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Coinvariant Algebras of Finite Subgroups of $\mathrm{SL}(3, C)$

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Abstract. For most of the finite subgroups of $\mathrm{SL}(3, \mathbf{C})$ we give explicit formulae for the Molien series of the coinvariant algebras, generalizing McKay's formulae [McKay99] for subgroups of $\mathrm{SU}(2)$. We also study the G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^3)$ for any finite subgroup G of $\mathrm{SO}(3)$, which is known to be a minimal (crepant) resolution of the orbit space $\mathbf{C}^3/G$. In this case the fiber over the origin of the Hilbert-Chow morphism from $\mathrm{Hilb}^G(\mathbf{C}^3)$ to $\mathbf{C}^3/G$ consists of finitely many smooth rational curves, whose planar dual graph is identified with a certain subgraph of the representation graph of G. This is an $\mathrm{SO}(3)$ version of the McKay correspondence in the $\mathrm{SU}(2)$ case.

0 Introduction

Let G be a finite subgroup of $\mathrm{SL}(n, \mathbf{C})$, S_G the coinvariant algebra of G, and $(S_G)_i$ the subspace of S_G of homogeneous degree i respectively. For each irreducible representation ρ of G, let $\langle \rho, (S_G)_i \rangle_G$ be the multiplicity of ρ in $(S_G)_i$ and define the Molien series $P_{S_G,\rho}(t)$ of S_G for ρ to be

$$P_{S_G,\rho}(t) = \sum \langle \rho, (S_G)_i \rangle_G t^i.$$

Since S_G is finite-dimensional, $P_{S_G,\rho}(t)$ is a polynomial of t. One can define similarly the Molien series $P_{M,\rho}(t)$ for an arbitrary graded G-module M with finite dimensional graded pieces. If M is the polynomial algebra S in two variables and if G is a subgroup of $\mathrm{SU}(2)$, then the Molien series $P_{S,\rho}(t)$ of S is a rational function of t by [Springer87] and it is well understood as is the connection with the Dynkin diagram corresponding to G (*cf.* [Springer87] and [McKay99]). In these cases the Molien series $P_{S_G,\rho}(t)$ of S_G is easily derived from the formula for $P_{S,\rho}(t)$.

The first purpose of this paper is to give an explicit formula for $P_{S_G,\rho}$ when G is one of the exceptional finite subgroups of $\mathrm{SL}(3, \mathbf{C})$ of type from (E) to (L) in the notation of [YY93]. Using the Koszul complex with G-action, we derive a certain system of equations analogous to the $\mathrm{SU}(2)$ case [McKay99] satisfied by the Molien series $P_{S,\rho}$. The equations are obtained just by taking alternating sums of componentwise generating functions of G-modules in the Koszul complex. They are given explicitly in

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terms of irreducible decompositions of tensor products with the natural representation ρ_{nat} and its second exterior product $\bigwedge^2 \rho_{\mathrm{nat}}$. This will be discussed in Section 2. The consequence of this section enables us to compute $P_{S,\rho}$ explicitly later. However the calculation of $P_{S_G,\rho}$ in the exceptional cases (E)–(L) is much harder, which will be discussed in Sections 4 and 5. This study of the Molien series $P_{S_G,\rho}$ was in fact motivated by the study of the G-orbit Hilbert scheme explained below, in particular by the study of $\pi^{-1}(0)$.

For a positive integer N, $\mathrm{Hilb}^N(\mathbf{C}^n)$ is the universal scheme which parametrizes all zero-dimensional subschemes of $\mathbf{C}^3$ of length N. For a finite subgroup G of $\mathrm{GL}(n,\mathbf{C})$, we choose $N = |G|$, the order of G. Then the group G acts in the natural manner on $\mathrm{Hilb}^{|G|}(\mathbf{C}^n)$. The G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^n)$ is by definition the unique irreducible component of the G-invariant part of $\mathrm{Hilb}^{|G|}(\mathbf{C}^n)$ dominating $\mathbf{C}^n/G$, the G-invariant part of the corresponding Chow scheme of $|G|$ points. In other words, $\mathrm{Hilb}^G(\mathbf{C}^n)$ is the universal subscheme of the Hilbert scheme $\mathrm{Hilb}^{|G|}(\mathbf{C}^n)$ which parametrizes all smoothable scheme-theoretic G-orbits of length $|G|$. The G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^n)$ is a fairly natural algebro-geometric object which incorporates all representation-theoretic information about G as a subgroup of $\mathrm{GL}(n,\mathbf{C})$. It has already been studied in detail in the $\mathrm{SU}(2)$ case [IN99] and in the case where G is a noncommutative simple subgroup A_5 or $\mathrm{PSL}(2,7)$ of $\mathrm{SL}(3,\mathbf{C})$ [GNS00]. The scheme $\mathrm{Hilb}^N(\mathbf{C}^n)$ is known to be very singular if $n \geq 3$. However for a finite subgroup G of $\mathrm{SL}(3,\mathbf{C})$, $\mathrm{Hilb}^G(\mathbf{C}^3)$ is known to be nonsingular by [N01] in the abelian case and by [BKR01] in the general case.

The second purpose of the article is to study $\mathrm{Hilb}^G(\mathbf{C}^3)$, among other things, the fiber $\pi^{-1}(0)$ of the Hilbert-Chow morphism $\pi\colon \mathrm{Hilb}^G(\mathbf{C}^3) \to \mathbf{C}^3/G$ when G is a finite subgroup of $\mathrm{SO}(3)$. This will be discussed in Section 3.

It is well known that there is a surjective homomorphism from $\mathrm{SU}(2)$ onto $\mathrm{SO}(3)$ having ± 1 as its kernel, by which non-abelian subgroups of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ correspond bijectively. For a subgroup G of $\mathrm{SO}(3)$ we define the representation graph $R(G)$ of G by using the irreducible decompositions of tensor products with ρ_{nat} in the same manner as in the $\mathrm{SU}(2)$ case. First we observe that $\pi^{-1}(0)$ is a union of finitely many smooth rational curves. So we define, in the same way as in the $\mathrm{SU}(2)$ case, the planar dual graph $\bar{R}(G)$ of $\pi^{-1}(0)$ by associating a vertex to each rational curve in $\pi^{-1}(0)$, and by associating an edge connecting a pair of the vertices to each intersection point of the corresponding curves. Then it turns out that the planar dual graph $\bar{R}(G)$ is identified with a particular subgraph of $R(G)$. In other words, every irreducible rational curve in $\pi^{-1}(0)$ is labeled by one of the nontrivial irreducible representations of G and vice versa, whose intersections are described purely in terms of irreducible decompositions of tensor products with ρ_{nat} in a manner similar to the $\mathrm{SU}(2)$ case. Thus we have a complete description of $\pi^{-1}(0)$ in the $\mathrm{SO}(3)$ case. However in almost all cases other than (A), (H) and (I) in the notation of [YY93] the precise structure of $\pi^{-1}(0)$ is yet to be determined.

This paper is organized as follows. In Section 1, we explain basic lemmas necessary for computing $P_{S_G,\rho}$. In Section 2, we first recall the Koszul complex over S and show that any alternating sum of componentwise generating functions of the G-modules in the Koszul complex is equal to zero, which yields a Springer-McKay type identity of $P_{S,\rho}$. In Section 3, we describe $\pi^{-1}(0)$ completely when G is a subgroup of $\mathrm{SO}(3)$. In Sections 4 and 5 we give tables of $P_{S_G,\rho}$ for every finite subgroup G of $\mathrm{SL}(3,\mathbf{C})$ of type from (E) to (L) and every non-trivial representation ρ of G.

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1 The Coinvariant Algebra for a Finite Subgroup G of $\mathrm{SL}(3,\mathbf{C})$

1.1 The Molien Series

Let V be an n-dimensional complex vector space, $V^\vee$ the dual of V and G a finite subgroup of $\mathrm{GL}(V)$. We denote by ρ the matrix representation of G afforded by the natural inclusion of G into $\mathrm{GL}(V)$ and by $\rho^\vee$ its contragredient representation. As usual we call ρ the natural representation of G. We use the same notation as in [GNS00]; in particular we denote by $S = S(V^\vee)$, $\mathfrak{m} = S_+$, S^G and S^G_+ respectively the symmetric algebra of $V^\vee$ over $\mathbf{C}$, the maximal ideal of S of the origin, the invariant algebra of G, and the maximal ideal of S^G of the origin. Let $\mathfrak{n}$ be the ideal of S generated by S^G_+ and $S_G := S/\mathfrak{n}$ the coinvariant algebra of G. Since $\mathfrak{n}$ is a graded ideal of S, S_G is a graded algebra, too.

By the Noether normalization lemma, we can take a minimal system of homogeneous parameters $f_1, f_2, \dots, f_n$ of S^G so that S^G is a finite module over $\mathbf{C}[f_1, \dots, f_n]$. Extending them we choose a minimal system of homogeneous generators $f_1, f_2, \dots, f_r$ of S^G and fix them once for all. The ideal $\mathfrak{n}$ of S is generated by $f_1, f_2, \dots, f_r$.

Let $\hat{G} = \{\rho_0 = 1, \rho_1, \dots, \rho_s\}$ be the set of representatives of equivalence classes of all irreducible representations of G and χ_i the character of ρ_i for $0 \le i \le s$. For an arbitrary graded $\mathbf{C}G$-module $M = \bigoplus_{i\ge 0} M_i$ with $\dim M_i < \infty$, we define the Molien series of M for ρ_j by

$$P_{M,\rho_j}(t) = \sum_{i\ge 0} \langle M_i, \rho_j\rangle_G t^i,$$

where

$$\langle M_i, \rho_j\rangle_G = \dim \mathrm{Hom}_G(\rho_j, M_i) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_j(g)}\,\mathrm{Tr}_{M_i}(g).$$

The following is derived easily from the formula in [Bourbaki, Lemme 2, p. 110]

$$P_{S,\rho_j}(t) = \frac{1}{|G|}\sum_{g\in G} \frac{\overline{\chi_j(g)}}{\det\bigl(1-\rho^\vee(g)t\bigr)}. \tag{1}$$

Now we recall from [Stanley79, (4.9)].

Theorem 1.2 *Let $f_1, f_2, \dots, f_r$ be homogeneous generators of S^G chosen as above, $d_i = \deg f_i$, $(f_1, f_2, \dots, f_n)$ the ideal of S generated by $f_1, f_2, \dots, f_n$, and let $R = S/(f_1, f_2, \dots, f_n)$. Then as $\mathbf{C}G$-modules we have*

$$S \simeq R \otimes \mathbf{C}[f_1, f_2, \dots, f_n] \quad \text{and} \quad R \simeq (\mathbf{C}G)^e$$

where $e = |G|^{-1} d_1 d_2 \cdots d_n$.

Proposition 1.3 *Keeping the notations as above, we have*

(i)
$$P_{R,\rho_j}(t) = \frac{\prod_{i=1}^n (1-t^{d_i})}{|G|} \sum_{g\in G} \frac{\overline{\chi_j(g)}}{\det(1-\rho^\vee(g)t)}.$$

(ii) $P_{R,\rho_j}(t) - P_{S_G,\rho_j}(t)$ *is a polynomial with non-negative integer coefficients.*

(iii)
$$\sum_{j=0}^{s} (\deg \rho_j) P_{S_G,\rho_j}(t) = \sum_{j\geq 0} \dim(S_G)_i t^i.$$

Proof (i) It follows from Theorem 1.2 that $P_{S,\rho_j}(t) = P_{R,\rho_j}(t)/\prod_{i=1}^n (1-t^{d_i})$. From Molien's formula (1), we infer (i).

(ii) Since we have a canonical surjection from R to S_G, $P_{R,\rho_j}(t) - P_{S_G,\rho_j}(t)$ has non-negative integer coefficients.

(iii) Let $S_G = \bigoplus_{j=0}^s (S_G)_{\rho_j}$ be the decomposition into homogeneous components, namely ρ_j-factors $(S_G)_{\rho_j}$ of S_G. Since $\dim(S_G)_{\rho_j} = (\deg \rho_j)\langle S_G, \rho_j\rangle_G$, the above equation is clear from the definition of $P_{S_G,\rho_j}(t)$. ∎

We note that if there exists a complex reflection group $\tilde{G}$ of $\mathrm{GL}(V)$ containing G with $[\tilde{G}:G] = 2$, then it is easier to calculate $P_{S_G,\rho_j}(t)$ by using the following

Theorem 1.4 ([Bourbaki] or [GNS00, 1.6]) *Assume that there exists a complex reflection subgroup $\tilde{G}$ of $\mathrm{GL}(V)$ containing G with $[\tilde{G}:G] = 2$.*

(i) *There exist n homogeneous $\tilde{G}$-invariants $f_1, f_2, \dots, f_n$ such that as $\mathbf{C}\tilde{G}$-modules $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, \dots, f_n]$ and $S_{\tilde{G}} = S/(f_1, f_2, \dots, f_n) \simeq \mathbf{C}\tilde{G}$.*

(ii) *Let $f_{n+1} = \mathrm{Jac}(f_1, f_2, \dots, f_n)$. Then we have*
$$S^G = \mathbf{C}[f_1, f_2, \dots, f_n, f_{n+1}] \quad \text{and} \quad S_{\tilde{G}} \simeq S_G \oplus \mathbf{C} f_{n+1}.$$

Moreover
$$(S_{\tilde{G}})_k \simeq \begin{cases} (S_G)_k, & \text{if } k < d_{n+1}, \\ \mathbf{C} f_{n+1}, & \text{if } k = d_{n+1}, \\ 0, & \text{if } k > d_{n+1}, \end{cases}$$

where $d_{n+1} = \deg f_{n+1} = \sum_{i=1}^n (d_i - 1)$.

Corollary 1.5 *Under the same assumptions in Theorem 1.4*

$$P_{S_G,\rho_j}(t) = P_{S_{\tilde{G}},\rho_j}(t) = \prod_{i=1}^n (1-t^{d_i}) P_{S,\rho_j}(t),$$

$$P_{S_G,\rho_0}(t) = P_{S_{\tilde{G}},\rho_0}(t) + t^{n+1} = \prod_{i=1}^n (1-t^{d_i}) P_{S,\rho_j}(t) + t^{d_{n+1}}.$$

Proof Immediate from Theorem 1.4. ■

Remark 1.6 Let G be a finite subgroup of $\mathrm{SL}(3,\mathbf{C})$ of exceptional type (E)–(L). Then homogeneous generators of S^G are known explicitly in [YY93]. Moreover, since $(S_G)_i \simeq S_i/(\mathfrak{n})_i$ and $(\mathfrak{n})_i = V^\vee \cdot (\mathfrak{n})_{i-1} + \sum_{\deg f_j = i} \mathbf{C} f_j$, we can calculate $(\mathfrak{n})_i$ inductively. Thus all the informations of Proposition 1.3 are available, which turns out to be sufficient to determine $P_{S_G,\rho_j}(t)$ by the case-by-case examination. The results are summarized in Sections 4 and 5.

Either of the groups of type (H), (I) and (L) is a subgroup of some complex reflection group of index two, while the group of type (E), (F) or (J) is a subgroup of some complex reflection group of index 6, 3 or 12 respectively. In these cases we can apply [Steinberg64] and [Stanley79] to describe $R := S/(f_1, f_2, f_3)$ in some detail. However no group of type (G) or (K) is a subgroup of a complex reflection group. Nevertheless in any case from (E) to (L) the algebra R has a remarkable duality as in the cases of complex reflection groups. We will discuss it elsewhere.

2 Koszul Complex and Springer-McKay Identities of Molien Series

We keep the previous notation. We start with the Koszul complex for the symmetric algebra $S = S(V^\vee)$ (*cf.* [Lang84, XVI Section 10]).

Lemma 2.1 *Let $\bigwedge^k V^\vee$ be the k-th alternating product of $V^\vee$.*

(i) *There is a unique homomorphism*

$$d_k \colon \bigwedge^k V^\vee \otimes S \to \bigwedge^{k-1} V^\vee \otimes S$$

such that for $x_i \in V^\vee$ and $y \in S$

$$d_k\big((x_1 \wedge x_2 \wedge \cdots \wedge x_k) \otimes y\big) = \sum_{i=1}^{k} (-1)^{i-1} (x_1 \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k) \otimes (x_i \cdot y).$$

(ii) *There is an exact sequence with d_k given by* (i)

$$0 \to \bigwedge^n V^\vee \otimes S \xrightarrow{d_n} \bigwedge^{n-1} V^\vee \otimes S \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} V^\vee \otimes S \xrightarrow{d_1} S \xrightarrow{d_0} \mathbf{C} \to 0.$$

(iii) *For each integer $m \geq 1$ we have an exact sequence*

$$0 \to \bigwedge^n V^\vee \otimes S_{m-n} \to \bigwedge^{n-1} V^\vee \otimes S_{m-n+1} \to \cdots \to S_m \to 0,$$

where $S_j = 0$ for $j < 0$.

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(iv) *For each integer $m \geq 1$ and for each irreducible representation ρ_j $(0 \leq j \leq s)$, we have an exact sequence*

$$0 \to \Big(\bigwedge^{n} V^{\vee} \otimes S_{m-n}\Big)_{\rho_j} \to \Big(\bigwedge^{n-1} V^{\vee} \otimes S_{m-n+1}\Big)_{\rho_j} \to \cdots \to (S_m)_{\rho_j} \to 0.$$

Proof For a proof of (i), (ii) and (iii), see [Lang84, (10.13) and (10.14)]. Since d_k is a G-homomorphism, we decompose the exact sequence of (iii) into ρ_j-components, which proves (iv). ■

We denote by $\rho^{(k)}$ (resp. $\rho^{\vee(k)}$) the $\mathbf{C}G$-module $\bigwedge^k V$ (resp. $\bigwedge^k V^{\vee}$). Note that $\rho^{(0)} = 1$, $\rho^{(1)} = \rho$, $\rho^{(n)} = \det$, and $\rho^{\vee(k)}$ is the dual $\mathbf{C}G$-module of $\rho^{(k)}$. Define non-negative integers $a_{i,j}^{(k)}$ by

$$\rho^{(k)} \otimes \rho_i = \sum_{j=0}^{s} a_{ij}^{(k)} \rho_j, \quad \text{for } 0 \leq i \leq s \text{ and } 0 \leq k \leq n. \tag{2}$$

Theorem 2.2 *The Molien series $P_{S,\rho_j}(t)$ satisfy the following equations:*

$$\sum_{k=0}^{n} \sum_{j=0}^{s} (-1)^k a_{ij}^{(k)} t^k P_{S,\rho_j}(t) = \delta_{i,0} \quad \textit{for } i = 0, 1, \ldots, s.$$

Proof We see

$$\begin{aligned} \dim\Big(\bigwedge^{k} V^{\vee} \otimes S_{m-k}\Big)_{\rho_i} &= \deg(\rho_i) \dim \operatorname{Hom}_G(\rho_i, \rho^{\vee(k)} \otimes S_{m-k}) \\ &= \deg(\rho_i) \dim \operatorname{Hom}_G(\rho^{(k)} \otimes \rho_i, S_{m-k}) \\ &= \deg(\rho_i) \sum_{j=0}^{s} a_{ij}^{(k)} \dim \operatorname{Hom}_G(\rho_j, S_{m-k}). \end{aligned}$$

Thus we obtain

$$\sum_{m \geq 0} \left(\dim\Big(\bigwedge^{k} V^{\vee} \otimes S_{m-k}\Big)_{\rho_i} \right) t^m = \deg(\rho_i) \sum_{j=0}^{s} a_{ij}^{(k)} t^k P_{S,\rho_j}(t).$$

Hence our theorem follows from Lemma 2.1 (ii) and (iv). ■

Remark 2.3 This proposition can be proved directly by using (1).

Corollary 2.4 *Keep the same notation in Theorem 2.2.*

(i) *If G is a subgroup of* $\mathrm{SL}(V)$*, then*

$$\sum_{k=1}^{n-1}\sum_{j=0}^{s}(-1)^k a_{ij}^{(k)} t^k P_{S,\rho_j}(t) = \bigl(-1-(-1)^n t^n\bigr) P_{S,\rho_i}(t) + \delta_{i,0},$$

(ii) *If G is a subgroup of* $\mathrm{SL}(2,\mathbf{C})$*, then*

$$\sum_{j=0}^{s} a_{ij}^{(1)} P_{S,\rho_j}(t) = (t+t^{-1})P_{S,\rho_i}(t) - t^{-1}\delta_{i,0},$$

(iii) *If G is a subgroup of* $\mathrm{SL}(3,\mathbf{C})$ *and if $\rho^\vee = \rho$, then*

$$\sum_{j=0}^{s} a_{ij}^{(1)} P_{S,\rho_j}(t) = (t+1+t^{-1})P_{S,\rho_i}(t) + (t^2-t)^{-1}\delta_{i,0}.$$

(The assumption in (iii) is satisfied if $G \subset \mathrm{SO}(3)$.)

Proof If G is a finite subgroup of $\mathrm{SL}(V)$, then $\rho^{(0)}$ and $\rho^{(n)}$ are trivial. So (i) follows at once from Theorem 2.2. If $\dim V = 2$, we obtain (ii) by dividing both sides of (i) by $-t$. Under the assumption of (iii), we have $\rho^{(1)} = \rho^{(2)} = \rho$. Dividing both sides of (i) by (t^2-t), we obtain (iii). ■

Put $F_j(t) = P_{S,\rho_j}(t)\prod_{i=1}^{n}(1-t^{d_i})$ for $0 \le j \le s$. By Theorem 1.4

$$F_j(t) = \begin{cases} 1+t^{d_{n+1}} & \text{if } j=0 \\ P_{S_G,\rho_j}(t) & \text{if } j \neq 0. \end{cases}$$

The next corollary is immediate from Corollary 2.4.

Corollary 2.5 *Keep the notation as above. Let $0 \le i \le s$. Then*

(i) *If G is a finite subgroup of* $\mathrm{SL}(2,\mathbf{C})$*, then*

$$\sum_{j=0}^{s} a_{ij}^{(1)} F_j(t) = (t+t^{-1})F_i(t) - \frac{(1-t^{d_1})(1-t^{d_2})}{t}\delta_{i,0}.$$

(ii) *If G is a finite subgroup of* $\mathrm{SO}(3)$*, then*

$$\sum_{j=0}^{s} a_{ij}^{(1)} F_j(t) = (t+1+t^{-1})F_i(t) + \frac{(1-t^{d_1})(1-t^{d_2})(1-t^{d_3})}{(t^2-t)}\delta_{i,0}.$$

Remark 2.6 The system of equations in Corollary 2.4(ii) were given in [Springer87] and [McKay99] by using corresponding Coxeter-Dynkin diagrams, or McKay's semi-affine graphs. Corollary 2.4(i) claims, roughly speaking, that one can calculate all the Molien series once one knows $a_{ij}^{(k)}$, in particular only $a_{ij}^{(1)}$ when $G \subset \mathrm{SL}(2,\mathbf{C})$ or $G \subset \mathrm{SO}(3)$. In this sense the representation graph (or rather the indices $a_{ij}^{(1)}$) of a subgroup G of $\mathrm{SO}(3)$ plays the same role in calculating Molien series as the Coxeter-Dynkin diagram for a finite subgroup of $\mathrm{SL}(2,\mathbf{C})$.

2.7 Complex Reflection Groups

If G is a finite subgroup of $\mathrm{SL}(2,\mathbf{C})$ or $\mathrm{SO}(3)$, there exists a complex reflection group $\tilde{G}$ containing G with $[\tilde{G}:G]=2$. We list all such pairs G and $\tilde{G}$ in Table 1 and Table 2. We use the notation in [Cohen76]; the group G_i is the complex reflection group with Shephard-Todd number i. The symbol $W(A)$ stands for the Weyl group of type A. The integer d_i in the tables is the degree of f_i defined in Theorem 1.4.

G in $\mathrm{SL}(2,\mathbf{C})$	order	$\tilde{G}$	d_1, d_2
cyclic	l	$W(I_2^{(l)})$	$2, l$
binary dihedral	$4l$	$G(2l, l, 2)$	$4, 2l$
binary tetrahedral	24	G_{12}	$6, 8$
binary octahedral	48	G_{13}	$8, 12$
binary icosahedral	120	G_{22}	$12, 20$

Table 1: Subgroups of $\mathrm{SL}(2,\mathbf{C})$

G in $\mathrm{SO}(3)$	order	$\tilde{G}$	d_1, d_2, d_3
cyclic	l	$W(I_2^{(l)})$	1,2,l
dihedral	2l	$W(I_2^{(l)} \times A_1)$	2,2,l
tetrahedral $(\simeq A_4)$	12	$W(A_3)$	2,3,4
octahedral $(\simeq S_4)$	24	$W(B_3)$	2,4,6
icosahedral $(\simeq A_5)$	60	$W(H_3)$	2,6,10

Table 2: Subgroups of $\mathrm{SO}(3)$

3 Geometric McKay Correspondence for Subgroups of $\mathrm{SO}(3)$

Let $\pi\colon \operatorname{Hilb}^G(\mathbf{C}^3) \to \mathbf{C}^3/G$ be the Hilbert-Chow morphism for $G \subset \mathrm{SO}(3)$.

Theorem 3.1 *Let G be a finite subgroup of $\mathrm{SO}(3)$. For $I \in \operatorname{Hilb}^G(\mathbf{C}^3)$ with $I \subset \mathfrak{m}$, we define $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$. For $1 \le i \le s$, we define $C_j = \{I \in \operatorname{Hilb}^G(\mathbf{C}^3)\ ;\ V(I) \supset \rho_j, I \subset \mathfrak{m}\}$. Then*

(i) $C_j \simeq \mathbf{P}^1$ *and* $\pi^{-1}(0) = \bigcup_{j=1}^{s} C_j$.
(ii) *If $I \in C_j$ and $I \notin C_i$ for any $j \neq i$, then $V(I) \simeq \rho_j$ as G-modules.*
(iii) *If only two rational curves C_i and C_j meet at $I \in \pi^{-1}(0)$, then C_i and C_j intersect at I transversally and $V(I) \simeq \rho_i + \rho_j$.*
(iv) *If G is either cyclic, A_4 or D_{4m+2}, then there are no three rational curves meeting at a point of $\pi^{-1}(0)$.*
(v) *If $G = D_{4m}, S_4$ or A_5, then there is a unique $I \in \pi^{-1}(0)$ such that $\{I\} = C_i \cap C_j \cap C_k$ for $\rho_i, \rho_j, \rho_k \in \hat{G}$ all distinct. In this case $V(I) \simeq \rho_i + \rho_j + \rho_k$ and the curves C_i, C_j, C_k meet transversally at I as coordinate axes of $(\mathbf{C}^3, 0)$.*
(vi) *No four rational curves C_i meet at a point of $\pi^{-1}(0)$.*

Our proof of Theorem 3.1 is carried out by the case by case examination. When G is abelian, our theorem is proved by the same argument as in the two dimensional case. When G is isomorphic to the alternating group A_4 or A_5, our theorem has been proved in [GNS00]. So we only need to prove our theorem when G is a dihedral group or $G = S_4$. We will give a proof of it in the subsections 3.4, 3.5 and 3.6.

3.2 Graphs of G

Here we define three graphs for a finite subgroup G of $\mathrm{SO}(3)$.

First we define the planar dual graph $\bar{R}(G)$ of $\pi^{-1}(0)$ as follows: the set of vertices of $\bar{R}(G)$ is $\{C_j\}_{1\le j\le s}$; C_i and C_j are joined by a single edge if and only if $C_i \cap C_j \neq \phi$. We note that in Theorem 3.1 there are three rational curves C_i, C_j and C_k in $\pi^{-1}(0)$ meeting at a point, for which we define a planar triangle in $\bar{R}(G)$ with three vertices C_i, C_j and C_k instead of a two cell. See Table 3.

Next we define the (unoriented) representation graph $R(G)$ of G as follows: the set of vertices is $\hat{G}$; let $a_{i,j}^{(1)}$ be the integer defined in (2); ρ_i and ρ_j are joined by an edge of multiplicity $a_{i,j}^{(1)}$ if $a_{i,j}^{(1)} \neq 0$, where if $i = j$ the edge joining ρ_i with itself is understood as a loop of multiplicity $a_{i,i}^{(1)}$. We note $a_{i,j}^{(1)} = 0$ or 1 for $i \neq j$, while $a_{i,i}^{(1)} = 0, 1,$ or 2. We also note that $a_{i,j}^{(1)} = a_{i,j}^{(2)}$ for any finite subgroup G of $\mathrm{SO}(3)$.

Finally we define a subgraph $R_0(G)$ of $R(G)$ as follows: the set of vertices is $\{\rho_j\}_{1\le j\le s}$ and ρ_i and ρ_j are joined by a single edge if and only if $i \neq j$ and $a_{i,j}^{(1)} \neq 0$. In other words, $R_0(G)$ is the subgraph of $R(G)$ obtained from $R(G)$ by removing the vertex ρ_0, all the edges starting from ρ_0 and all the loops in $R(G)$.

The following theorem is a corollary to the proof of Theorem 3.1 once we calculate the representation graph $R(G)$.

Theorem 3.3 *$\bar{R}(G)$ is isomorphic to $R_0(G)$ under the map $C_i \mapsto \rho_i$ $(1 \le i \le s)$. The graphs $\bar{R}(G)$ and $R(G)$ are given in Table 3.*

In the rest of this section we give proofs of Theorem 3.1 in the cases where G is a dihedral group or G is isomorphic to S_4.

3.4 Proof of Theorem 3.1—The Dihedral Group of Order $2\ell = 4m$

Let G be the dihedral group of order 2ℓ:

$$G = \left\langle \sigma = \begin{pmatrix} \varepsilon^{-1} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle, \quad \text{where } \varepsilon = e^{2\pi i/\ell}.$$

We define

$$f_1 = z^2, \ f_2 = xy, \ f_3 = x^\ell + y^\ell, \ f_4 = z(x^\ell - y^\ell).$$

Then we see $\{f_1, f_2, f_3, f_4\}$ is a system of generators of S^G which satisfies

$$f_4^2 - f_1 f_3^2 + 4 f_1 f_2^\ell = 0,$$

regardless of the parity of ℓ.

First in this subsection we consider the case where ℓ is even. So we write $\ell = 2m$, $|G| = 4m$. The character table of G is given in Table 4.

The coinvariant algebra S_G splits into irreducible components as in Table 5. Using Table 5 we define ideals in $\mathrm{Hilb}^G(\mathbf{C}^3)$ $[a\!:\!b] \in \mathbf{P}^1$ as in [GNS00].

$$\begin{aligned} I([a\!:\!b]_{1_2}) &= \left(az + b(x^{2m} - y^{2m}), xz, yz\right) + \mathfrak{n}, \\ I([a\!:\!b]_{1_3}) &= \left(a(x^m + y^m) + b(x^m - y^m)z, x^{m+1}, y^{m+1}, (x^m + y^m)z\right) + \mathfrak{n}, \\ I([a\!:\!b]_{1_4}) &= \left(a(x^m - y^m) + b(x^m + y^m)z, x^{m+1}, y^{m+1}, (x^m - y^m)z\right) + \mathfrak{n}, \\ I([a\!:\!b]_{2_j}) &= S[G] \cdot (ax^j z + by^{2m-j}, x^{j+1}z, x^{2m-j+1}) + \mathfrak{n}, \quad (i = 1, 2, \dots, m-1). \end{aligned}$$

It is clear that $V\left(I([a\!:\!b]_\rho)\right) \simeq \rho$ as G-modules. We note that the following exhaust all the possible cases of coincidence between $I([a\!:\!b]_\rho)$.

$$\begin{aligned} I([0\!:\!1]_{1_2}) &= I([1\!:\!0]_{2_1}), \\ I([0\!:\!1]_{2_j}) &= I([1\!:\!0]_{2_{j+1}}), \quad \text{for } j = 1, 2, \dots, m-2, \\ I([0\!:\!1]_{2_{m-1}}) &= I([0\!:\!1]_{1_3}) = I([0\!:\!1]_{1_4}). \end{aligned}$$

Now we prove

$$\pi^{-1}(0) = \bigcup_{\rho \in \hat{G} \setminus \{1_1\}} I([a\!:\!b]_\rho).$$

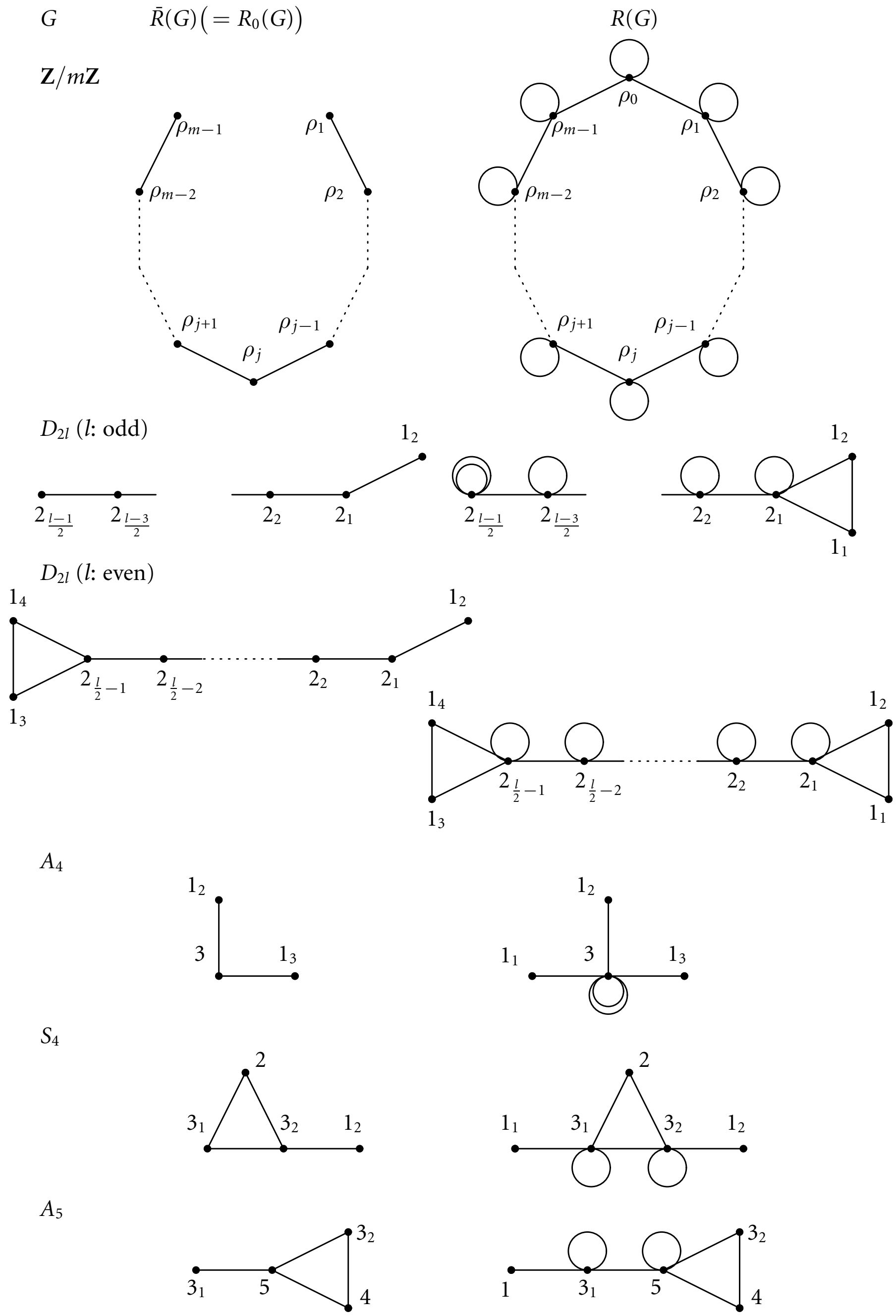

Table 3: Graphs of subgroups of SO(3)

c.c	1	-1	τ	$\tau\sigma$	σ^i
age	0	1	1	1	1
$\sharp$	1	1	m	m	2
1_1	1	1	1	1	1
1_2	1	1	-1	-1	1
1_3	1	$(-1)^m$	1	-1	$(-1)^i$
1_4	1	$(-1)^m$	-1	1	$(-1)^i$
2_j	2	$(-1)^j 2$	0	0	$\varepsilon^{ij}+\varepsilon^{-ij}$
					$(1 \le i, j \le m-1)$

Table 4: Characters of $G(D_{2\ell})$, $\ell = 2m$:even.

degree	$(S_G)_j$	irred. factors
1	$\langle x, y\rangle \oplus \langle z\rangle$	$2_1 + 1_2$
$2 \le j \le m-1$	$\langle x^j, y^j\rangle \oplus \langle x^{j-1}z, -y^{j-1}z\rangle$	$2_j + 2_{j-1}$
m	$\langle x^m + y^m\rangle \oplus \langle x^m - y^m\rangle$	
	$\oplus\langle x^{m-1}z, -y^{m-1}z\rangle$	$1_3 + 1_4 + 2_{m-1}$
$m+1$	$\langle y^{m+1}, x^{m+1}\rangle \oplus \langle (x^m - y^m)z\rangle$	
	$\oplus\langle (x^m + y^m)z\rangle$	$2_{m-1} + 1_3 + 1_4$
$m+2 \le j \le 2m-1$	$\langle y^j, x^j\rangle \oplus \langle y^{j-1}z, -x^{j-1}z\rangle$	$2_{2m-j} + 2_{2m-j+1}$
$2m$	$\langle x^{2m} - y^{2m}\rangle \oplus \langle y^{2m-1}z, -x^{2m-1}z\rangle$	$1_2 + 2_1$

Table 5: The coinvariant algebra of $G(D_{2\ell})$, $\ell = 2m$: even.

It is immediate from the definition and the Diagram D_{4m} (see 3.7) that $I([a\colon b]_\rho)$ are contained in $\pi^{-1}(0)$. Conversely let I be an ideal contained in $\pi^{-1}(0)$, that is, $\mathfrak{n} \subset I \subset \mathfrak{m}$ and $S/I \simeq \mathbf{C}[G]$. By the Diagram D_{4m}, it is easy to see that $x^{2m-j}z$, $y^{2m-j}z \in I$ for all $j = 1, 2, \dots, m-1$ and that $x^j + ax^jz + by^{2m-j} \notin I$ for any $a, b \in \mathbf{C}$ and $j = 1, 2, \dots, m-1$. If $x^jz + by^{2m-j} \in I$ for some $b \neq 0$ and some $j = 1, 2, \dots, m-1$, then we have $I([1\colon b]_{2_j}) \subset I$ which implies $I([1\colon b]_{2_j}) = I$.

Now we assume the contrary, that is, that $x^jz + by^{2m-j} \notin I$ for any nonzero b and any $j = 1, \dots, m-1$. Then by the condition $S/I \simeq \mathbf{C}[G]$ we have either $x^jz \in I$ or $y^{2m-j} \in I$. If there is $j \ge 2$ such that $x^jz \in I$, $x^{j-1}z \notin I$, then $y^{2m-j+1} \in I$. It follows that $I = I([1\colon 0]_{2_j})$. If $xz \in I$, then $I = I([a\colon b]_{1_2})$.

It remains to consider the case where there is no j such that $x^jz \in I$. Hence $y^{m+1} \in I$. If $x^m + y^m + b(x^m - y^m)z \in I$ (resp. $x^m - y^m + b(x^m + y^m)z \in I$) for some $b \in \mathbf{C}$, then $I = I([1\colon b]_{1_3})$(resp. $I([1\colon b]_{1_4})$). Otherwise I contains $(x^m - y^m)z$

and $(x^m + y^m)z$ and then we have $I = I([0:1]_{1_3})$. Thus we complete the proof of Theorem 3.1 when G is a dihedral group of order $4m$.

3.5 Proof of Theorem 3.1—The Dihedral Group of Order $4m + 2$

Now we consider the second case where G is a dihedral group of order $2\ell = 4m + 2$. Table 6 is the character table of G. The coinvariant algebra S_G splits into irreducible components as in Table 7.

c.c	1	τ	σ^i
age	0	1	1
$\sharp$	1	$2m+1$	2
1_1	1	1	1
1_2	1	-1	1
2_j	2	0	$\varepsilon^{ij} + \varepsilon^{-ij}$
			$(1 \le i, j \le m)$

Table 6: Characters of $G(D_{2\ell})$, $\ell = 2m + 1$:odd.

degree	$(S_G)_j$	irred. factors
1	$\langle x, y\rangle \oplus \langle z\rangle$	$2_1 + 1_2$
j	$\langle x^j, y^j\rangle \oplus \langle x^{j-1}z, -y^{j-1}z\rangle$	$2_j + 2_{j-1}$
		$(2 \le j \le m-1)$
m	$\langle x^m, y^m\rangle \oplus \langle x^{m-1}z, -y^{m-1}z\rangle$	$2_m + 2_{m-1}$
$m+1$	$\langle y^{m+1}, x^{m+1}\rangle \oplus \langle x^m z, -y^m z\rangle$	$2_m + 2_m$
$m+2$	$\langle y^{m+2}, x^{m+2}\rangle \oplus \langle x^{m+1}z, -y^{m+1}z\rangle$	$2_{m-1} + 2_m$
j	$\langle y^j, x^j\rangle \oplus \langle y^{j-1}z, -x^{j-1}z\rangle$	$2_{2m-j+1} + 2_{2m-j+2}$
		$(m+3 \le j \le 2m)$
$2m+1$	$\langle x^{2m+1} - y^{2m+1}\rangle \oplus \langle y^{2m}z, -x^{2m}z\rangle$	$1_2 + 2_1$

Table 7: The coinvariant algebra of $G(D_{2\ell})$, $\ell = 2m + 1$: odd.

We define

$$I([a:b]_{1_2}) = \big(az + b(x^{2m+1} - y^{2m+1}), xz, yz\big) + \mathfrak{n},$$

$$I([a:b]_{2_j}) = S[G] \cdot (ax^j z + by^{2m-j+1}, x^{j+1}z, x^{2m-j+2}) + \mathfrak{n}, \quad j = 1, 2, \dots, m,$$

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where

$$I([0:1]_{1_2}) = I([1:0]_{2_1}),$$
$$I([0:1]_{2_j}) = I([1:0]_{2_{j+1}}), \quad \text{for } j = 1, 2, \dots, m-1.$$

We see $\pi^{-1}(0) = \bigcup_{\rho \in \hat{G} \setminus \{1_1\}} I([a:b]_\rho)$ in the same manner as in the case of even ℓ. As before we see that $x^{2m-j}z$, $y^{2m-j}z \in I$ for any $j = 1, 2, \dots, m-1$ and that $x^j + ax^jz + by^{2m-j} \notin I$ for any $a, b \in \mathbf{C}$ and $j = 1, 2, \dots, m-1$. If $x^jz + by^{2m-j} \in I$ for some $b \neq 0$ and some $j = 1, 2, \dots, m-1$, then $I = I([1:b]_{2_j})$. If there is $j \geq 2$ such that $x^jz \in I$, $x^{j-1}z \notin I$, then $I = I([1:0]_{2_j})$. If $xz \in I$, then $I = I([a:b]_{1_2})$. If $x^mz \notin I$, then $y^{m+1}, x^{m+1} \in I$ so that $I = I([0:1]_{2_m})$.

3.6 Proof of Theorem 3.1—The Symmetry Group $G = S_4$

Let

$$G = \left\langle \sigma = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

We define

$$f_1 = xyz, \ f_2 = x^2 + y^2 + z^2, \ f_3 = x^4 + y^4 + z^4,$$
$$f_4 = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2).$$

Then $\{f_1^2, f_2, f_3, f_1 f_4\}$ is a system of generators of S^G which satisfies

$$4(f_1 f_4)^2 + 108 f_1^6 - 20 f_1^4 f_2^3 + 36 f_1^4 f_2 f_3 + f_1^2 f_2^6 - 4 f_1^2 f_2^4 f_3 + 5 f_1^2 f_2^2 f_3^2 - 2 f_1^2 f_3^3 = 0.$$

The following is the character table of G.

c.c	1	σ^2	τ	σ	$\sigma\tau\sigma^2$
age	0	1	1	1	1
$\sharp$	1	3	8	6	6
1_1	1	1	1	1	1
1_2	1	1	1	-1	-1
2	2	2	-1	0	0
3_1	3	-1	0	1	-1
3_2	3	-1	0	-1	1

Table 8: Characters of S_4.

The decomposition of the coinvariant algebra S_G into irreducible components is given in Table 9 where

$$g = x^2 + \omega y^2 + \omega^2 z^2,\ \bar{g} = x^2 + \omega^2 y^2 + \omega z^2, \quad \omega = e^{2\pi\sqrt{-1}/3}.$$

d	$(S_G)_d$	irred. factors
1	$\langle x, y, z\rangle$	3_1
2	$\langle g, \bar{g}\rangle \oplus \langle yz, zx, xy\rangle$	$2 + 3_2$
3	$\langle f_1\rangle \oplus \langle x^3, y^3, z^3\rangle$	
	$\oplus\langle (y^2 - z^2)x, (z^2 - x^2)y, (x^2 - y^2)z\rangle$	$1_2 + 3_1 + 3_2$
4	$\langle \bar{g}^2, g^2\rangle \oplus \langle (y^2 - z^2)yz, (z^2 - x^2)zx, (x^2 - y^2)xy\rangle$	
	$\oplus\langle f_1x, f_1y, f_1z\rangle$	$2 + 3_1 + 3_2$
5	$\langle f_1g, -f_1\bar{g}\rangle \oplus \langle f_1yz, f_1zx, f_1xy\rangle$	
	$\oplus\langle (y^2 - z^2)x^3, (z^2 - x^2)y^3, (x^2 - y^2)z^3\rangle$	$2 + 3_1 + 3_2$
6	$\langle f_4\rangle \oplus \langle f_1(y^2 - z^2)x, f_1(z^2 - x^2)y, f_1(x^2 - y^2)z\rangle$	
	$\oplus\langle f_1x^3, f_1y^3, f_1z^3\rangle$	$1_2 + 3_1 + 3_2$
7	$\langle f_1\bar{g}^2, -f_1g^2\rangle$	
	$\oplus\langle f_1(y^2 - z^2)yz, f_1(z^2 - x^2)zx, f_1(x^2 - y^2)xy\rangle$	$2 + 3_2$
8	$\langle f_1(y^2 - z^2)x^3, f_1(z^2 - x^2)y^3, f_1(x^2 - y^2)z^3\rangle$	3_1

Table 9: The coinvariant algebra of S_4.

We define

$$I([a\!:\!b]_{1_2}) = S\cdot(af_1 + bf_4, f_1x, f_1y, f_1z) + \mathfrak{n},$$

$$I([a\!:\!b]_2) = S[G]\cdot\big(ag^2 + bf_1\bar{g}, (y^2 - z^2)x^3, f_1yz\big) + \mathfrak{n},$$

$$I([a\!:\!b]_{3_1}) = S[G]\cdot\big(a(y^2 - z^2)yz + bf_1yz, f_1g, (y^2 - z^2)x^3\big) + \mathfrak{n},$$

$$I([a\!:\!b]_{3_2}) = S[G]\cdot\big(af_1x + b(y^2 - z^2)x^3, f_1g, f_1yz\big) + \mathfrak{n}.$$

Let $\bar{S}_d = (S_G)_d$, the degree d part of S_G. Let $I \in \mathrm{Hilb}^G(\mathbf{C}^3)$ such that $\mathfrak{n} \subset I \subset \mathfrak{m}$. First we note by using the quiver diagram of S_4 as before that I does not contain the elements whose projections to $\bar{S}_1 \oplus \bar{S}_2$ (the degree one and two parts of S_G) are nonzero. We note also that I contains $\bar{S}_7 \oplus \bar{S}_8$.

Assume that I contains an element $af_1 + bf_4$ for $a \neq 0$. Then by the quiver diagram of S_4, we see easily that $I = I([a\!:\!b]_{1_2})$.

Now we consider the case I contains no element $af_1 + bf_4$ for $a \neq 0$. Since $S_G/I = \mathbf{C}[G]$, $f_4 \in I$, that is $\bar{S}_6(1_2) \subset I$. If I contains an element $af_1x + b(y^2 - z^2)x^3$ for

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$a \neq 0$, then $I = I([a{:}b]_{3_2})$. If I contains an element $a(y^2 - z^2)yz + bf_1yz$ for $a \neq 0$, then $I = I([a{:}b]_{3_1})$.

Now we consider the remaining cases. By the quiver diagram of S_4, we see $\bar{S}_5(3_1) \oplus \bar{S}_5(3_2) \subset I$ and $\bar{S}_6 \subset I$. If $a\bar{g}^2 + bf_1g \in I$ for $a \neq 0$, then $I = I([a{:}b]_2)$. If I contains no element $a\bar{g}^2 + bf_1g$ for $a \neq 0$, then $f_1g \in \bar{S}_5(2) \subset I$ because I contains no elements with nonzero projections to $\bar{S}_1 \oplus \bar{S}_2$. Hence $\bar{S}_5 \subset I$, and $I = I([0{:}1]_2) = I([0{:}1]_{3_1}) = I([0{:}1]_{3_2})$.

The following exhaust all the possible cases of coincidence between $I([a{:}b]_\rho)$.

$$I([0{:}1]_{1_2}) = I([1{:}0]_{3_1}),$$
$$I([0{:}1]_2) = I([0{:}1]_{3_1}) = I([0{:}1]_{3_2}).$$

This completes the proof of Theorem 3.1. ∎

3.7 Quiver Diagrams

The diagrams $D_{2\ell}$ and S_4 on the next page are drawn in the same manner as in [GNS00]. They express the quiver structure of S_G, that is the decomposition of $S_1 \cdot ((S_G)_d)_{\rho_j}$. The rows are indexed by degrees and the columns by irreducible representations. Each irreducible factor ρ_j of $(S_G)_d$ has multiplicity one except when $G = D_{4m+2}$, $d = m+1$, $\rho_j = 2_m$ and $(S_G)_{m+1} = \langle y^{m+1}, x^{m+1}\rangle \oplus \langle x^m z, -y^m z\rangle = 2 \cdot 2_m$. Each vertex in the diagram stands for nonzero $((S_G)_d)_{\rho_j}$ and we join $((S_G)_d)_{\rho_j}$ and $((S_G)_{d+1})_{\rho_k}$ with an edge when nonzero $((S_G)_{d+1})_{\rho_k}$ appears in $S_1 \cdot ((S_G)_d)_{\rho_j}$. In the unique exceptional case where $G = D_{4m+2}$, the diagram shows

$$S_1 \cdot \big((S_G)_m\big)_{2_{m-1}} = \langle x^m z, -y^m z\rangle, \quad S_1 \cdot \big((S_G)_m\big)_{2_m} = (S_G)_{m+1}.$$

4 The Molien Series P_{S_G,ρ_j}—The Case (E)

Finite subgroups of $\mathrm{SL}(3, \mathbf{C})$ are classified in [Blichfeldt17]. With the notation in [YY93], there are exactly 4 infinite series labeled by (A), (B), (C), (D), and 8 exceptional cases labeled by (E) through (L). Homogeneous generators of the invariant rings for the exceptional 8 groups, together with explicit descriptions of these groups, are given in [YY93], which we shall follow.[1] Since the character tables of these groups can be obtained by using, for example, GAP, we omitted them; instead we give short descriptions of irreducible characters. In what follows we denote by 1_0 the trivial character (or representation) of G.

In this and the next section we calculate P_{R,ρ_j} and P_{S_G,ρ_j} explicitly for (E)–(L). See also [GNS00]. In this section we discuss the case (E) in some detail as a prototype for all the other cases. In what follows in order to save space we will not explain the customary notation.

[1] Since our results use the results in [YY93], we mention here some of their misprints: page 34, line 1, $\frac{1}{\sqrt{-7}}$ should be $\frac{-1}{\sqrt{-7}}$, page 80, line 2, $(15 + 5\sqrt{15}i)x^3y^3$ should be $(15 + 5\sqrt{15}i)y^3z^3$.

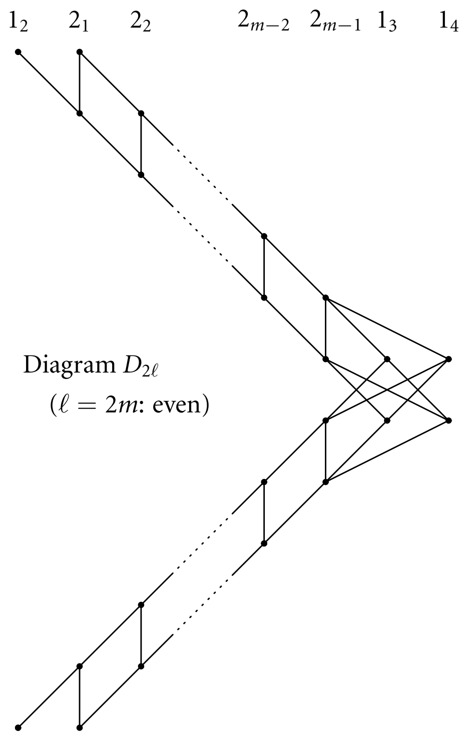


Diagram $D_{2\ell}$
($\ell = 2m$: even)

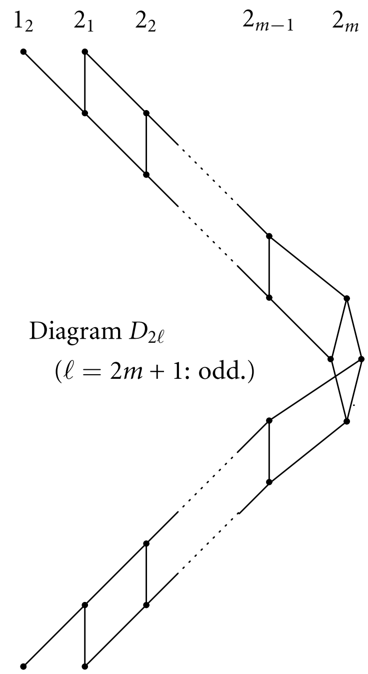
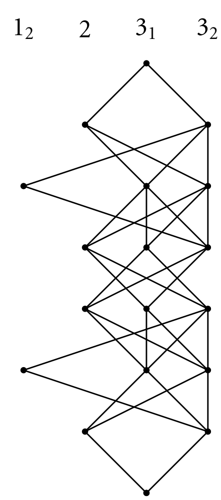


Diagram $D_{2\ell}$
($\ell = 2m + 1$: odd.)

Diagram S_4:



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Let

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \right\rangle,$$

where $\omega = e^{2\pi i/3}$. Then we have $|G| = 108$, and $\hat{G} = \{1_0, 1_1, 1_2, 1_3, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, 4_1, 4_2\}$, where $1_1(V) = \sqrt{-1}$, $1_2 = 1_1^2$, $1_3 = 1_1^3$, $3_1 = \rho$, $3_2 = 1_1\rho$, $3_3 = 1_2\rho$, $3_4 = 1_3\rho$, $3_5 = \rho^\vee$, $3_6 = 1_1\rho^\vee$, $3_7 = 1_2\rho^\vee$, $3_8 = 1_3\rho^\vee$, $4_1(T) = 1$ and $4_2(T) = -2$.

The decompositions of $\rho_i \otimes \rho$ are given in Appendix.

We also have $S^G = \mathbf{C}[f_1, f_2, f_3, f_4, f_5]$ with $\deg f_1 = 6$, $\deg f_2 = 6$, $\deg f_3 = 12$, $\deg f_4 = 12$, and $\deg f_5 = 9$.

Put $R = S/(f_1, f_2, f_3)$. Then we can easily compute $P_{R,\rho_j}(t)$ by applying Proposition 1.3. Thus we see that R splits into irreducible representations as in Table 10.

Next we calculate the Molien series $P_{S_G}(t)$ by the repeated use of the trivial relation $(\mathfrak{n})_i = V^\vee \cdot (\mathfrak{n})_{i-1} + (S^G)_i$ for any i. In the case (E) we need to compute only for $i \le 21$. What we do is not more than elementary linear algebra, so we omit the details of the computation. We see

$$\begin{aligned} P_R(t) &= \frac{(1-t^6)^2(1-t^{12})}{(1-t)^3} \\ &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 \\ &\quad + 33t^8 + 35t^9 + 36t^{10} + 36t^{11} + 35t^{12} + 33t^{13} + 30t^{14} \\ &\quad + 26t^{15} + 21t^{16} + 15t^{17} + 10t^{18} + 6t^{19} + 3t^{20} + t^{21}, \end{aligned}$$

$$\begin{aligned} P_{S_G}(t) &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 \\ &\quad + 33t^8 + 34t^9 + 33t^{10} + 30t^{11} + 24t^{12} + 15t^{13} + 6t^{14}. \end{aligned}$$

Then in view of Proposition 1.3 we can compute the Molien series $P_{S_G,\rho_j}(t)$. Summarizing the computation we see S_G splits as in Table 11.

In other words,

$$\begin{aligned} P_{S_G,1_0}(t) &= 1, \\ P_{S_G,1_1}(t) &= t^3 + t^9, \\ P_{S_G,1_2}(t) &= 2t^6, \\ P_{S_G,1_3}(t) &= t^3 + t^9, \\ P_{S_G,3_1}(t) &= 3t^5 + 3t^8 + 2t^{11}, \\ P_{S_G,3_2}(t) &= t^2 + t^5 + 3t^8 + 2t^{11}, \\ P_{S_G,3_3}(t) &= 2t^5 + 2t^8 + 4t^{11} + 2t^{14}, \end{aligned}$$

deg	1_0	1_1	1_2	1_3	3_1	3_2	3_3	3_4	3_5	3_6	3_7	3_8	4_1	4_2	$\dim R_d$
1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	3
2	0	0	0	0	0	1	0	1	0	0	0	0	0	0	6
3	0	1	0	1	0	0	0	0	0	0	0	0	1	1	10
4	0	0	0	0	0	0	0	0	1	2	0	2	0	0	15
5	0	0	0	0	3	1	2	1	0	0	0	0	0	0	21
6	0	0	2	0	0	0	0	0	0	0	0	0	3	3	26
7	0	0	0	0	0	0	0	0	2	2	4	2	0	0	30
8	0	0	0	0	3	3	2	3	0	0	0	0	0	0	33
9	1	1	0	1	0	0	0	0	0	0	0	0	4	4	35
10	0	0	0	0	0	0	0	0	2	3	4	3	0	0	36
11	0	0	0	0	2	3	4	3	0	0	0	0	0	0	36
12	1	1	0	1	0	0	0	0	0	0	0	0	4	4	35
13	0	0	0	0	0	0	0	0	3	3	2	3	0	0	33
14	0	0	0	0	2	2	4	2	0	0	0	0	0	0	30
15	0	0	2	0	0	0	0	0	0	0	0	0	3	3	26
16	0	0	0	0	0	0	0	0	3	1	2	1	0	0	21
17	0	0	0	0	1	2	0	2	0	0	0	0	0	0	15
18	0	1	0	1	0	0	0	0	0	0	0	0	1	1	10
19	0	0	0	0	0	0	0	0	0	1	0	1	0	0	6
20	0	0	0	0	1	0	0	0	0	0	0	0	0	0	3
21	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 10: The decomposition of R of type (E).

deg	1_0	1_1	1_2	1_3	3_1	3_2	3_3	3_4	3_5	3_6	3_7	3_8	4_1	4_2	$\dim(S_G)_d$
1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	3
2	0	0	0	0	0	1	0	1	0	0	0	0	0	0	6
3	0	1	0	1	0	0	0	0	0	0	0	0	1	1	10
4	0	0	0	0	0	0	0	0	1	2	0	2	0	0	15
5	0	0	0	0	3	1	2	1	0	0	0	0	0	0	21
6	0	0	2	0	0	0	0	0	0	0	0	0	3	3	26
7	0	0	0	0	0	0	0	0	2	2	4	2	0	0	30
8	0	0	0	0	3	3	2	3	0	0	0	0	0	0	33
9	0	1	0	1	0	0	0	0	0	0	0	0	4	4	34
10	0	0	0	0	0	0	0	0	1	3	4	3	0	0	33
11	0	0	0	0	2	2	4	2	0	0	0	0	0	0	30
12	0	0	0	0	0	0	0	0	0	0	0	0	3	3	24
13	0	0	0	0	0	0	0	0	1	1	2	1	0	0	15
14	0	0	0	0	0	0	2	0	0	0	0	0	0	0	6

Table 11: The decomposition of S_G of type (E).

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$$P_{S_G,3_4}(t) = t^2 + t^5 + 3t^8 + 2t^{11},$$
$$P_{S_G,3_5}(t) = t + t^4 + 2t^7 + t^{10} + t^{13},$$
$$P_{S_G,3_6}(t) = 2t^4 + 2t^7 + 3t^{10} + t^{13},$$
$$P_{S_G,3_7}(t) = 4t^7 + 4t^{10} + 2t^{13},$$
$$P_{S_G,3_8}(t) = 2t^4 + 2t^7 + 3t^{10} + t^{13},$$
$$P_{S_G,4_1}(t) = t^3 + 3t^6 + 4t^9 + 3t^{12},$$
$$P_{S_G,4_2}(t) = t^3 + 3t^6 + 4t^9 + 3t^{12}.$$

As a consequence we see

$$P_{S_G,\rho_j}(t) = [(1-t^9)(1-t^{12})P_{R,\rho_j}(t)]_+$$

where $[f(t)]_+ = \sum_{d=0}^{21} \max\{a_d, 0\}t^d$ for $f(t) = \sum a_d t^d \in \mathbf{Z}[t]$. Note that this formula does not imply a similar formula for ρ_{S_G}.

5 The Molien Series P_{S_G,ρ_j}

In this section we report the results for the other types (F)–(L). For the sake of the reader's convenience we list the decompositions of $\rho_i \otimes \rho$ in the appendix.

5.1 The Group of Type (F)

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix} \right\rangle,$$

where $\omega = e^{2\pi i/3}$. $|G| = 216$. $\hat{G} = \{1_0, 1_1, 1_2, 1_3, 2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, 6_1, 6_2, 8\}$, where $1_3 = 1_1 1_2$, $3_1 = \rho$, $3_2 = 1_1\rho$, $3_3 = 1_2\rho$, $3_4 = 1_3\rho$, $3_5 = \rho^\vee$, $3_6 = 1_1\rho^\vee$, $3_7 = 1_2\rho^\vee$, $3_8 = 1_3\rho^\vee$, $6_1 = \rho^2 - \rho^\vee$, $6_2 = \rho^{\vee 2} - \rho$. $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$, with $\deg f_1 = 6$, $\deg f_2 = 9$, $\deg f_3 = 12$, $\deg f_4 = 12$. Let $R = S/(f_1, f_2, f_3)$. Then we have

$$P_{R,1_0}(t) = 1 + t^{12} + t^{24},$$
$$P_{R,1_1}(t) = P_{R,1_2} = P_{R,1_3} = t^6 + t^{12} + t^{18},$$
$$P_{R,2}(t) = t^3 + 2t^9 + 2t^{15} + t^{21},$$
$$P_{R,3_1}(t) = 2t^5 + 2t^8 + 2t^{11} + t^{17} + t^{20} + t^{23},$$

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$$
\begin{aligned}
P_{R,3_2}(t) &= P_{R,3_3} = P_{R,3_4} = t^5 + t^8 + 3t^{11} + 2t^{14} + 2t^{17},\\
P_{R,3_5}(t) &= t + t^4 + t^7 + 2t^{13} + 2t^{16} + 2t^{19},\\
P_{R,3_6}(t) &= P_{R,3_7} = P_{R,3_8} = 2t^7 + 2t^{10} + 3t^{13} + t^{16} + t^{19},\\
P_{R,6_1}(t) &= 2t^4 + 2t^7 + 5t^{10} + 3t^{13} + 4t^{16} + t^{19} + t^{22},\\
P_{R,6_2}(t) &= t^2 + t^5 + 4t^8 + 3t^{11} + 5t^{14} + 2t^{17} + 2t^{20},\\
P_{R,8}(t) &= t^3 + 3t^6 + 5t^9 + 6t^{12} + 5t^{15} + 3t^{18} + t^{21}.
\end{aligned}
$$

We see that

$$
\begin{aligned}
P_{S_G}(t) = 1 &+ 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 27t^6 + 33t^7 + 39t^8 + 44t^9 + 48t^{10}\\
&+ 51t^{11} + 51t^{12} + 48t^{13} + 42t^{14} + 34t^{15} + 24t^{16} + 15t^{17} + 8t^{18} + 3t^{19}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= P_{S_G,1_2} = P_{S_G,1_3} = t^6 + t^{12},\\
P_{S_G,2}(t) &= t^3 + 2t^9 + t^{15},\\
P_{S_G,3_1}(t) &= 2t^5 + 2t^8 + 2t^{11},\\
P_{S_G,3_2}(t) &= P_{S_G,3_3} = P_{S_G,3_4} = t^5 + t^8 + 3t^{11} + 2t^{14} + t^{17},\\
P_{S_G,3_5}(t) &= t + t^4 + t^7 + t^{13} + t^{16} + t^{19},\\
P_{S_G,3_6}(t) &= P_{S_G,3_7} = P_{S_G,3_8} = 2t^7 + 2t^{10} + 3t^{13} + t^{16},\\
P_{S_G,6_1}(t) &= 2t^4 + 2t^7 + 5t^{10} + 3t^{13} + 2t^{16},\\
P_{S_G,6_2}(t) &= t^2 + t^5 + 4t^8 + 3t^{11} + 4t^{14} + t^{17},\\
P_{S_G,8}(t) &= t^3 + 3t^6 + 5t^9 + 6t^{12} + 4t^{15} + t^{18}.
\end{aligned}
$$

5.2 The Group of Type (G)

$$
G = \left\langle \begin{pmatrix} 1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{pmatrix}, \right.
$$

$$
\left. U = \begin{pmatrix} \varepsilon^2 & 0 & 0\\ 0 & \varepsilon^2 & 0\\ 0 & 0 & \varepsilon^5 \end{pmatrix}, \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega \end{pmatrix} \right\rangle,
$$

where $\varepsilon = e^{2\pi i/9}$, $\omega = e^{2\pi i/3}$. $|G| = 648$. $\hat{G} = \{1_0, 1_1, 1_2, 2_1, 2_2, 2_3, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 6_1, 6_2, 6_3, 6_4, 6_5, 6_6, 8_1, 8_2, 8_3, 9_1, 9_2\}$, where 2_1 and 3_7 are rational valued characters and $1_1(U) = \omega$, $1_2 = 1_1^2$, $2_2 = 1_1 2_1$, $2_3 = 1_2 2_1$, $3_1 = \rho$, $3_2 = 1_1 3_1$, $3_3 = 1_2 3_1$,

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$3_4 = \rho^\vee$, $3_5 = 1_1 3_4$, $3_6 = 1_2 3_4$, $6_1 = \rho^2 - \rho^\vee$, $6_2 = 1_1 6_1$, $6_3 = 1_2 6_1$, $6_4 = \rho^{\vee 2} - \rho$, $6_5 = 1_1 6_4$, $6_6 = 1_2 6_4$, $8_1 = \rho\rho^\vee - 1_0$, $8_2 = 1_1 8_1$, $8_3 = 1_2 8_1$, $9_1 = 3_7\rho$, $9_2 = 3_7\rho^\vee$. $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$, with $\deg f_1 = 9, \deg f_2 = 12, \deg f_3 = 18, \deg f_4 = 18$. $R = S/(f_1, f_2, f_3)$. Then we have

$$P_{R,1_0}(t) = 1 + t^{18} + t^{36},$$
$$P_{R,1_1}(t) = 2t^{12} + t^{30},$$
$$P_{R,1_2}(t) = t^6 + 2t^{24},$$
$$P_{R,2_1}(t) = 3t^{15} + 3t^{21},$$
$$P_{R,2_2}(t) = 2t^9 + 2t^{15} + t^{27} + t^{33},$$
$$P_{R,2_3}(t) = t^3 + t^9 + 2t^{21} + 2t^{27},$$
$$P_{R,3_1}(t) = t^8 + 2t^{11} + 3t^{17} + t^{20} + t^{26} + t^{35},$$
$$P_{R,3_2}(t) = t^5 + 2t^{11} + t^{14} + 2t^{20} + t^{23} + 2t^{29},$$
$$P_{R,3_3}(t) = t^5 + t^8 + t^{14} + 2t^{17} + 3t^{23} + t^{32},$$
$$P_{R,3_4}(t) = t + t^{10} + t^{16} + 3t^{19} + 2t^{25} + t^{28},$$
$$P_{R,3_5}(t) = 2t^7 + t^{13} + 2t^{16} + t^{22} + 2t^{25} + t^{31},$$
$$P_{R,3_6}(t) = t^4 + 3t^{13} + 2t^{19} + t^{22} + t^{28} + t^{31},$$
$$P_{R,3_7}(t) = t^6 + 2t^{12} + 3t^{18} + 2t^{24} + t^{30},$$
$$P_{R,6_1}(t) = t^4 + t^7 + t^{10} + 2t^{13} + 4t^{16} + t^{19} + 5t^{22} + t^{25} + t^{28} + t^{31},$$
$$P_{R,6_2}(t) = t^4 + 3t^{10} + 2t^{13} + 2t^{16} + 3t^{19} + 3t^{22} + t^{25} + 3t^{28},$$
$$P_{R,6_3}(t) = t^7 + 3t^{10} + t^{13} + 5t^{16} + 2t^{19} + 2t^{22} + 2t^{25} + t^{28} + t^{34},$$
$$P_{R,6_4}(t) = t^5 + t^8 + t^{11} + 5t^{14} + t^{17} + 4t^{20} + 2t^{23} + t^{26} + t^{29} + t^{32},$$
$$P_{R,6_5}(t) = t^2 + t^8 + 2t^{11} + 2t^{14} + 2t^{17} + 5t^{20} + t^{23} + 3t^{26} + t^{29},$$
$$P_{R,6_6}(t) = 3t^8 + t^{11} + 3t^{14} + 3t^{17} + 2t^{20} + 2t^{23} + 3t^{26} + t^{32},$$
$$P_{R,8_1}(t) = 3t^9 + 3t^{12} + 3t^{15} + 6t^{18} + 3t^{21} + 3t^{24} + 3t^{27},$$
$$P_{R,8_2}(t) = t^3 + t^6 + t^9 + 4t^{12} + 3t^{15} + 3t^{18} + 5t^{21} + 2t^{24} + 2t^{27} + 2t^{30},$$
$$P_{R,8_3}(t) = 2t^6 + 2t^9 + 2t^{12} + 5t^{15} + 3t^{18} + 3t^{21} + 4t^{24} + t^{27} + t^{30} + t^{33},$$
$$P_{R,9_1}(t) = t^5 + t^8 + 4t^{11} + 3t^{14} + 6t^{17} + 3t^{20} + 5t^{23} + 2t^{26} + 2t^{29},$$
$$P_{R,9_2}(t) = 2t^7 + 2t^{10} + 5t^{13} + 3t^{16} + 6t^{19} + 3t^{22} + 4t^{25} + t^{28} + t^{31}.$$

We see that

$$\begin{aligned} P_{S_G}(t) = {} & 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 + 36t^7 + 45t^8 + 54t^9 \\ & + 63t^{10} + 72t^{11} + 80t^{12} + 87t^{13} + 93t^{14} + 98t^{15} + 102t^{16} \\ & + 105t^{17} + 105t^{18} + 102t^{19} + 96t^{20} + 88t^{21} + 78t^{22} \\ & + 66t^{23} + 52t^{24} + 36t^{25} + 21t^{26} + 10t^{27} + 3t^{28}, \end{aligned}$$

and

$$\begin{aligned}
P_{S_G,1_0}(t) &= 1, \\
P_{S_G,1_1}(t) &= 2t^{12}, \\
P_{S_G,1_2}(t) &= t^6 + t^{24}, \\
P_{S_G,2_1}(t) &= 3t^{15} + 3t^{21}, \\
P_{S_G,2_2}(t) &= 2t^9 + 2t^{15}, \\
P_{S_G,2_3}(t) &= t^3 + t^9 + t^{21} + t^{27}, \\
P_{S_G,3_1}(t) &= t^8 + 2t^{11} + 3t^{17} + t^{20}, \\
P_{S_G,3_2}(t) &= t^5 + 2t^{11} + t^{14} + 2t^{20}, \\
P_{S_G,3_3}(t) &= t^5 + t^8 + t^{14} + 2t^{17} + 2t^{23}, \\
P_{S_G,3_4}(t) &= t + t^{10} + t^{16} + 2t^{19} + 2t^{25}, \\
P_{S_G,3_5}(t) &= 2t^7 + t^{13} + 2t^{16} + t^{22}, \\
P_{S_G,3_6}(t) &= t^4 + 3t^{13} + 2t^{19} + t^{28}, \\
P_{S_G,3_7}(t) &= t^6 + 2t^{12} + 3t^{18} + t^{24}, \\
P_{S_G,6_1}(t) &= t^4 + t^7 + t^{10} + 2t^{13} + 4t^{16} + t^{19} + 4t^{22}, \\
P_{S_G,6_2}(t) &= t^4 + 3t^{10} + 2t^{13} + 2t^{16} + 3t^{19} + 2t^{22} + t^{25}, \\
P_{S_G,6_3}(t) &= t^7 + 3t^{10} + t^{13} + 5t^{16} + 2t^{19} + 2t^{22} + t^{25}, \\
P_{S_G,6_4}(t) &= t^5 + t^8 + t^{11} + 5t^{14} + t^{17} + 4t^{20} + t^{23}, \\
P_{S_G,6_5}(t) &= t^2 + t^8 + 2t^{11} + 2t^{14} + 2t^{17} + 4t^{20} + t^{23} + 2t^{26}, \\
P_{S_G,6_6}(t) &= 3t^8 + t^{11} + 3t^{14} + 3t^{17} + 2t^{20} + 2t^{23}, \\
P_{S_G,8_1}(t) &= 3t^9 + 3t^{12} + 3t^{15} + 6t^{18} + 3t^{21} + 3t^{24}, \\
P_{S_G,8_2}(t) &= t^3 + t^6 + t^9 + 4t^{12} + 3t^{15} + 3t^{18} + 4t^{21} + t^{24} + t^{27}, \\
P_{S_G,8_3}(t) &= 2t^6 + 2t^9 + 2t^{12} + 5t^{15} + 3t^{18} + 3t^{21} + 2t^{24}, \\
P_{S_G,9_1}(t) &= t^5 + t^8 + 4t^{11} + 3t^{14} + 6t^{17} + 3t^{20} + 4t^{23} + t^{26}, \\
P_{S_G,9_2}(t) &= 2t^7 + 2t^{10} + 5t^{13} + 3t^{16} + 6t^{19} + 3t^{22} + 2t^{25}.
\end{aligned}$$

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5.3 The Group of Type (H)

$$G=\left\langle \begin{pmatrix}1&0&0\\0&\varepsilon^{-1}&0\\0&0&\varepsilon\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}1&1&1\\2&s&t\\2&t&s\end{pmatrix}\right\rangle,$$

where $\varepsilon = e^{2\pi i/5}, \omega = e^{2\pi i/3}, s = \varepsilon^2+\varepsilon^3$ and $t = \varepsilon+\varepsilon^5$. $|G| = 60$. $\hat{G} = \{1_0, 3_1 = \rho = \rho^\vee, 3_2, 4, 5\}$. Let $\tilde{G}$ be a group generated by G and $-I$ where I is the identity matrix of degree 3. Then $\tilde{G}$ is a Coxeter group of type H_3 and there exist three homogeneous invariants f_1, f_2, f_3 with $\deg f_1 = 2$, $\deg f_2 = 6$, $\deg f_3 = 10$ such that $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, f_3]$ and $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \mathrm{Jac}(f_1, f_2, f_3)$. Hence we have

$$\begin{aligned}
P_{S_G,1_0} &= 1,\\
P_{S_G,3_1} &= t^3+t^5+t^7+t^8+t^{10}+t^{12},\\
P_{S_G,3_2} &= t+t^5+t^6+t^9+t^{10}+t^{14},\\
P_{S_G,4} &= t^3+t^4+t^6+t^7+t^8+t^9+t^{11}+t^{12},\\
P_{S_G,5} &= t^2+t^4+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{13}.
\end{aligned}$$

5.4 The Group of Type (I)

$$G=\left\langle \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}\varepsilon&0&0\\0&\varepsilon^2&0\\0&0&\varepsilon^4\end{pmatrix}, \frac{-1}{\sqrt{-7}}\begin{pmatrix}\varepsilon^4-\varepsilon^3&\varepsilon^2-\varepsilon^5&\varepsilon-\varepsilon^6\\ \varepsilon^2-\varepsilon^5&\varepsilon-\varepsilon^6&\varepsilon^4-\varepsilon^3\\ \varepsilon-\varepsilon^6&\varepsilon^4-\varepsilon^3&\varepsilon^2-\varepsilon^5\end{pmatrix}\right\rangle,$$

where $\varepsilon = e^{2\pi i/7}$. $|G| = 168$. $\hat{G} = \{1_0, 3_1 = \rho, 3_2 = \rho^\vee, 6, 7, 8\}$, Let $\tilde{G}$ be a group generated by G and $-I$ where I is the identity matrix of degree 3. Then $\tilde{G}$ is a complex reflection group of type $J_3(4)$ (*cf.* [Cohen76]) and there exist three homogeneous invariants f_1, f_2, f_3 with $\deg f_1 = 4$, $\deg f_2 = 6$, $\deg f_3 = 14$ such that $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, f_3]$ and $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \mathrm{Jac}(f_1, f_2, f_3)$. Hence we have

$$\begin{aligned}
P_{S_G,1_0} &= 1,\\
P_{S_G,3_1} &= t^3+t^5+t^{10}+t^{12}+t^{13}+t^{20},\\
P_{S_G,3_2} &= t+t^8+t^9+t^{11}+t^{16}+t^{18},\\
P_{S_G,6} &= t^2+t^4+t^6+t^8+t^9+t^{10}+t^{11}+t^{12}+t^{13}+t^{15}+t^{17}+t^{19},\\
P_{S_G,7} &= t^3+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{12}+t^{13}+t^{14}+t^{15}+t^{16}+t^{18},\\
P_{S_G,8} &= t^4+t^5+t^6+2t^7+t^8+t^9+t^{10}+t^{11}+t^{12}+t^{13}+2t^{14}+t^{15}+t^{16}+t^{17}.
\end{aligned}$$

5.5 The Group of Type (J)

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \right.$$

$$\left. \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}, W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \right\rangle,$$

where $\varepsilon = e^{2\pi i/5}$, $\omega = e^{2\pi i/3}$, $s = \varepsilon^2 + \varepsilon^3$, and $t = \varepsilon + \varepsilon^4$. $|G| = 180$. $\hat{G} = \{1_0, 1_1, 1_2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 4_1, 4_2, 4_3, 5_1, 5_2, 5_3\}$, where 4_1 and 5_1 are rational valued characters and $1_1(W) = \omega$, $1_2 = 1_1^2$, $3_1 = \rho$, $3_2 = \rho^\vee = 1_1 3_1$, $3_3 = 1_2 3_1$, $3_4(x) = 3_1(x^7)$, $\forall x \in G$, $3_5 = 1_1 3_4$, $3_6 = 1_2 3_4$, $4_2 = 1_1 4_1$, $4_3 = 1_2 4_1$, $5_2 = 1_1 5_1$ and $5_3 = 1_2 5_1$. $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$, with $\deg f_1 = 6$, $\deg f_2 = 6$, $\deg f_3 = 15$, $\deg f_4 = 12$. Put $R = S/(f_1, f_2, f_3)$. Then we have

$$\begin{aligned}
P_{R,1_0}(t) &= 1 + t^{12} + t^{24}, \\
P_{R,1_1}(t) &= t^2 + t^{14} + t^{20}, \\
P_{R,1_2}(t) &= t^4 + t^{10} + t^{22}, \\
P_{R,3_1}(t) &= 2t^5 + t^8 + 2t^{11} + 2t^{14} + t^{17} + t^{23}, \\
P_{R,3_2}(t) &= t + t^7 + 2t^{10} + 2t^{13} + t^{16} + 2t^{19}, \\
P_{R,3_3}(t) &= t^3 + t^6 + 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{21}, \\
P_{R,3_4}(t) &= 2t^5 + t^8 + t^{11} + 2t^{14} + 3t^{17}, \\
P_{R,3_5}(t) &= 3t^7 + 2t^{10} + t^{13} + t^{16} + 2t^{19}, \\
P_{R,3_6}(t) &= t^3 + 2t^9 + 3t^{12} + 2t^{15} + t^{21}, \\
P_{R,4_1}(t) &= t^3 + 2t^6 + 2t^9 + 2t^{12} + 2t^{15} + 2t^{18} + t^{21}, \\
P_{R,4_2}(t) &= t^5 + 3t^8 + 3t^{11} + 2t^{14} + 2t^{17} + t^{20}, \\
P_{R,4_3}(t) &= t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + t^{19}, \\
P_{R,5_1}(t) &= 3t^6 + 3t^9 + 3t^{12} + 3t^{15} + 3t^{18}, \\
P_{R,5_2}(t) &= t^2 + t^5 + 3t^8 + 3t^{11} + 3t^{14} + 2t^{17} + 2t^{20}, \\
P_{R,5_3}(t) &= 2t^4 + 2t^7 + 3t^{10} + 3t^{13} + 3t^{16} + t^{19} + t^{22}.
\end{aligned}$$

We see that

$$\begin{aligned}
P_{S_G}(t) = 1 &+ 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 + 33t^8 \\
&+ 35t^9 + 36t^{10} + 36t^{11} + 35t^{12} + 33t^{13} + 30t^{14} + 25t^{15} \\
&+ 19t^{16} + 12t^{17} + 5t^{18} + 3t^{19} + t^{20}.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= t^2 + t^{20},\\
P_{S_G,1_2}(t) &= t^4 + t^{10},\\
P_{S_G,3_1}(t) &= 2t^5 + t^8 + 2t^{11} + 2t^{14},\\
P_{S_G,3_2}(t) &= t + t^7 + 2t^{10} + t^{13} + t^{16} + t^{19},\\
P_{S_G,3_3}(t) &= t^3 + t^6 + 2t^9 + t^{12} + t^{15},\\
P_{S_G,3_4}(t) &= 2t^5 + t^8 + t^{11} + 2t^{14} + t^{17},\\
P_{S_G,3_5}(t) &= 3t^7 + 2t^{10} + t^{13} + t^{16},\\
P_{S_G,3_6}(t) &= t^3 + 2t^9 + 3t^{12} + t^{15},\\
P_{S_G,4_1}(t) &= t^3 + 2t^6 + 2t^9 + 2t^{12} + t^{15},\\
P_{S_G,4_2}(t) &= t^5 + 3t^8 + 3t^{11} + 2t^{14} + t^{17},\\
P_{S_G,4_3}(t) &= t^4 + 2t^7 + 2t^{10} + 3t^{13} + 2t^{16},\\
P_{S_G,5_1}(t) &= 3t^6 + 3t^9 + 3t^{12} + 3t^{15} + t^{18},\\
P_{S_G,5_2}(t) &= t^2 + t^5 + 3t^8 + 3t^{11} + 2t^{14} + t^{17},\\
P_{S_G,5_3}(t) &= 2t^4 + 2t^7 + 3t^{10} + 3t^{13} + t^{16}.
\end{aligned}
$$

5.6 The Group of Type (K)

$$
G = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}, \frac{1}{\sqrt{-7}} \begin{pmatrix} \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 \\ \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 \\ \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 \end{pmatrix}, \right.
$$
$$
\left. W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \right\rangle,
$$

where $\varepsilon = e^{2\pi i/7}$ and $\omega = e^{2\pi i/3}$. $|\hat{G}| = 504$. $\hat{G} = \{1_0, 1_1, 1_2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 6_1, 6_2, 6_3, 7_1, 7_2, 7_3, 8_1, 8_2, 8_3\}$, where 6_1, 7_1 and 8_1 are rational valued characters and $1_1(W) = \omega$, $1_2 = 1_1^2$, $3_1 = \rho$, $3_2 = 1_1 3_1$, $3_3 = 1_2 3_1$, $3_4 = \rho^\vee$, $3_5 = 1_1 3_4$, $3_6 = 1_2 3_4$, $6_2 = 1_1 6_1$, $6_3 = 1_2 6_1$, $7_2 = 1_1 7_1$, $7_3 = 1_2 7_1$, $8_2 = 1_1 8_1$ and $8_3 = 1_2 8_1$. $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$, with $\deg f_1 = 6$, $\deg f_2 = 12$, $\deg f_3 = 21$, $\deg f_4 = 18$. Put $R = S/(f_1, f_2, f_3)$. Then we have

$$
\begin{aligned}
P_{R,1_0}(t) &= 1 + t^{18} + t^{36},\\
P_{R,1_1}(t) &= t^8 + t^{14} + t^{32},
\end{aligned}
$$

$$P_{R,1_2}(t) = t^4 + t^{22} + t^{28},$$
$$P_{R,3_1}(t) = t^5 + t^{11} + t^{14} + 2t^{17} + 2t^{20} + t^{23} + t^{35},$$
$$P_{R,3_2}(t) = t^7 + t^{10} + 2t^{13} + t^{16} + t^{19} + t^{25} + t^{28} + t^{31},$$
$$P_{R,3_3}(t) = t^3 + t^9 + t^{12} + t^{18} + 2t^{21} + t^{24} + 2t^{27},$$
$$P_{R,3_4}(t) = t + t^{13} + 2t^{16} + 2t^{19} + t^{22} + t^{25} + t^{31},$$
$$P_{R,3_5}(t) = 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{24} + t^{27} + t^{33},$$
$$P_{R,3_6}(t) = t^5 + t^8 + t^{11} + t^{17} + t^{20} + 2t^{23} + t^{26} + t^{29},$$
$$P_{R,6_1}(t) = 2t^6 + t^9 + 3t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + 3t^{24} + t^{27} + 2t^{30},$$
$$P_{R,6_2}(t) = t^2 + 2t^8 + t^{11} + 2t^{14} + 3t^{17} + 3t^{20} + 2t^{23} + 3t^{26} + t^{32},$$
$$P_{R,6_3}(t) = t^4 + 3t^{10} + 2t^{13} + 3t^{16} + 3t^{19} + 2t^{22} + t^{25} + 2t^{28} + t^{34},$$
$$P_{R,7_1}(t) = t^3 + t^6 + 2t^9 + 2t^{12} + 3t^{15} + 3t^{18} + 3t^{21} + 2t^{24} + 2t^{27} + t^{30} + t^{33},$$
$$P_{R,7_2}(t) = t^5 + t^8 + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 2t^{26} + 2t^{29},$$
$$P_{R,7_3}(t) = 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + 3t^{25} + t^{28} + t^{31},$$
$$P_{R,8_1}(t) = t^6 + 2t^9 + 3t^{12} + 4t^{15} + 4t^{18} + 4t^{21} + 3t^{24} + 2t^{27} + t^{30},$$
$$P_{R,8_2}(t) = t^5 + 2t^8 + 3t^{11} + 4t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 2t^{26} + 2t^{29} + t^{32},$$
$$P_{R,8_3}(t) = t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 4t^{22} + 3t^{25} + 2t^{28} + t^{31}.$$

We see that

$$\begin{aligned} P_{S_G}(t) = 1 &+ 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 27t^6 + 33t^7 + 39t^8 + 45t^9 \\ &+ 51t^{10} + 57t^{11} + 62t^{12} + 66t^{13} + 69t^{14} + 71t^{15} + 72t^{16} + 72t^{17} \\ &+ 71t^{18} + 69t^{19} + 66t^{20} + 61t^{21} + 54t^{22} + 45t^{23} + 35t^{24} + 24t^{25} \\ &+ 13t^{26} + 3t^{27} + t^{28}. \end{aligned}$$

Hence we have

$$P_{S_G,1_0}(t) = 1,$$
$$P_{S_G,1_1}(t) = t^8 + t^{14},$$
$$P_{S_G,1_2}(t) = t^4 + t^{28},$$
$$P_{S_G,3_1}(t) = t^5 + t^{11} + t^{14} + 2t^{17} + 2t^{20},$$
$$P_{S_G,3_2}(t) = t^7 + t^{10} + 2t^{13} + t^{16} + t^{19},$$
$$P_{S_G,3_3}(t) = t^3 + t^9 + t^{12} + t^{18} + t^{21} + t^{24} + t^{27},$$

$$P_{S_G,3_4}(t) = t + t^{13} + 2t^{16} + t^{19} + t^{22} + t^{25},$$
$$P_{S_G,3_5}(t) = 2t^{9} + t^{12} + 2t^{15} + t^{18} + t^{24},$$
$$P_{S_G,3_6}(t) = t^{5} + t^{8} + t^{11} + t^{17} + t^{20} + t^{23},$$
$$P_{S_G,6_1}(t) = 2t^{6} + t^{9} + 3t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + t^{24},$$
$$P_{S_G,6_2}(t) = t^{2} + 2t^{8} + t^{11} + 2t^{14} + 3t^{17} + 2t^{20} + 2t^{23} + t^{26},$$
$$P_{S_G,6_3}(t) = t^{4} + 3t^{10} + 2t^{13} + 3t^{16} + 3t^{19} + t^{22} + t^{25},$$
$$P_{S_G,7_1}(t) = t^{3} + t^{6} + 2t^{9} + 2t^{12} + 3t^{15} + 3t^{18} + 2t^{21} + t^{24},$$
$$P_{S_G,7_2}(t) = t^{5} + t^{8} + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 2t^{23} + t^{26},$$
$$P_{S_G,7_3}(t) = 2t^{7} + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + t^{25},$$
$$P_{S_G,8_1}(t) = t^{6} + 2t^{9} + 3t^{12} + 4t^{15} + 4t^{18} + 4t^{21} + 2t^{24},$$
$$P_{S_G,8_2}(t) = t^{5} + 2t^{8} + 3t^{11} + 4t^{14} + 3t^{17} + 3t^{20} + 2t^{23},$$
$$P_{S_G,8_3}(t) = t^{4} + 2t^{7} + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + t^{25}.$$

5.7 The Group of Type (L)

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \right.$$
$$\left. \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{pmatrix} \right\rangle,$$

where $\varepsilon = e^{2\pi i/5}$, $s = \varepsilon^2 + \varepsilon^3$, $t = \varepsilon + \varepsilon^4$, $\lambda_1 = -\frac{1-\sqrt{-15}}{4}$ and $\lambda_2 = -\frac{1+\sqrt{-15}}{4}$. $|G| = 1080$. $\hat{G} = \{1_0, 3_1, 3_2, 3_3, 3_4, 5_1, 5_2, 6_1, 6_2, 8_1, 8_2, 9_1, 9_2, 9_3, 10, 15_1, 15_2\}$, where $3_1 = \rho$, $3_2 = \rho^\vee$, $3_3(x) = 3_1(x^7)$ for all $x \in G$, $3_4(x) = 3_2(x^7)$ for all $x \in G$, $6_1 = \rho^2 - \rho^\vee$, $6_2 = \rho^{\vee 2} - \rho$, $8_1 = 3_1 3_2 - 1_0$, $8_2 = 3_3 3_4 - 1_0$, $9_1 = 3_1 3_4$, $9_2 = 3_1 3_3$, $9_3 = 3_2 3_4$, $15_1 = 3_1 5_1$ and $15_2 = 3_2 5_1$. Let $\tilde{G}$ be a group generated by G and $-I$ where I is the identity matrix of degree 3. Then $\tilde{G}$ is a complex reflection group of type $J_3(5)$ (*cf.* [Cohen76]) and there exist three homogeneous invariants f_1, f_2, f_3 with $\deg f_1 = 6$, $\deg f_2 = 12$, $\deg f_3 = 30$ such that $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, f_3]$ and $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \mathrm{Jac}(f_1, f_2, f_3)$. Hence we have

$$P_{S_G,1_0} = 1,$$
$$P_{S_G,3_1} = t^{5} + t^{11} + t^{20} + t^{26} + t^{29} + t^{44},$$
$$P_{S_G,3_2} = t + t^{16} + t^{19} + t^{25} + t^{34} + t^{40},$$
$$P_{S_G,3_3} = t^{5} + t^{17} + t^{20} + t^{23} + t^{32} + t^{38},$$

$$P_{S_G,3_4} = t^7 + t^{13} + t^{22} + t^{25} + t^{28} + t^{40},$$
$$P_{S_G,5_1} = t^6 + t^{12} + t^{15} + t^{18} + t^{21} + t^{24} + t^{27} + t^{30} + t^{33} + t^{39},$$
$$P_{S_G,5_2} = t^6 + t^{12} + t^{15} + t^{18} + t^{21} + t^{24} + t^{27} + t^{30} + t^{33} + t^{39},$$
$$P_{S_G,6_1} = t^4 + 2t^{10} + t^{16} + t^{19} + t^{22} + 2t^{25} + t^{28} + t^{31} + t^{37} + t^{43},$$
$$P_{S_G,6_2} = t^2 + t^8 + t^{14} + t^{17} + 2t^{20} + t^{23} + t^{26} + t^{29} + 2t^{35} + t^{41},$$
$$P_{S_G,8_1} = t^6 + t^9 + t^{12} + 2t^{15} + t^{18} + 2t^{21} + 2t^{24} + t^{27} + 2t^{30} + t^{33} + t^{36} + t^{39},$$
$$P_{S_G,8_2} = t^9 + 2t^{12} + 2t^{15} + 2t^{18} + t^{21} + t^{24} + 2t^{27} + 2t^{30} + 2t^{33} + t^{36},$$
$$P_{S_G,9_1} = t^6 + t^9 + 2t^{12} + t^{15} + 2t^{18} + 2t^{21} + 2t^{24} + 2t^{27} + t^{30} + 2t^{33} + t^{36} + t^{39},$$
$$P_{S_G,9_2} = t^4 + t^{10} + 2t^{13} + 2t^{16} + 2t^{19} + 2t^{22} + t^{25} + 2t^{28} + 2t^{31} + t^{34} + 2t^{37},$$
$$P_{S_G,9_3} = 2t^8 + t^{11} + 2t^{14} + 2t^{17} + t^{20} + 2t^{23} + 2t^{26} + 2t^{29} + 2t^{32} + t^{35} + t^{41},$$
$$P_{S_G,10} = t^3 + 2t^9 + t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + 2t^{24} + 2t^{27} + 2t^{30} + t^{33} + 2t^{36} + t^{42},$$
$$P_{S_G,15_1} = t^5 + t^8 + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 3t^{26} + 3t^{29} + 3t^{32} + 2t^{35} + 2t^{38},$$
$$P_{S_G,15_2} = 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + 3t^{25} + 3t^{28} + 3t^{31} + 3t^{34} + t^{37} + t^{40}.$$

5.8 Summary

Here we list the invariants for the subgroups of type (E)–(L) where $d_{\max} = d_1 + d_2 + d_3 - 3$:

type	d_1, d_2, d_3	d_4, d_5	$d_{\max}$	$\lvert G\rvert$	e
E	6, 6, 12	12, 9	21	108	4
F	6, 9, 12	12	24	216	3
G	9, 12, 18	18	36	648	3
H	2, 6, 10	15	15	60	2
I	4, 6, 14	21	21	168	2
J	6, 6, 15	12	24	180	3
K	6, 12, 21	18	36	504	3
L	6, 12, 30	45	45	1080	2

Table 12: Groups (E)–(L)

Summarizing the calculation in the previous subsections we infer

Theorem 5.9 *Let G be a subgroup of* $\mathrm{SL}(3,\mathbf{C})$ *of type from* (E) *to* (L). *Let f_i be generators of the invariant ring S^G and $d_i = \deg f_i$ $(1 \le i \le n)$, $d_{\max} = d_1 + d_2 + d_3 - 3$ as in Table 12, and S_G the coinvariant algebra. Then for any irreducible representation ρ_j of G the Molien series P_{S_G,ρ_j} is given by the formula*

$$P_{S_G,\rho_j}(t) = \Big[\prod_{i=4}^{n}(1 - t^{d_i})P_{R,\rho_j}(t)\Big]_+ + \begin{cases} t^{18}(\delta_{j,8} + \delta_{j,5_1}) & \text{if } G = \text{(F) or (J)},\\ 0 & \text{otherwise.}\end{cases}$$

where $[f(t)]_+ := \sum_{d=0}^{d_{\max}} \max\{a_d, 0\}t^d$ *for* $f(t) = \sum a_d t^d \in \mathbf{Z}[t]$.

Remark 5.10 Theorem 5.9 implies the following. Suppose $j \neq 8$ if G is type (F) or $j \neq 5_1$ if G is of type (J). For any fixed irreducible representation ρ_j multiplication by f_α $(\alpha = 4, 5)$ is a homomorphism ϕ^α_{d,ρ_j} from $(R_d)_{\rho_j}$ to $(R_{d+d_\alpha})_{\rho_j}$. Then ϕ^α_{d,ρ_j} is surjective if $\dim(R_d)_{\rho_j} \ge \dim(R_{d+d_\alpha})_{\rho_j}$, while it is injective if $\dim(R_d)_{\rho_j} \le \dim(R_{d+d_\alpha})_{\rho_j}$. In other words, rank ϕ^α_{d,ρ_j} is equal to $\min\{\dim(R_d)_{\rho_j}, \dim(R_{d+d_\alpha})_{\rho_j}\}$. Moreover $f_4R \cap f_5R = f_4f_5R \simeq f_4f_5\mathbf{C}$. In the exceptional case, for instance, of type (J) and $j = 5_1$, the nonzero coefficient of t^{19} in $P_{S_G,3_2}$ explains nonvanishing of the coefficient of t^{18} in $P_{S_G,5_1}$. We note that the above theorem does not imply $P_{S_G}(t) = [\prod_{i=4}^{n}(1 - t^{d_i})P_R(t)]_+$ even in the case other than (F) and (J).

6 Appendix

In this appendix we list the decompositions of irreducible representations tensored with the natural representation ρ.

6.1 Type (E)

$$1_0 \otimes \rho = 3_1, \qquad 1_1 \otimes \rho = 3_2,$$
$$1_2 \otimes \rho = 3_3, \qquad 1_3 \otimes \rho = 3_4,$$
$$3_1 \otimes \rho = 3_5 + 3_6 + 3_8, \qquad 3_2 \otimes \rho = 3_5 + 3_6 + 3_7,$$
$$3_3 \otimes \rho = 3_6 + 3_7 + 3_8, \qquad 3_4 \otimes \rho = 3_5 + 3_7 + 3_8,$$
$$3_5 \otimes \rho = 1_0 + 4_1 + 4_2, \qquad 3_6 \otimes \rho = 1_1 + 4_1 + 4_2,$$
$$3_7 \otimes \rho = 1_2 + 4_1 + 4_2, \qquad 3_8 \otimes \rho = 1_3 + 4_1 + 4_2,$$
$$4_1 \otimes \rho = 3_1 + 3_2 + 3_3 + 3_4, \qquad 4_2 \otimes \rho = 3_1 + 3_2 + 3_3 + 3_4.$$

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6.2 Type (F)

$$
\begin{aligned}
&1_0 \otimes \rho = 3_1, && 1_1 \otimes \rho = 3_2,\\
&1_2 \otimes \rho = 3_3, && 1_3 \otimes \rho = 3_4,\\
&2 \otimes \rho = 6_2, && 3_1 \otimes \rho = 3_5 + 6_1\\
&3_2 \otimes \rho = 3_6 + 6_1, && 3_3 \otimes \rho = 3_7 + 6_1,\\
&3_4 \otimes \rho = 3_8 + 6_1, && 3_5 \otimes \rho = 1_0 + 8,\\
&3_6 \otimes \rho = 1_1 + 8, && 3_7 \otimes \rho = 1_2 + 8,\\
&3_8 \otimes \rho = 1_3 + 8 && 6_1 \otimes \rho = 2 + 2 \cdot 8,\\
&6_2 \otimes \rho = 3_5 + 3_6 + 3_7 + 3_8 + 6_1, && 8 \otimes \rho = 3_1 + 3_2 + 3_3 + 3_4 + 2 \cdot 6_2.
\end{aligned}
$$

6.3 Type (G)

$$
\begin{aligned}
&1_0 \otimes \rho = 3_1, && 1_1 \otimes \rho = 3_2,\\
&1_2 \otimes \rho = 3_3, && 2_1 \otimes \rho = 6_5,\\
&2_2 \otimes \rho = 6_6, && 2_3 \otimes \rho = 6_4,\\
&3_1 \otimes \rho = 3_4 + 6_1, && 3_2 \otimes \rho = 3_5 + 6_2,\\
&3_3 \otimes \rho = 3_6 + 6_3, && 3_4 \otimes \rho = 1_0 + 8_1,\\
&3_5 \otimes \rho = 1_1 + 8_2, && 3_6 \otimes \rho = 1_2 + 8_3,\\
&3_7 \otimes \rho = 9_1, && 6_1 \otimes \rho = 2_2 + 8_1 + 8_3,\\
&6_2 \otimes \rho = 2_3 + 8_1 + 8_2, && 6_3 \otimes \rho = 2_1 + 8_2 + 8_3,\\
&6_4 \otimes \rho = 3_4 + 6_2 + 9_2, && 6_5 \otimes \rho = 3_5 + 6_3 + 9_2,\\
&6_6 \otimes \rho = 3_6 + 6_1 + 9_2, && 8_1 \otimes \rho = 3_1 + 6_4 + 6_6 + 9_1,\\
&8_2 \otimes \rho = 3_2 + 6_4 + 6_5 + 9_1, && 8_3 \otimes \rho = 3_3 + 6_5 + 6_6 + 9_1,\\
&9_1 \otimes \rho = 6_1 + 6_2 + 6_3 + 9_2, && 9_2 \otimes \rho = 3_7 + 8_1 + 8_2 + 8_3.
\end{aligned}
$$

6.4 Type (H)

$$
\begin{aligned}
&1_0 \otimes \rho = 3_1, && 3_1 \otimes \rho = 1_0 + 3_1 + 5,\\
&3_2 \otimes \rho = 4 + 5, && 4 \otimes \rho = 3_2 + 4 + 5,\\
&5 \otimes \rho = 3_1 + 3_2 + 4 + 5.
\end{aligned}
$$

6.5 Type (I)

$$1_0 \otimes \rho = 3_1, \qquad 3_1 \otimes \rho = 3_2 + 6,$$
$$3_2 \otimes \rho = 1_0 + 8, \qquad 6 \otimes \rho = 3_2 + 7 + 8,$$
$$7 \otimes \rho = 6 + 7 + 8, \qquad 8 \otimes \rho = 3_1 + 6 + 7 + 8.$$

6.6 Type (J)

$$1_0 \otimes \rho = 3_1, \qquad 1_1 \otimes \rho = 3_2,$$
$$1_2 \otimes \rho = 3_3, \qquad 3_1 \otimes \rho = 1_2 + 3_2 + 5_3,$$
$$3_2 \otimes \rho = 1_0 + 3_3 + 5_1, \qquad 3_3 \otimes \rho = 1_1 + 3_1 + 5_2,$$
$$3_4 \otimes \rho = 4_3 + 5_3, \qquad 3_5 \otimes \rho = 4_1 + 5_1,$$
$$3_6 \otimes \rho = 4_2 + 5_2, \qquad 4_1 \otimes \rho = 3_4 + 4_2 + 5_2,$$
$$4_2 \otimes \rho = 3_5 + 4_3 + 5_3, \qquad 4_3 \otimes \rho = 3_6 + 4_1 + 5_1,$$
$$5_1 \otimes \rho = 3_1 + 3_4 + 4_2 + 5_2, \qquad 5_2 \otimes \rho = 3_2 + 3_5 + 4_3 + 5_3,$$
$$5_3 \otimes \rho = 3_3 + 3_6 + 4_1 + 5_1.$$

6.7 Type (K)

$$1_0 \otimes \rho = 3_1, \qquad 1_1 \otimes \rho = 3_2,$$
$$1_2 \otimes \rho = 3_3, \qquad 3_1 \otimes \rho = 3_4 + 6_3,$$
$$3_2 \otimes \rho = 3_5 + 6_1, \qquad 3_3 \otimes \rho = 3_6 + 6_2,$$
$$3_4 \otimes \rho = 1_0 + 8_1, \qquad 3_5 \otimes \rho = 1_1 + 8_2,$$
$$3_6 \otimes \rho = 1_2 + 8_3, \qquad 6_1 \otimes \rho = 3_6 + 7_2 + 8_2,$$
$$6_2 \otimes \rho = 3_4 + 7_3 + 8_3, \qquad 6_3 \otimes \rho = 3_5 + 7_1 + 8_1,$$
$$7_1 \otimes \rho = 6_2 + 7_2 + 8_2, \qquad 7_2 \otimes \rho = 6_3 + 7_3 + 8_3,$$
$$7_3 \otimes \rho = 6_1 + 7_1 + 8_1, \qquad 8_1 \otimes \rho = 3_1 + 6_2 + 7_2 + 8_2,$$
$$8_2 \otimes \rho = 3_2 + 6_3 + 7_3 + 8_3, \qquad 8_3 \otimes \rho = 3_3 + 6_1 + 7_1 + 8_1.$$

6.8 Type (L)

$$1_0 \otimes \rho = 3_1, \qquad 3_1 \otimes \rho = 3_2 + 6_1,$$
$$3_2 \otimes \rho = 1_0 + 8_1, \qquad 3_3 \otimes \rho = 9_2,$$

$$3_4 \otimes \rho = 9_1, \qquad 5_1 \otimes \rho = 15_1,$$
$$5_2 \otimes \rho = 15_1, \qquad 6_1 \otimes \rho = 8_1 + 10,$$
$$6_2 \otimes \rho = 3_2 + 15_2, \qquad 8_1 \otimes \rho = 3_1 + 6_2 + 15_1,$$
$$8_2 \otimes \rho = 9_3 + 15_1, \qquad 9_1 \otimes \rho = 3_3 + 9_3 + 15_1,$$
$$9_2 \otimes \rho = 8_2 + 9_1 + 10, \qquad 9_3 \otimes \rho = 3_4 + 9_2 + 15_2,$$
$$10 \otimes \rho = 6_2 + 9_3 + 15_1, \qquad 15_1 \otimes \rho = 6_1 + 9_2 + 2 \cdot 15_2,$$
$$15_2 \otimes \rho = 5_1 + 5_2 + 8_1 + 8_2 + 9_1 + 10.$$

6.9 Addendum

In [GNS00, p. 52, p. 53] there are a few errors in notation and formulation, though harmless for the consequences in the subsequent sections of [GNS00]. As the arguments in this article are entirely independent from [GNS00] we would like to correct the errors in [GNS00] in the paper [GNS3] much closer to [GNS00].

We acknowledge Professor Li Chiang for pointing out the following errors in [GNS00] (different from the above) to us. The fourth line of [GNS00, p. 57] must be replaced by

$$f^3 + \bar{f}^3 = \prod_{i=0}^{2}(f + \omega^i \bar{f}) = 27 f_3^2 - 9 f_2 f_4 + 2 f_2^3.$$

The fifth column of $S_d[\rho]$ of [GNS00, p. 57, Table 2.2] must be replaced by

$$\{\bar{f}^2\} + \{f^2\} + \{yzf, \omega^2 zxf, \omega xyf\}.$$

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