta}\frac{\partial V}{\partial\theta} \\ &\quad + U\frac{\partial z}{\partial\theta} - \left(\frac{\partial}{\partial\theta} - \mathrm{i}nq\right)\left[T^* \frac{\partial z}{\partial\theta}\right],\end{aligned} \tag{2.2}$$

where the equilibrium quantities, Q, S, T, U, V, W, X are defined in equation (A7) of Connor *et al.* (1988),

$$S = \mathrm{i}n\rho/R_0, \tag{2.3}$$

$$Q = -\mathrm{i}\frac{R_0}{n\rho}\frac{1}{|\nabla\rho|^2}, \tag{2.4}$$

$$T = \frac{\nabla\theta \cdot \nabla\rho}{|\nabla\rho|^2} + \mathrm{i}\frac{R_0 g'}{n\rho f}\frac{1}{|\nabla\rho|^2}, \tag{2.5}$$

$$U = \frac{\mu_0 p'}{B_0^2 f^2}\frac{R^2}{R_0^2|\nabla\rho|^2}, \tag{2.6}$$

$$V = \mathrm{i}\frac{n}{\rho R_0}\frac{R_0^2}{R^2|\nabla\rho|^2} - \mathrm{i}\frac{R_0}{\rho n}\left(\frac{g'}{f}\right)^2\frac{1}{|\nabla\rho|^2} \tag{2.7}$$

[1]The behaviour of ψ near the resonant surface at $m = nq(\rho)$, is a combination of large and small solutions (Glasser *et al.* 1975), and it is this combination that must be matched to the solution in the inner resonant layer.

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$$W = \frac{2\mu_0 p' g'}{B_0^2 f^3} \frac{R^2}{R_0^2 |\nabla\rho|^2} - \frac{\mathrm{d}}{\mathrm{d}\rho} \frac{g'}{f} \tag{2.8}$$

$$X = \mathrm{i}\frac{n\mu_0 p'\rho}{B_0^2 f^2 R_0} \left[\frac{\partial}{\partial\theta} \left(T^* \frac{R^2}{R_0^2} \right) + \frac{\partial}{\partial\rho} \frac{R^2}{R_0^2} - \frac{R^2}{R_0^2} \left(\frac{f'}{f} - \frac{1}{\rho} \right) - \frac{\mu_0 p'}{B_0^2 f^2} \frac{R^4}{R_0^4 |\nabla\rho|^2} \right], \tag{2.9}$$

with $'$ now representing the radial derivative with respect to ρ. We note here that the above expressions were derived for equilibria of arbitrary aspect ratio, cross-sectional shape and β. The method employed in the following analysis is rather general and does not assume that the second dependent variable, z, is also of single poloidal harmonic structure.

To simplify the analysis we neglect the term involving S, relative to $m^2 Q$ in (2.1). This is equivalent to reducing the field line bending energy in a circular cylinder from the $(m^2 + k_z^2 r^2)$ of the Newcomb (1960) analysis of stability in a linear pinch, to m^2. In a torus this is equivalent to an assumption that $(\epsilon/q_s)^2 \ll 1$, where ϵ is the local aspect ratio and $q_s = m/n$ is the value of the safety factor at the resonance. Since our focus will be on tearing modes which are resonant in the pedestal region of a tokamak of aspect ratio around $1/3$, q_s may be of order 4 or greater, so this approximation would appear to introduce errors of only approximately 1 %.

The required 1-D tearing equation is now obtained by solving (2.1) for $\partial z/\partial\theta$, inserting the result in (2.2) and taking the flux surface average. Thus:

$$\frac{\partial z}{\partial\theta} = \frac{\mathrm{e}^{\mathrm{i}m\theta}}{Q} \left[\frac{1}{m} \frac{\mathrm{d}\psi}{\mathrm{d}\rho} + \psi \left(\mathrm{i}T + \frac{U}{m - nq} \right) \right] - \frac{K(\rho)}{Q}, \tag{2.10}$$

where $K(\rho)$ is a flux surface dependent constant of integration to be determined by a periodicity constraint on $z(\rho, \theta)$. Thus

$$K(\rho) = \frac{1}{m} \frac{\mathrm{d}\psi}{\mathrm{d}\rho} \alpha_m + \psi \left[\gamma_m + \frac{\delta_m}{m - nq} \right], \tag{2.11}$$

with

$$\alpha_m = \frac{\langle \mathrm{e}^{\mathrm{i}m\theta} |\nabla\rho|^2 \rangle}{\langle |\nabla\rho|^2 \rangle} = \frac{\langle \cos(m\theta) |\nabla\rho|^2 \rangle}{\langle |\nabla\rho|^2 \rangle} \tag{2.12}$$

$$\gamma_m = \mathrm{i}\frac{\langle \mathrm{e}^{\mathrm{i}m\theta} T |\nabla\rho|^2 \rangle}{\langle |\nabla\rho|^2 \rangle} = -\frac{\langle \sin(m\theta) \nabla\theta \cdot \nabla\rho \rangle}{\langle |\nabla\rho|^2 \rangle} \tag{2.13}$$

$$\delta_m = \frac{\langle \mathrm{e}^{\mathrm{i}m\theta} U |\nabla\rho|^2 \rangle}{\langle |\nabla\rho|^2 \rangle} = \frac{\langle \cos(m\theta) R^2 \rangle}{R_0^2 \langle |\nabla\rho|^2 \rangle} \frac{\mu_0 p'}{B_0^2 f^2}, \tag{2.14}$$

where the second form in (2.12)–(2.14) applies for equilibria which are symmetric above and below the median plane. Now, since the m number for tearing modes which are resonant in the pedestal region of a tokamak is likely to be moderately large, the coefficients defined by α_m, γ_m and δ_m in (2.12)–(2.14) may be very small unless there is strong shaping. Consequently, we can normally neglect the integration constant, $K(\rho)$ defined in (2.11). In § 2.1 we will investigate the consequences of retaining finite $K(\rho)$.

Inserting the expression for $(\partial z/\partial\theta)$ (in the $K(\rho) = 0$ limit) into (2.2) and multiplying by the factor $\mathrm{e}^{-\mathrm{i}m\theta}$, we take the flux surface average to obtain a 1-D tearing mode equation. Expressed in terms of the equilibrium quantities, Q, T, U, V, W

and X, this takes the form:

$$\frac{(m-nq)}{m^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\left\langle\frac{1}{Q}\right\rangle\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\right]+\psi\frac{(m-nq)}{m}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\mathrm{i}\left\langle\frac{T}{Q}\right\rangle+\frac{1}{(m-nq)}\left\langle\frac{U}{Q}\right\rangle\right]$$
$$=\psi\left[(m-nq)\langle V\rangle+\mathrm{i}\langle W\rangle+\frac{\langle X\rangle}{(m-nq)}+(m-nq)\left\langle\frac{T\ T^*}{Q}\right\rangle+\frac{1}{(m-nq)}\left\langle\frac{U^2}{Q}\right\rangle\right.$$
$$\left.+\mathrm{i}\left\langle\frac{U(T-T^*)}{Q}\right\rangle\right]. \tag{2.15}$$

Now, writing $1/Q=\lambda\rho|\nabla\rho|^2$, where $\lambda=\mathrm{i}n/R_0$ and dividing through by $\lambda(m-nq)/m^2$, equation (2.15) takes the form of the second order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\langle|\nabla\rho|^2\rangle\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\right]+m\psi\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\mathrm{i}\rho\langle T|\nabla\rho|^2\rangle+\frac{\rho}{(m-nq)}\langle U|\nabla\rho|^2\rangle\right]$$
$$=\psi\frac{m^2}{\lambda}\left[\langle V\rangle+\mathrm{i}\frac{\langle W\rangle}{(m-nq)}+\frac{\langle X\rangle}{(m-nq)^2}\right]$$
$$+\psi m^2\rho\left[\langle TT^*|\nabla\rho|^2\rangle+\frac{\mathrm{i}\langle U(T-T^*)|\nabla\rho|^2\rangle}{(m-nq)}+\frac{\langle U^2|\nabla\rho|^2\rangle}{(m-nq)^2}\right], \tag{2.16}$$

which is of the same structure as the equation derived by Hegna & Callen (1994), namely

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left[A(\rho)\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\right]-\left[B(\rho)+\frac{mC(\rho)}{(m-nq)}+\frac{m^2D(\rho)}{(m-nq)^2}\right]\psi=0, \tag{2.17}$$

where, on inserting the definitions (2.4)–(2.9),

$$A=\rho\langle|\nabla\rho|^2\rangle \tag{2.18}$$

$$B=\frac{m^2}{\rho}\left[\left\langle\frac{R_0^2}{R^2}\frac{1}{|\nabla\rho|^2}\right\rangle+\rho^2\left\langle\frac{|\nabla\theta\boldsymbol{\cdot}\nabla\rho|^2}{|\nabla\rho|^2}\right\rangle\right]=m^2\rho\langle|\nabla\theta|^2\rangle \tag{2.19}$$

$$C=-q\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{R_0g'}{f}+\frac{R_0}{fg}\frac{\mu_0p'}{B_0^2}\left\langle\frac{R^2}{R_0^2}\right\rangle\right] \tag{2.20}$$

$$D=\frac{\mu_0p'}{B_0^2f^2}\left[\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left\langle\frac{R^2}{R_0^2}\right\rangle-\left\langle\frac{R^2}{R_0^2}\right\rangle\left(\frac{\rho g'}{g}\right)\right]. \tag{2.21}$$

Some details of the derivation of (2.18)–(2.21) are given in appendix A, and in appendix B we express equation (2.17) in terms of the variables used in equation (26) of Hegna & Callen (1994).

2.1. *Consequences of finite K*

We now return to (2.10) and (2.11) and construct the additional terms that will appear in the tearing equation when we retain terms with finite $K(\rho)$. After lengthy, but straightforward, further analysis, we find that each of the coefficients $A(\rho)$, $B(\rho)$, $C(\rho)$ and $D(\rho)$ is modified by an additional contribution, which we shall denote by a circumflex. Thus

$$\left.\begin{aligned}A&\rightarrow A(\rho)-\hat{A}(\rho),\\ B&\rightarrow B(\rho)-\hat{B}(\rho),\\ C&\rightarrow C(\rho)-\hat{C}(\rho),\\ D&\rightarrow D(\rho)-\hat{D}(\rho),\end{aligned}\right\} \tag{2.22}$$

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with

$$\hat{A}(\rho) = A(\rho)|\alpha_m|^2, \tag{2.23}$$

$$\hat{B}(\rho) = m^2 A(\rho)\gamma_m^2 - m\frac{\mathrm{d}}{\mathrm{d}\rho}[A(\rho)\gamma_m\alpha_m], \tag{2.24}$$

$$\hat{C}(\rho) = -q\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{\alpha_m\delta_m A(\rho)}{q}\right], \tag{2.25}$$

$$\hat{D}(\rho) = A(\rho)\delta_m\left(\delta_m - \frac{s\alpha_m}{\rho}\right) \tag{2.26}$$

where $s = \rho q'/qs = \rho q'/q$ is the magnetic shear. $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are small in the large m limit because the numerators in the definitions of α_m, δ_m, γ_m (see (2.12)–(2.14)) must vanish both at high m, or with weak shaping. At a fixed finite m these terms can, however, become more important with stronger shaping (e.g. as one approaches the separatrix).

2.2. *Comparison with earlier results*

The Hegna–Callen equation represented a significant advance on earlier work by making possible a simple 1-D tearing analysis of large aspect ratio toroidal equilibria with weakly shaped poloidal cross-sections. Our derivation has not only extended the validity of the 1-D equation to finite aspect ratio equilibria, subject to $(\epsilon/q_s)^2 \ll 1$, with arbitrary poloidal shaping, but it has also revealed the presence of new terms arising from finite values of the integration constant $K(\rho)$. These additional terms of (2.23)–(2.26) have no counterpart in Hegna & Callen (1994) or Nishimura *et al.* (1998), but they are small unless there is strong shaping containing poloidal harmonics that couple to the mode number, m.

We now compare our tearing equation (2.17) with Hegna & Callen (1994) and Nishimura *et al.* (1998). We begin by transforming from the Hegna–Callen equilibrium variables, I and ψ, to the g, f, ρ variables of the present work. Thus:

$$I \to R_0 B_0 g(\rho), \tag{2.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\psi} \to \frac{1}{\psi'}\frac{\mathrm{d}}{\mathrm{d}\rho}, \tag{2.28}$$

$$\psi'(\rho) \to R_0 B_0 f(\rho). \tag{2.29}$$

The coefficients A, B, C and D can then be identified in equation (26) of Hegna & Callen (1994) and compared to (2.18)–(2.21). This shows agreement in the expressions for A and B, close agreement on C, but not for D. Since

$$\sigma = \frac{1}{f}\left(g' + \frac{g\mu_0 p'}{B^2}\right), \tag{2.30}$$

one can indeed write $C \propto (\partial\langle\sigma\rangle/\partial\rho)$ if $B \simeq B_\phi$, as in Hegna & Callen (1994). There is some similarity with the expression for D that appears in equation (19) of Nishimura *et al.* (1998), where special equilibria with $g =$constant were studied so that the last term in (2.21) is absent, but nevertheless their $D \propto n^2q^2$ rather than m^2, and so it differs away from the resonance.

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As noted by Hegna, Callen and Nishimura, there is an important comparison for the expression given in (2.21) for $D(\rho)$. This is associated with the Mercier stability criterion, $D_M < 0$, for the ideal MHD stability of a mode localised around a rational surface (Mercier 1960). Glasser *et al.* (1975) showed D_M[2] plays an important role in the theory of tearing mode stability in a torus. They found the asymptotic form of the ideal MHD solutions as $\rho \to \rho_s$ is

$$\psi \sim c_0 \, |x-1|^{\nu_-} + c_1 \, |x-1|^{\nu_+}, \tag{2.31}$$

where $x = \rho/\rho_s$, constants c_0 and c_1 have different values to the left and right of the resonance, and the Mercier indices $\nu_\pm$ have values:

$$\nu_\pm = \tfrac{1}{2} \pm \sqrt{-D_M}. \tag{2.32}$$

This serves to define a generalised $\varDelta'$

$$\varDelta' = \left.\frac{c_1}{c_0}\right|_R + \left.\frac{c_1}{c_0}\right|_L, \tag{2.33}$$

where R and L denote locations immediately to the right and left of the resonance, respectively. This expression, obtained from the ideal MHD solution, must be matched to the analogous quantity arising from the inner resonant layer solution, to obtain the tearing mode dispersion relation.

Using the results in Glasser *et al.* (1975) and Connor *et al.* (1988)[3] we find that, at the tearing mode resonance, D of (2.21) should be compared to $-(As^2/\rho^2)((1/4) + D_M)$, where

$$\begin{aligned} D_M &= -\frac{1}{4} + E + F + H \\ &= -\frac{1}{4} + \frac{q}{q'}\frac{\mu_0 p'}{B_0^2 f^2}\left\langle \frac{R^2}{R_0^2}\frac{1}{|\nabla\rho|^2}\right\rangle - \frac{q^2}{q'^2}\left(\frac{\mu_0 p'}{B_0^2 f^2}\right)^2 \left\langle \frac{R^2}{R_0^2}\frac{1}{|\nabla\rho|^2}\right\rangle^2 \\ &\quad - \left(\frac{\mu_0 p'}{B_0^2 f^2}\right)\frac{1}{q'^2}\left(\frac{\rho}{R_0 f}\right)^2 \times \left\langle \frac{\partial}{\partial\rho}\left(\frac{R^2}{R_0^2}\right) - \frac{R^2}{R_0^2}\frac{\rho}{f}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{f}{\rho}\right)\right\rangle \left\langle \frac{B^2 R^2}{B_0^2 R_0^2 |\nabla\rho|^2}\right\rangle \\ &\quad + \left(\frac{\mu_0 p'}{B_0^2 f^2 q'}\right)^2 \left(\frac{\rho}{R_0 f}\right)^2 \left\langle \frac{R^4}{R_0^4|\nabla\rho|^2}\right\rangle \left\langle \frac{B^2 R^2}{B_0^2 R_0^2 |\nabla\rho|^2}\right\rangle, \end{aligned} \tag{2.34}$$

and the quantities E, F and H are defined in Glasser *et al.* (1975). (In a later paper, Glasser, Greene & Johnson (1976) showed that for a large aspect ratio circular cross-section plasma:

$$E + F + H = \frac{2\rho\mu_0 p'}{B_0^2}\frac{q^2-1}{s^2}, \tag{2.35}$$

where the important factor $q^2 - 1$ removes, for $q > 1$, the possibility of the instability predicted by Suydam (1958) in a straight cylinder.) Thus we can write:

$$\frac{1}{4} + D_M \propto \frac{\rho\mu_0 p'}{B_0^2}\frac{\kappa_{\mathrm{eff}}}{s^2}, \tag{2.36}$$

[2]The quantity labelled D_M here is precisely the object denoted by D_I in Glasser *et al.* (1975).

[3]A factor $1/f^2$ was missed from the final term of equation (B3) of Connor *et al.* (1988) due to a typographical error, and this is correctly included here.

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with the 'effective' curvature, κ_{eff}, deduced from (2.34). However, Hegna & Callen (1994), perhaps seeking a D consistent with this argument, assumed κ_{eff} was the surface-averaged normal curvature, κ_n, and, furthermore, that $\kappa_n \propto V'' = (\mathrm{d}\langle J\rangle/\mathrm{d}\rho)$, where $\langle J\rangle = (\langle R^2\rangle q/R_0B_0g)$, to obtain the following result for D:

$$D_{HC} \propto \frac{\rho\mu_0 p'}{B_0^2 s^2}\frac{\mathrm{d}\langle J\rangle}{\mathrm{d}\rho} \propto \frac{\rho\mu_0 p'}{B_0^2 s^2}\frac{1}{R_0}\frac{q}{g}\left(\frac{\mathrm{d}\langle R^2\rangle}{\mathrm{d}\rho} + \langle R^2\rangle\left(\frac{q'}{q} - \frac{g'}{g}\right)\right). \tag{2.37}$$

However, at low β and with $B_\phi \simeq B$ (e.g. at large aspect ratio),

$$\kappa_n \propto V'' - \frac{\langle R^2\rangle}{R_0B_0}\frac{q'}{g} \tag{2.38}$$

(Connor, Hastie & Helander 2009), so that their argument should have implied

$$D_{HC} \rightarrow D \propto \frac{\rho\mu_0 p'}{B_0^2 s^2}\frac{1}{R_0}\frac{q}{g}\left(\frac{\mathrm{d}\langle R^2\rangle}{\mathrm{d}\rho} - \langle R^2\rangle\frac{g'}{g}\right). \tag{2.39}$$

Equation (2.39) is indeed consistent with our expression for D in (2.21), and also with the work of Nishimura *et al.* (1998) in the special case $g' = 0$ that they considered. Equations (2.21), (2.39) are not, however, consistent with $D = -(As^2/\rho^2)((1/4) + D_M)$, since $\kappa_{\mathrm{eff}} \neq \kappa_n$.[4] We should not expect D to be exactly equal to $-(As^2/\rho^2)((1/4) + D_M)$, because the ideal instability investigated by Mercier, and later by Greene & Johnson (1962) using Hamada coordinates, is a mode with a range of coupled poloidal harmonics, whereas ψ of the envisaged tearing mode, has an isolated single poloidal harmonic.

It would be inconsistent with the 'single poloidal harmonic' assumption to simply replace $D(\rho)$ by the value corresponding to D_M in (2.17); although the use of D_M would capture the poloidal mode coupling effects close to the singular surface that can have a profound effect on the Mercier indices, which in turn influence the value of the generalised Δ' stability parameter (Glasser *et al.* 1975).

3. Conclusions

Within the foregoing sections we have assumed that the perturbed poloidal flux function, $\psi(\rho, \theta)$, contained only one poloidal harmonic, $\mathrm{e}^{\mathrm{i}m\theta}$. However our solution for the variable z, equation (2.10), contains a full spectrum of poloidal harmonics. Under these assumptions we have extended the validity of the tearing equation proposed by Hegna & Callen (1994) to axisymmetric equilibria of arbitrary aspect ratio and arbitrary β. In doing so we have only made use of the approximation, $\epsilon^2/q_s^2 \ll 1$. This would certainly rule out the use of the resulting 1-D equation for studying internal kink type disruptions in tokamaks (where the $m = n = 1$ harmonic plays a crucial role), but should prove to be an accurate approximation for modes which are resonant in the pedestal region of a tokamak in H-mode. An unexpected result of this calculation has been the appearance of a new set of terms arising from the effect of the integration constant K (denoted by $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$). However, it appears unlikely that such terms will play a significant role in determining tearing stability since they are normally negligibly small, except perhaps in very strongly

[4] A more detailed discussion of the relation of D_M to κ_n is given by Johnson & Greene (1967).

shaped cross-sections or, e.g. in the vicinity of a separatrix boundary. For simplicity we ignored these extra terms in (2.17).

It is also clear from the foregoing derivation of a 1-D equation that the pressure gradient term, $D(\rho)$ of (2.17), differs from the quantity $-(As^2/\rho^2)((1/4)+D_M)$ that would be expected in general tearing mode theory, as the singular surface is approached. The difference arises because the derivation of (2.17) is based on a single poloidal harmonic assumption, whereas retention of the coupled poloidal harmonics is required to capture the true value in the limit as $\rho \rightarrow \rho_s$. The approach outlined in Fitzpatrick *et al.* (1993) retains seven coupled poloidal harmonics, but its restrictions to weak shaping and low β severely impede its application to the pedestal. The single poloidal harmonic approach outlined in this paper accommodates strong shaping and β effects, but neglects poloidal mode coupling that is needed to describe D_M at the resonance and that may be important more globally. Nevertheless, for Δ' calculations at the foot of the pedestal where s^2 becomes large near a separatrix boundary, both the exact Mercier indices and the approximate (1-D) ones return to similar, low β, values (of 0 and 1), and the 1-D approximation may give a good indication of tearing instability in a rather simple manner.

Numerical investigations of H-mode equilibria are presently underway.

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Appendix A

We can generate unique expressions for the coefficients B, C and D, by exploiting the fact that all toroidal mode number dependencies in the 1-D tearing (2.17) can be expressed as powers, up to quadratic, of $m/(m-nq)$.

First, we collect all three terms in (2.16), that include parts proportional to $(m/(m-nq))^2$ and contribute to the coefficient $D(\rho)$ in (2.17), namely;

$$\frac{1}{\lambda}\langle X\rangle + \rho\langle U^2|\nabla\rho|^2\rangle - \rho\frac{n}{m}q'\langle U|\nabla\rho|^2\rangle. \tag{A 1}$$

Now replacing n by the identity $(nq-m+m)/q$ and using $s=\rho q'/q$, this expression becomes

$$\frac{1}{\lambda}\langle X\rangle + \rho\langle U^2|\nabla\rho|^2\rangle - s\langle U|\nabla\rho|^2\rangle + \frac{(m-nq)}{m}s\langle U|\nabla\rho|^2\rangle, \tag{A 2}$$

where the first three terms yield (2.21) for D and the last term now contributes to the expression for the coefficient C, rather than D. Three different terms from (2.16) and the final term of equation (A 2), contribute the term in (2.17) that is proportional to $m/(m-nq)$, with the following factor in the coefficient:

$$\frac{\mathrm{i}m}{\lambda}\langle W\rangle + \mathrm{i}m\rho\langle U(T-T^*)|\nabla\rho|^2\rangle - \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\langle U|\nabla\rho|^2\rangle\right) + s\langle U|\nabla\rho|^2\rangle, \tag{A 3}$$

where the last term is the contribution from equation (A 2) above. Using (2.5), (2.6) and (2.8) for T, U and W, the expression in (A 3) becomes:

$$-\frac{m}{n}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{R_0 g'}{f}\right]-q\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{R_0}{fg}\frac{\mu_0 p'}{B_0^2}\left\langle\frac{R^2}{R_0^2}\right\rangle\right]. \tag{A 4}$$

Now, on replacing m by the identity $m-nq+nq$, we obtain the following expression:

$$C=-q\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{R_0 g'}{f}+\frac{R_0}{fg}\frac{\mu_0 p'}{B_0^2}\left\langle\frac{R^2}{R_0^2}\right\rangle\right]-\frac{(m-nq)}{n}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{R_0 g'}{f}\right), \tag{A 5}$$

where the first two terms coincide with (2.20) for $C(\rho)$, and the third term contributes to the coefficient $B(\rho)$ and exactly cancels the remaining n dependence in B, leading to (2.19) for $B(\rho)$.

To demonstrate the second equality in (2.19) we consider cylindrical toroidal coordinates R, Z, ϕ. The Jacobian for the transformation $(R, Z)\rightarrow(\rho,\theta)$ is:

$$J=\frac{\rho R}{R_0}=\frac{\partial R}{\partial\theta}\frac{\partial Z}{\partial\rho}-\frac{\partial R}{\partial\rho}\frac{\partial Z}{\partial\theta}. \tag{A 6}$$

We can obtain $\nabla\rho$ and $\nabla\theta$, using

$$\left.\begin{aligned}\nabla R&=\frac{\partial R}{\partial\rho}\nabla\rho+\frac{\partial R}{\partial\theta}\nabla\theta\\ \nabla Z&=\frac{\partial Z}{\partial\rho}\nabla\rho+\frac{\partial Z}{\partial\theta}\nabla\theta\end{aligned}\right\} \tag{A 7}$$

and deduce:

$$\left.\begin{aligned}J^2|\nabla\rho|^2&=\left(\frac{\partial R}{\partial\theta}\right)^2+\left(\frac{\partial Z}{\partial\theta}\right)^2\\ J^2|\nabla\theta|^2&=\left(\frac{\partial R}{\partial\rho}\right)^2+\left(\frac{\partial Z}{\partial\rho}\right)^2\\ J^2\nabla\rho\boldsymbol{\cdot}\nabla\theta&=-\left[\frac{\partial R}{\partial\theta}\frac{\partial R}{\partial\rho}+\frac{\partial Z}{\partial\theta}\frac{\partial Z}{\partial\rho}\right].\end{aligned}\right\} \tag{A 8}$$

Squaring and adding (A 6) and (A 7) using (A 8) one finds:

$$\frac{R^2}{R_0^2|\nabla\rho^2|}+\rho^2\frac{\nabla\theta\boldsymbol{\cdot}\nabla\rho}{|\nabla\rho^2|}=\rho^2|\nabla\theta|^2. \tag{A 9}$$

Appendix B

The 1-D tearing (2.17) is expressed in terms of equilibrium variables ρ, g and f. More familiar variables are the equilibrium poloidal flux ψ and $I(\psi)$ as used by Hegna & Callen (1994). These are related by (2.27)–(2.29). In this appendix we give the form that (2.17) takes when expressed in these Hegna–Callen variables. Of the four terms in (2.17) we find:

$$\mathit{Term}\ 1\rightarrow\frac{I\rho}{q}\frac{\mathrm{d}}{\mathrm{d}\psi}\left[\frac{q}{I}\langle|\nabla\psi|^2\rangle\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\psi}\right], \tag{B 1}$$

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$$Term\ 2 \rightarrow -m^2\rho\langle|\nabla\theta|^2\rangle\tilde{A}, \tag{B 2}$$

$$Term\ 3 \rightarrow \frac{m\rho I}{(m-nq)}\frac{\mathrm{d}}{\mathrm{d}\psi}\left[I'(\psi) + \frac{\mu_0 p'(\psi)}{I(\psi)}\langle R^2\rangle\right]\tilde{A}, \tag{B 3}$$

$$Term\ 4 \rightarrow -\frac{\rho I m^2\mu_0 p'}{(m-nq)^2}\left[\frac{\mathrm{d}}{\mathrm{d}\psi}\left\langle\frac{R^2}{I}\right\rangle\right]\tilde{A}, \tag{B 4}$$

where, as in the work of Hegna and Callen, the dependent variable $\tilde{A}$ is the, single poloidal harmonic, tearing mode eigenfunction and $'$ denotes the radial derivative with respect to ψ. Finally, on multiplying through by the factor $q/\rho I$ we obtain the 1-D tearing equation in a rather simple form:

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\left[\frac{q}{I}\langle|\nabla\psi|^2\rangle\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\psi}\right] - \left\{\frac{m^2 q}{I}\langle|\nabla\theta|^2\rangle - \frac{mq}{(m-nq)}\frac{\mathrm{d}}{\mathrm{d}\psi}\left[I' + \frac{\mu_0 p'}{I}\langle R^2\rangle\right]\right.$$
$$\left. + \frac{m^2 q\mu_0 p'}{(m-nq)^2}\left[\frac{\mathrm{d}}{\mathrm{d}\psi}\left\langle\frac{R^2}{I}\right\rangle\right]\right\}\tilde{A} = 0. \tag{B 5}$$

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