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KLEENE ALGEBRAS ARE ALMOST UNIVERSAL

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This paper studies endomorphism monoids of Kleene algebras. The main result is that these algebras form an almost universal variety $\underset{\sim}{K}$, from which it follows that for a given monoid M there is a proper class of non-isomorphic Kleene algebras with endomorphism monoid M^{+} (where M^{+} denotes the extension of M by a single element that is a right zero in M^{+}). Kleene algebras form a subvariety of de Morgan algebras containing Boolean algebras. Previously it has been shown the latter are uniquely determined by their endomorphisms, while the former constitute a universal variety, containing, in particular, arbitrarily large finite rigid algebras. Non-trivial algebras in $\underset{\sim}{K}$ always have non-trivial endomorphisms (so that universality of $\underset{\sim}{K}$ is ruled out) and unlike the situation for de Morgan algebras the size of End(L) for a finite Kleene algebra L necessarily increases as $|L|$ does. The paper concludes with results on endomorphism monoids of algebras in subvarieties of the variety of MS-algebras.

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1. Introduction.

This paper concerns endomorphism monoids of Kleene algebras and complements [3], which deals with endomorphism monoids of de Morgan algebras.

A de Morgan algebra is an algebra $(L;\vee,\wedge,\sim,0,1)$ of type $(2,2,1,0,0)$ such that $(L;\vee,\wedge,0,1)$ is a distributive $(0,1)$-lattice, $\sim$ is a dual $(0,1)$-lattice endomorphism (so that $\sim(a\vee b) = \sim a\wedge\sim b$, $\sim(a\wedge b) = \sim a\vee\sim b$, $\sim 0 = 1$, and $\sim 1 = 0$) and $\sim\sim a = a$.

A Kleene algebra is a de Morgan algebra in which the inequality

$$a\wedge\sim a\leq b\vee\sim b$$

holds.

We denote the varieties of de Morgan and Kleene algebras by $\underset{\sim}{M}$ and $\underset{\sim}{K}$, respectively.

It was shown by Kalman [21] that the non-trivial subvarieties of $\underset{\sim}{M}$ form a 3-element chain $\underset{\sim}{B} \subset \underset{\sim}{K} \subset \underset{\sim}{M}$ where $\underset{\sim}{B}$ denotes the variety of Boolean algebras. Here $\underset{\sim}{B}$ is generated by the 2-element chain B, $\underset{\sim}{K}$ by the 3-element chain $K = \{0,a,1\}$ in which $\sim a = a$, and $\underset{\sim}{M}$ by the 4-element complemented lattice $\{0,a,b,1\}$ with $\sim a = a$ and $\sim b = b$.

For further references on Kleene and de Morgan algebras, see Balbes and Dwinger [4] and the bibliography of Goldberg [16].

As regards endomorphisms, $\underset{\sim}{B}$ and $\underset{\sim}{M}$ display opposite extremes of behaviour, so that it is of interest to investigate the intermediate variety of Kleene algebras. We shall show that $\underset{\sim}{K}$ has much closer affinities with $\underset{\sim}{M}$ than with $\underset{\sim}{B}$. It is well-known that a Boolean algebra is uniquely determined by its endomorphism monoid; this was proved, independently, by Magill [24], Maxson [25], and Schein [30]. By contrast, given any monoid M, it is possible to find a proper class of non-isomorphic de Morgan algebras each member of which has M as its endomorphism monoid [3].

Given a variety $\underset{\sim}{V}$, the existence or otherwise of many non-isomorphic algebras with the same endomorphism monoid is closely related to the existence or otherwise of suitable category embeddings into $\underset{\sim}{V}$;

specifically, any universal variety is such that there is a proper class (to isomorphism) of its algebras having any prescribed monoid as an endomorphism monoid. We recall that a category $\underset{\sim}{C}$ is said to be universal if any algebraic category fully embeds in $\underset{\sim}{C}$, or, equivalently, if the category $\underset{\sim}{G}$ of undirected graphs and their compatible mappings fully embeds in $\underset{\sim}{C}$; see Pultr and Trnková [29] for a full account of universality and its consequences. (An undirected graph (V,E) is a set V together with a collection E of 2-element subsets. A map ϕ between graphs G and G' is said to be compatible if, for $x,y \in V$, $\{\phi(x),\phi(y)\} \in E'$ whenever $\{x,y\} \in E$.) A variety $\underset{\sim}{V}$ is said to be universal if the category with the algebras of $\underset{\sim}{V}$ as objects and with all homomorphisms between them as morphisms is universal. The main theorem of [3] asserts that de Morgan algebras form a universal variety.

Any universal variety has arbitrarily large rigid algebras, that is, algebras with no non-trivial endomorphisms. An easy lemma, proved in Section 2 (Lemma 4) shows that the only rigid Kleene algebras are the 2-element algebra B and the 3-element algebra K. Thus $\underset{\sim}{K}$ cannot be universal. However, this observation gives a somewhat misleading impression. As we shall show, there exist in profusion Kleene algebras whose only endomorphisms are constants, an endomorphism being called constant if its image consists of the constants of the algebra, namely 0 and 1. A variety $\underset{\sim}{V}$ is called almost universal if there exists an embedding Φ of $\underset{\sim}{G}$ into $\underset{\sim}{V}$ which is almost full in the sense that every non-constant homomorphism between objects of $\Phi(\underset{\sim}{G})$ is in the image of Φ. Our main result, proved in Section 3, is:

THEOREM 1. *The variety of Kleene algebras is almost universal.*

If End(A) denotes the endomorphism monoid of an algebra A, then, as a Corollary, we obtain (see Pultr and Trnková [29]):

THEOREM 2. *For any monoid M and infinite cardinal $\kappa \geq |M|$, there exists a family of Kleene algebras $(L_i : i<2^\kappa)$ such that, for $i<2^\kappa$, $|L_i| = \kappa$, $End(L_i) \cong M^+$ where M^+ denotes the monoid M with the* *addition of one new element that is a right zero in* M^+, *and, for distinct* $i,j<2^\kappa$, *there exists precisely one homomorphism* $f:L_i \longrightarrow L_j$ *(which is a constant).*

The embedding we construct of $\mathcal{G}$ into $\mathcal{K}$ is such that every algebra in the image of $\mathcal{G}$ is infinite. This is only to be expected since the size of the endomorphism monoid of a finite Kleene algebra increases with the size of the algebra, a fact we establish in Lemma 5. This implies that we could not hope to find a countably infinite family of finite non-isomorphic Kleene algebras having a given finite monoid as endomorphism monoid. Theorem 3 of [3] shows that a family of de Morgan algebras of this kind does exist.

2. Preliminaries.

Our construction in Section 3 uses topological duality. We summarize the basic information about duality which we require. For a more detailed account consult [26], [27], or the survey papers Davey and Duffus [12] and [28].

For a poset P , a sequence $x_0,\ldots,x_{n-1}$ in P is a path of length $n-1$ connecting x_0 to x_{n-1} if x_i is comparable with x_{i+1} for $i<n-1$. A path is a fence if there are no other comparabilities between its elements. A poset is connected if, for all $x,y\in P$, there is a path connecting x to y . Clearly, connectivity is an equivalence relation, the equivalence classes of which are referred to as components. The distance from x to y is the length of a path of minimal size connecting x and y if x and y are connected, and is undefined otherwise. For $Q\subseteq P$, let $[Q)$ and $(Q]$ denote the order filter and ideal generated by Q . Then Q is increasing (decreasing) providing $Q=[Q)$ $(Q=(Q])$. The set of minimal (maximal) elements of P is denoted $\mathrm{Min}(P)$ ($\mathrm{Max}(P)$). For posets P and P' , a mapping $\phi:P\to P'$ is order-preserving if $\phi(x)\le\phi(y)$ whenever $x\le y$.

By an ordered topological space (P,τ) we mean an ordered set P with a topology τ defined on it. The space is totally order-disconnected provided that for any $x\nleq y$ in P , there exists a clopen increasing subset X of P such that $x\in X$ and $y\notin X$. If, in addition, τ is compact, then (P,τ) is called a Priestley space. Then there exists a dual category equivalence between the category of distributive $(0,1)$-lattices with $(0,1)$-lattice homomorphisms and the category of Priestley spaces with continuous order-preserving maps. The equivalence can be defined in such a way that if L and (P,τ) are associated under it, then the elements of L correspond to the clopen increasing subsets of P. Further, if $f:L \to L'$ is associated with the continuous order-preserving map $\phi:P' \to P$, then $f(a) = b$ if and only if $\phi^{-1}(A) = B$ where A and B are the clopen increasing sets that correspond to a and b.

The dual spaces of Kleene algebras were characterized in Cornish and Fowler [10]; for a more direct proof see Goldberg [17]. Let $\underset{\sim}{S}_K$ denote the following category of Priestley spaces: the objects of $\underset{\sim}{S}_K$ are spaces (P,τ,ζ) where (P,τ) is a Priestley space and ζ is a continuous order-reversing involution on P such that, for each $x \in P$, x and $\zeta(x)$ are comparable; the morphisms of $\underset{\sim}{S}_K$ are the continuous order-preserving maps which commute with the ζ-map.

PROPOSITION 3. ([10] or see Section 2 of [17]). The category $\underset{\sim}{K}$ of Kleene algebras in dually equivalent to $\underset{\sim}{S}_K$. Under this duality, if L and (P,τ,ζ) correspond, then, for $a \in L$, $\sim a$ is represented by $P \setminus \zeta^{-1}(A)$ where A represents a.

We can now use duality to establish easily the two lemmas on the existence of endomorphisms referred to in the introduction. Both lemmas can alternatively be proved algebraically.

LEMMA 4. *The only non-trivial rigid Kleene algebras are the 2-element algebra* B *and the 3-element algebra* K.

Proof. The duals of the two exceptional algebras cited are a 1-point space and a 2-point chain. It will therefore be sufficient to show that if $(P,\tau,\zeta) \in \underset{\sim}{S}_K$ is such that P is not of this form then there exists a continuous order-preserving map on P which commutes with ζ and which is not the identity. There are two cases to consider.

First suppose $p = \zeta(p)$ for some $p \in P$. Then the constant map with image p provides the required morphism.

Now assume that for every $x \in P$, either $x > \zeta(x)$ or $x < \zeta(x)$. Suppose, without loss of generality, that $p > \zeta(p)$ for some $p \in P$. Define $\phi: P \to P$ by $\phi(x) = p$ if $x > \zeta(x)$ and $\zeta(p)$ otherwise. Then ϕ is a morphism in $\mathcal{S}_K$. □

Our second lemma shows that the size of the endomorphism monoid of a finite Kleene algebra increases with the size of the algebra.

LEMMA 5. *Let L be a finite Kleene algebra with (P,τ,ζ) as its dual in $\mathcal{S}_K$. Then $|End(L)| \geq |P|/2$.*

Proof. For $p \in P$ we shall define a morphism $\phi_p: P \to P$ in $\mathcal{S}_K$ such that ϕ_p and ϕ_q are distinct whenever p and q are and $q \neq \zeta(p)$.

If $p = \zeta(p)$, let $\phi_p(x) = p$ for all $x \in P$. Clearly ϕ_p is continuous, order-preserving, and commutes with ζ.

For $p \neq \zeta(p)$, we may assume without loss of generality that $p > \zeta(p)$. There are two cases to consider.

Suppose that for some $q_0, q_1 \in P$, $q_0 = \zeta(q_0)$ and $q_0 < q_1$. Let

$$\phi_p(x) = \begin{cases} q_1 & \text{for } x \geq p, \\ \zeta(q_1) & \text{for } x \leq \zeta(p), \\ q_0 & \text{otherwise.} \end{cases}$$

Then it is easily checked that ϕ_p is a morphism in $\mathcal{S}_K$.

If no such pair q_0 and q_1 exists then let

$$\phi_p(x) = \begin{cases} x & \text{for } \zeta(x) = x, \\ p & \text{for } x > \zeta(x), \\ \zeta(p) & \text{otherwise.} \end{cases}$$

□

As can be seen from the proof, the lower bound on the number of endomorphisms in the preceding lemma is a conservative one. A more careful analysis (not justified here) reveals that the number of non-constant endomorphisms of a finite Kleene algebra increases with the size of the algebra.

Any space (P,τ,ζ) in $\underset{\sim}{S}_K$ is the union of subspaces

$$\vec{P} = \{x \in P: \ \ \zeta(x) \leq x\}$$

$$\text{and} \quad \overleftarrow{P} = \{x \in P: \ \ x \leq \zeta(x)\} \ .$$

The spaces $\vec{P}$ and $\overleftarrow{P}$ are homeomorphic, order anti-isomorphic, and respectively increasing and decreasing. They intersect in $\{x \in P: \zeta(x) = x\}$ which is a (discretely ordered) Boolean space contained in the minimal points of $\vec{P}$. We can define a relation R on $\vec{P}$ by $x - y$ if and only if $x \geq \zeta(y)$ (note that ζ maps $\vec{P}$ onto $\overleftarrow{P}$, so that R captures the order relations between elements of $\vec{P}$ and $\overleftarrow{P}$).

In the other direction, any space in $\underset{\sim}{S}_K$ (and, hence, any Kleene algebra) can be constructed from the following components: a Priestley space (Q,σ) (corresponding to $\vec{P}$ with the induced topology); a Boolean subspace Q_0 of the minimal points of (Q,σ) (corresponding to $\vec{P} \cap \overleftarrow{P})$; and a closed binary relation $-$ on Q satisfying

(i) for all $x \in Q$, $x - x$,

(ii) for all x, $y \in Q$, $x - y$ and $x \in Q_0$ imply $x \leq y$,

(iii) for all $x,y,z \in Q$, $x - y$ and $y \leq z$ imply $z - x$.

A space P in $\underset{\sim}{S}_K$ is then obtained as follows:

(i) P is the topological sum of Q and its order dual Q^d identified (via the identity map) along Q_0 and Q_0^d ;

(ii) $\zeta \restriction Q$ is the identity map from Q to Q^d and $\zeta \restriction Q^d$ is the identity map from Q^d to Q ;

(iii) the order on P is defined as follows: the spaces Q and Q^d have their original orders; for $x \in Q$ and $y \in Q^d$, $x \leq y$ if and only if $x = y$; and, for $x \in Q^d$ and $y \in Q$, $x \leq y$ if and only if $x - y$.

The spaces $(P,\tau,P_0,-)$ together with continuous order-preserving maps that also preserve P_0 and the $-$ relation are exactly those in the

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category dual to $\underset{\sim}{K}$ under the natural duality obtained by Davey and Werner [14] (see Clark and Krauss [9]), in which the dual of an algebra $L \in \underset{\sim}{K}$ is given by the homomorphisms from L into the 3-element generating algebra K, suitably structured. For details see [14] and for a fuller discussion of the relationship of the Priestley and natural duals, see [13].

As will be seen from the above remarks, the Priestley dual is a more complicated ordered set than the natural dual. Since the space we need to construct in Section 3 is very involved, we choose to work with the natural dual. To simplify notation a little further, we shall recast the description of Kleene algebra duals in a slightly different (but clearly equivalent) form.

Let $\underset{\sim}{T}_K$ denote the following category:

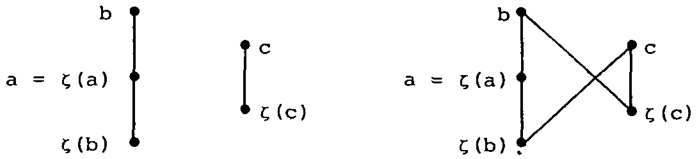
An object (P,τ,P_0,P_1) of $\underset{\sim}{T}_K$ (henceforth referred to as a k-space) is a quadruple such that (P,τ) is a Priestley space, $P_0 \subseteq P$, and $\{\{x\} : x \in P\} \subseteq P_1 \subseteq \{\{x,y\} : x,y \in P\}$. Both P_0 and P_1 are required to be closed, that is, P_0 is a closed subspace of P and $\{(x,y) : \{x,y\} \in P_1\}$ is a closed subspace of $P \times P$ with the product topology. Furthermore, we require:

(i) for $x,y,z \in P$, $\{x,y\} \in P_1$ and $y \leq z$ imply $\{x,z\} \in P_1$;

(ii) for $x,y \in P$, $x \in P_0$ and $\{x,y\} \in P_1$ imply $x \leq y$.

A morphism $\phi : (P,\tau,P_0,P_1) \to (P',\tau',P'_0,P'_1)$ of $\underset{\sim}{T}_K$ (subsequently referred to as a k-map) is a continuous order-preserving map $\phi : P \to P'$ that preserves P_0 and P_1: that is, for $x \in P_0$, $\phi(x) \in P'_0$ and, for $\{x,y\} \in P_1$, $\{\phi(x), \phi(y)\} \in P'_1$.

We then have, from [14], [13], and Proposition 3, that $\underset{\sim}{T}_K$ and $\underset{\sim}{S}_K$ are isomorphic categories and that $\underset{\sim}{T}_K$ is dually equivalent to $\underset{\sim}{K}$.



Q Q'

Figure 1

P P'

$P_0 = \{\{a\}\}$ $P'_0 = \{\{a\}\}$

$P_1 = \{\{a\},\{b\},\{c\},\{a,b\}\}$ $P'_1 = \{\{a\},\{b\},\{c\},\{a,b\},\{b,c\}\}$

Figure 2

Before we proceed to discuss almost universality we present two simple examples by way of illustration. Figure 1 gives the spaces in $\underset{\sim}{S}_K$ of two non-isomorphic Kleene algebras and Figure 2 gives the corresponding k-spaces (the topology in each case is discrete). Notice that their k-spaces P and P' are order-isomorphic. If $\phi : P \to P'$ denotes the order-isomorphism, then ϕ is a k-map, but ϕ^{-1} is not as it fails to map P'_1 into P_1.

To utilize $\underset{\sim}{T}_K$ in the verification that $\underset{\sim}{K}$ is almost universal, it is necessary to recognize those k-maps which correspond to constant homomorphisms. Suppose that $f : L \to L'$ in $\underset{\sim}{K}$ is associated with

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$\phi : P' \to P$ in $\underset{\sim}{T}_K$ and with $\psi : Q' \to Q$ in $\underset{\sim}{S}_K$. Then f is a constant homomorphism if and only if $\psi(Q') = \{x\}$ for some $x = \zeta(x) \in Q$. Thus f is a constant if and only if $\phi(P') = \{y\}$ for some $y \in P_0$.

To show that $\underset{\sim}{K}$ is almost universal, it will be sufficient to show that $\underset{\sim}{G}$ is dually isomorphic to a subcategory of $\underset{\sim}{T}_K^S$ of $\underset{\sim}{T}_K$ each k-space of which is such that $|P_0| = 1$, together with all k-maps except those with a one-element image. As in previous proofs of almost universality of varieties of distributive-lattice-ordered algebras, this is accomplished by making use of a dual embedding of $\underset{\sim}{G}$ as a full subcategory of a suitable category $\underset{\sim}{T}_5$ of ordered topological spaces, and then constructing an isomorphic embedding of $\underset{\sim}{T}_5$ into $\underset{\sim}{T}_K^S$.

The category $\underset{\sim}{T}_5$, which was introduced in [2], is defined as follows: objects (P, τ, P_0) of $\underset{\sim}{T}_5$ are compact totally order-disconnected spaces (P, τ) such that (i) P is a connected partially ordered set of height 2 with a 5-element subset of minimal isolated elements $P_0 = \{p_0, \ldots, p_4\} \subseteq P$, and (ii) if $x \in \mathrm{Max}(P)$, then there are distinct i, j such that $x \geq p_i, p_j$; morphisms of $\underset{\sim}{T}_5$ are continuous order-preserving maps $\phi : P \to P'$ that satisfy $\phi(p_i) = p_i'$ for $i < 5$.

In [2] (see also [1], Hedrlín and Pultr [19], [20], and recently, Koubek [22]), it was shown that $\underset{\sim}{G}$ is dually isomorphic to a full subcategory of $\underset{\sim}{T}_5$. Therefore, to establish that $\underset{\sim}{K}$ is almost universal, it remains to construct an isomorphic embedding Ψ of $\underset{\sim}{T}_5$ into $\underset{\sim}{T}_K^S$.

3. Proof of Theorem 1.

The object of this section is to establish that the variety of Kleene algebras $\underset{\sim}{K}$ is almost universal. As was indicated at the end of Section 2, this is achieved immediately we have defined a faithful functor $\Psi : \underset{\sim}{T}_5 \to \underset{\sim}{T}_K$ such that

(i) for any object (P, τ, P_0, P_1) of $\Psi(T_5)$, $|P_0| = 1$

(which ensures that each algebra in the dually equivalent subcategory of

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Kleene algebras has only one constant endomorphism) and

(ii) given objects (P,τ,P_0,P_1) and (P',τ',P_0',P_1') of $\Psi(\underset{\sim}{T}_5)$ and a k-map $\phi : P \to P'$ such that $\phi(P) \neq P_0'$, then there is a morphism ψ of $\underset{\sim}{T}_5$ such that $\Psi(\psi) = \phi$.

In other words, Ψ is an almost full functor (which implies that every non-constant homomorphism, and there is only one constant homomorphism, between any two algebras in the dually equivalent subcategory of Kleene algebras is a morphism in the image of Ψ).

Before proceeding to define Ψ it is appropriate to give some comments concerning its complexity. We have already observed in Section 2 that the number of non-constant endomorphisms of a finite Kleene algebra increases with the size of the algebra. Further consideration reveals that if the k-space of an infinite Kleene algebra has only a finite number of limit points, then its endomorphism monoid is infinite. In the light of these remarks it is to be expected that, for any $(P,\tau,P_0) \in \underset{\sim}{T}_5$, the k-space $\Psi(P,\tau,P_0)$ will contain infinitely many limit points.

Recalling Boolean algebras and Stone algebras and noting the abundance of endomorphisms of a finite Kleene algebra, one might suspect that such an algebra would be recoverable from its endomorphism monoid. It is worth remarking that this is not the case. In fact, it can be shown that, for any $n < \omega$, there is a family $(L_i : i < n)$ of non-isomorphic finite Kleene algebras such that $\mathrm{End}(L_i) \cong \mathrm{End}(L_j)$ for $i,j < n$. We omit the details.

We now define Ψ. This is a relatively simple matter for morphisms but not for objects. Thus, whilst defining Ψ on objects, we will take time to verify that, for $(P,\tau,P_0) \in \underset{\sim}{T}_5$, $\Psi(P,\tau,P_0)$ is indeed an object of $\underset{\sim}{T}_K$. Only once this has been done will we complete the definition of Ψ by extending it to morphisms.

Let $\underset{\sim}{Z}$ denote the set of integers. Then set

$$A_i = \{a_{i,j} : j \in \underset{\sim}{Z}\} \quad \text{for} \quad 0 < i ,$$
$$A_i' = \{a_{i,j}' : j \in \underset{\sim}{Z}\} \quad \text{for} \quad 0 < i ,$$

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$$B \;= \{b_i : i \in \mathbb{Z}\} ,$$
$$B' = \{b'_i : i \in \mathbb{Z}\} ,$$
$$C_i = \{c_{i,j} : 0 \le j < 9\} \quad \text{for} \quad 0 \le i < 5 ,$$
$$D \;= \{d_i : 0 < i\} ,$$
$$D' = \{d'_i : 0 < i\} ,$$
and $$E \;= \{e\} .$$

Define a partial order on

$$Q = \cup(A_i : 0 < i) \cup \cup(A'_i : 0 < i) \cup B \cup B' \cup \cup(C_i : 0 \le i < 5) \cup D \cup D' \cup E$$

as follows:

(i) $a_{i,2j} \le a_{i,2j-1}, a_{i,2j+1}$ for $0 < i$ and $j \in \mathbb{Z}$;

(ii) $a'_{i,2j} \le a'_{i,2j-1}, a'_{i,2j+1}$ for $0 < i$ and $j \in \mathbb{Z}$;

(iii) $b_{2i} \le b_{2i-1}, b_{2i+1}$ for $i \in \mathbb{Z}$;

(iv) $b'_{2i} \le b'_{2i-1}, b'_{2i+1}$ for $i \in \mathbb{Z}$;

(v) $c_{i,0} \le c_{i,1}$, $c_{i,2j} \le c_{i,2j-1}$, $c_{i,2j+1}$ for $0 < j < 4$, and

$c_{i,8} \le c_{i,7}$ for $0 \le i < 5$;

(vi) $c_{i,0} \le b_{2i+1}$ for $0 \le i < 5$;

(vii) $c_{i,8} \le b'_{2i+1}$ for $0 \le i < 5$.

Informally, for $0 < i$, A_i and A'_i are infinite fences as are B and B'. There are five finite fences C_i for $0 \le i < 5$ each of which acts as a bridge between distinct elements of B and B' . Elements of $D \cup D' \cup E$ are incomparable with all elements of Q other than themselves. A preliminary glance at Figure 3 will reveal the comparability graph of Q .

On Q define a topology σ in the following manner. For $0 < i$, let (D_i, σ_i) and (D'_i, σ'_i) be the one-point compactifications of D_i and D'_i by d_i and d'_i , respectively, where

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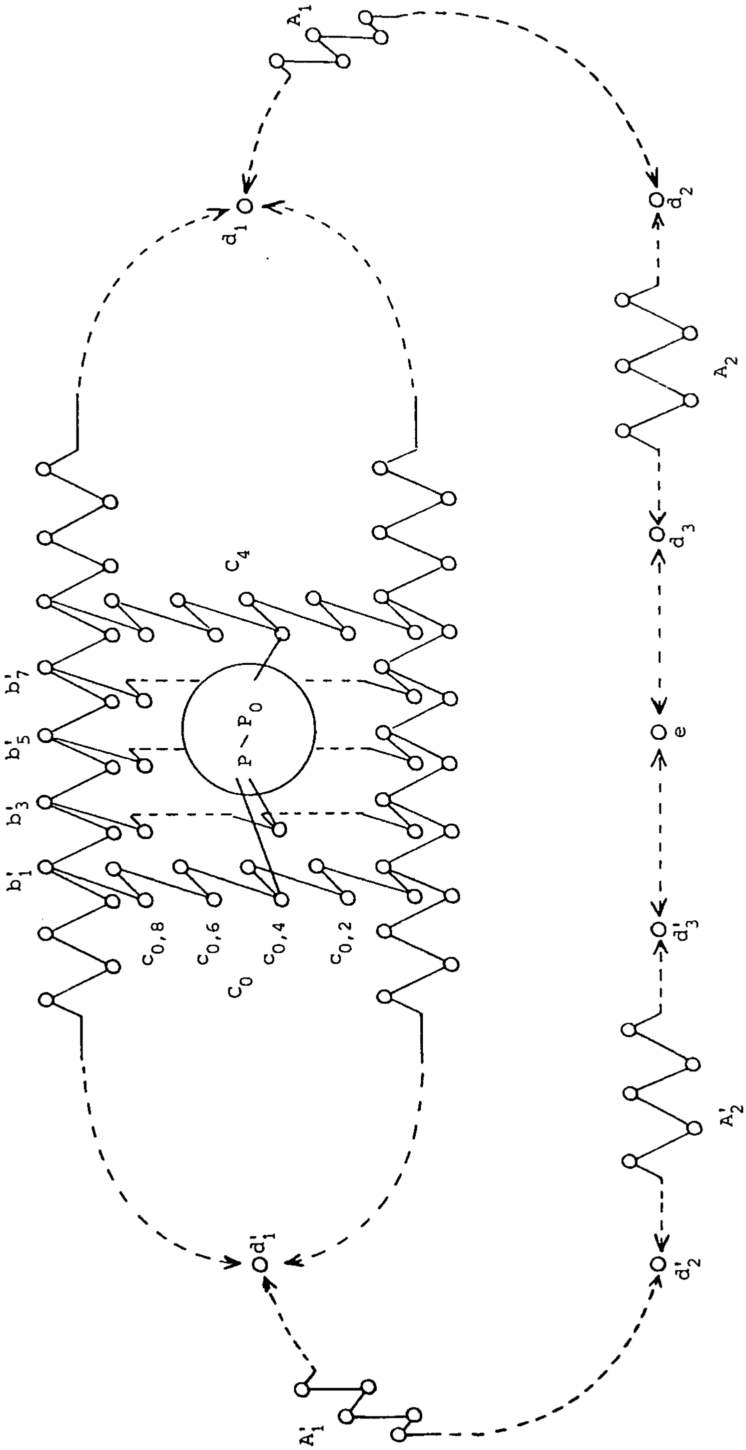

Figure 3

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$$D_1 = \{b_j : 0 < j\} \cup \{b'_j : 0 < j\} \cup \{d_1\} \cup \{a_{1,j} : j \le 0\} ,$$

$$D'_1 = \{a'_{1,j} : j \le 0\} \cup \{d'_1\} \cup \{b_j : \ j \le 0\} \cup \{b'_j : \ j \le 0\},$$

and, for $1 < i$,

$$D_i = \{a_{i-1,j} : 0 < j\} \cup \{d_i\} \cup \{a_{i,j} : j \le 0\}$$

and

$$D'_i = \{a'_{i-1,j} : 0 < j\} \cup \{d'_i\} \cup \{a'_{i,j} : j \le 0\} .$$

Then (Q,σ) is the one-point compactification of the topological sum of the spaces $(C_i : 0 \le i < 5)$, $((D_i,\sigma_i) : 0 < i)$, and $((D'_i,\sigma'_i) : 0 < i)$ by e . Since the one-point compactification of the sum of a family of compact totally disconnected spaces is a compact totally disconnected space, it is straightforward to verify that (Q,σ) is a Priestley space. As stated above, Figure 3 gives the comparability graph of (Q,σ): it has been drawn in such a way as to suggest σ .

We could at this stage define a k-space (Q,σ,Q_0,Q_1) such that the associated Kleene algebra has one non-trivial endomorphism that is also a constant. We shall not do this. However, we note that (Q,σ) will act as a cradle for each $(P,\tau,P_0) \in \underset{\sim}{T}_5$ and, as a k-subspace of $\Psi(P,\tau,P_0)$, the associated Kleene algebra will indeed have precisely one non-trivial endomorphism.

For $(P,\tau,P_0) \in \underset{\sim}{T}_5$, let $R = Q \cup (P \backslash P_0)$ where $P_0 = \{p_i : 0 \le i < 5\}$. Define a partial order on R as follows:

(i) for $x,y \in Q$, $x \le y$ in R if and only if $x \le y$ in Q ;

(ii) for $x,y \in P\backslash P_0$, $x \le y$ in R if and only if $x \le y$ in P ;

(iii) for $x \in Q$ and $y \in P\backslash P_0$, $x \le y$ in R if and only if, for some $0 \le i < 5$, $x = c_{i,4}$ and $p_i \le y$ in P .

Informally, R is the union of the posets P and Q where, for $0 \le i < 5$, $c_{i,4}$ is identified with p_i .

Since each element of $P_0 \subseteq P$ is isolated in the topology τ , R is the union of two compact totally disconnected spaces. Let ρ denote the sum of σ and the restriction of τ to the subspace $P\backslash P_0$ of P. The following is readily seen:

LEMMA 6. *For* $(P,\tau,P_0) \in \underset{\sim}{T}_5$, (R,ρ) *is a Priestley space.*

We will now define suitable R_0 and R_1 such that (R,ρ,R_0,R_1) is a k-space. We require that $R_0 \subseteq R$ and $\{\{x\}: x \in R\} \subseteq R_1 \subseteq \{\{x,y\}: x,y \in R\}$. In addition, both R_0 and R_1 are required to be closed: that is R_0 must be a closed subspace of R and $\{(x,y): \{x,y\} \in R_1\}$ must be a closed subspace of $R \times R$ with the product topology. Furthermore, it is necessary that

(i*) for $x,y,z \in R$, $\{x,y\} \in R_1$ and $y \leq z$ imply $\{x,z\} \in R_1$

and

(ii*) for $x,y \in R$, $x \in R_0$ and $\{x,y\} \in R_1$ imply $x \leq y$.

Let $R_0 = E$, so R_0 contains the single point e.

The definition of R_1 is quite complicated. We give it in its entirety and then seek to explain: it is probably advisable to read the definition and the explanation concurrently.

Let $(N_i,N_i' : 0 \leq i)$ be a partition of the positive integers into infinite sets and, for $0 \leq i$, let $\xi(i) = 4(i+1)(i+2)$ $(= \Sigma(8j: 0 \leq j \leq i+1))$.

R_1 contains the following elements:

(i) $\{u,v\}$ for any u and v with a common lower bound in R; and, for $0 < i$,

(ii) $\{a_{i,2j+1},a_{i,2k+1}\}$, $\{a'_{i,2j+1},a'_{i,2k+1}\}$, $\{b_{2j+1},b_{2k+1}\}$, and $\{b'_{2j+1},b'_{2k+1}\}$ for every $j \in \underset{\sim}{Z}$ and $k \neq j-2,j+2$;

(iii) $\{a_{i,2j+1},d_i\}$ and $\{a_{i,2j+1},d_{i+1}\}$,
$\{a'_{i,2j+1},d'_i\}$ and $\{a'_{i,2j+1},d'_{i+1}\}$,
$\{b_{2j+1},d_1\}$ and $\{b_{2j+1},d'_1\}$,

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and $\{b'_{2j+1}, d_1\}$ and $\{b'_{2j+1}, d'_1\}$ for every $i \in \mathbb{Z}$;

(iv) $\{d_1, d'_1\}$, $\{d_i, d_{i+1}\}$, and $\{d'_i, d'_{i+1}\}$;

(v) for $j \in N_i$, $\{a_{i,\xi(j)+8k}, a_{i,\xi(j)+8(k+1)}\}$ for $k = 0,\ldots,j-2$ and $\{a_{i,\xi(j)}, a_{i,\xi(j)+8(j-1)}\}$,

for $j \in N'_i$, $\{a'_{i,\xi(j)+8k}, a'_{i,\xi(j)+8(k+1)}\}$ for $k = 0,\ldots,j-2$ and $\{a'_{i,\xi(j)}, a'_{i,\xi(j)+8(j-1)}\}$,

for $j \in N_0$, $\{b_{\xi(j)+8k}, b_{\xi(j)+8(k+1)}\}$ for $k = 0,\ldots,j-2$ and $\{b_{\xi(j)}, b_{\xi(j)+8(j-1)}\}$,

and, for $j \in N'_0$, $\{b'_{\xi(j)+8k}, b'_{\xi(j)+8(k+1)}\}$ for $k = 0,\ldots,j-2$ and $\{b'_{\xi(j)}, b'_{\xi(j)+8(j-1)}\}$;

(vi) $\{a_{i,j}, a_{i,k\pm1}\}$ and $\{a_{i,j\pm1}, a_{i,k\pm1}\}$ for each $\{a_{i,j}, a_{i,k}\}$ added in (v),

$\{a'_{i,j}, a'_{i,k\pm1}\}$ and $\{a'_{i,j\pm1}, a'_{i,k\pm1}\}$ for each $\{a'_{i,j}, a'_{i,k}\}$ added in (v),

$\{b_j, b_{k\pm1}\}$ and $\{b_{j\pm1}, b_{k\pm1}\}$ for $\{b_j, b_k\}$ added in (v) ,

and $\{b'_j, b'_{k\pm1}\}$ and $\{b'_{j\pm1}, b'_{k\pm1}\}$ for $\{b'_j, b'_k\}$ added in (v).

There are two non-trivial conditions: (ii) and (v). Condition (i) ensures that R_1 contains the minimum number of elements consistent with condition (i*).

Condition (ii) places every maximal element in each of the infinite fences $(A_i\colon 0 < i)$, $(A'_i\colon 0 < i)$, B, and B', in relation with every other maximal element of its own fence except for the two points at distance four. As a direct result of (ii), conditions (iii) and (iv) are necessary additions in order that R_1 be closed. Figure 4, typical of any infinite fence, indicates all those points that are in relation to $a_{i,2j+1}$ at this stage.

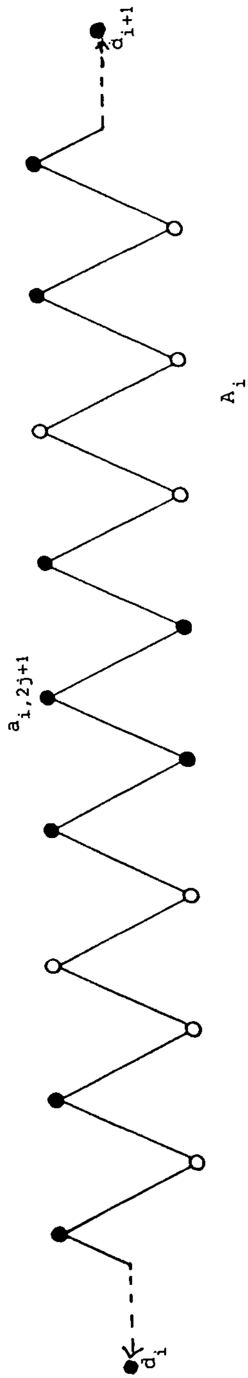

Figure 4

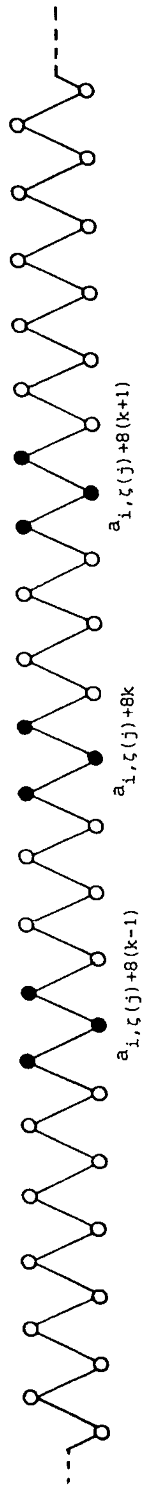

Figure 5

Condition (v) is to ensure that the minimal elements of each infinite fence behave like a graph with infinitely many isolated vertices and infinitely many disjoint cycles of different lengths. (It is a consequence of the function ξ that the cycles lie along the fences without overlapping.) To comply with (i*), it then becomes necessary to add condition (vi). The number 8 in condition (v) is chosen as the smallest number such that condition (vi) does not add any pair of maximal elements of distance four to R_1 (that is, those pairs deliberately omitted in (ii)). By way of example, Figure 5 shows all points in relation to $a_{i,\xi(j)+8k}$ in the event that $j \in N_i$.

The rôles of conditions (ii) and (v) will become apparent in due course.

LEMMA 7. *For* $(P,\tau,P_0) \in \underset{\sim}{T}_5$, (R,ρ,R_0,R_1) *satisfies (i*) and (ii*).*

Proof. We first show that condition (i*) holds. Suppose $\{x,y\} \in R_1$ and $y \le z$. Consider the various possibilities. If x and y are such that $\{x,y\}$ occurred in (i), then x and y have a common lower bound. Then so too do x and z and, hence, $\{x,z\} \in R_1$. If $\{x,y\}$ occurred in (ii), (iii), or (iv), then y is a maximal element of the partial order, in this case $\{x,z\} = \{x,y\}$. If $\{x,y\}$ occurred in (v), then $\{x,z\} \in R_1$ by (vi). The validity of (i*) is concluded by observing that $\{x,z\}$ occurs in (vi) whenever $\{x,y\}$ does.

Condition (ii*) holds vacuously since e is not present in any two-element member of R_1 . □

Obviously, $R_0 \subseteq R$ and $\{\{x\}: x \in R\} \subseteq R_1 \subseteq \{\{x,y\}: x,y \in R\}$. Thus, to show that (R,ρ,R_0,R_1) is a k-space, it remains to establish the following:

LEMMA 8. *For* $(P,\tau,P_0) \in \underset{\sim}{T}_5$, R_0 *and* R_1 *are closed.*

Proof. Clearly, as a one-element set, $R_0 = E$ is closed.

It is to be shown that $S = \{(x,y): \{x,y\} \in R_1\}$ is a closed subspace of $R \times R$ with the product topology.

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We first show that S_{CLB} is closed where S_{CLB} denotes those elements of S with a common lower bound, namely,

$$S_{CLB} = \{(x,y) \in S\colon z \le x,y \text{ for some } z \in R\} .$$

Suppose that $(x,y) \in C\ell(S_{CLB})$ where $C\ell(S_{CLB})$ denotes the closure of S_{CLB}. If $(x,y) \notin S_{CLB}$, then x and y have no common lower bound and there exist clopen decreasing sets $X,Y \subseteq R$ for which $x \in X$, $y \in Y$, and $X \cap Y = \emptyset$. Since $(x,y) \in X \times Y$ and $X \times Y$ is clopen, $S_{CLB} \cap (X \times Y) \ne \emptyset$. That is, $w \le u,v$ for some $(u,v) \in S_{CLB} \cap (X \times Y)$. Since $X \times Y$ is decreasing, $(w,w) \in X \times Y$. This is impossible as it implies that $w \in X \cap Y$. In conclusion, $(x,y) \in S_{CLB}$ and, hence, S_{CLB} is closed.

It remains to show that $C\ell(S_{NLB}) \subseteq S$ where $S_{NLB} = S \backslash S_{CLB}$ (that is, those elements of S with no lower bound).

Assume $(x,y) \in C\ell(S_{NLB})$. There are several possibilities:

If neither x nor y is in $D \cup D' \cup E$, then (x,y) is an isolated point in $R \times R$ and, hence, $(x,y) \in S_{NLB} \subseteq S$. Assume, therefore, that at least one of x or y is an element of $D \cup D' \cup E$.

Suppose $x = e$. If $y \ne e$, then choose a clopen neighbourhood Q of e such that $y \notin Q$. Observe that there is a neighbourhood $Q' \subseteq Q$ of e such that, for $u \in Q'$, $\{u,v\} \in R_1$ implies $v \in Q$. Since $Q' \times (R \backslash Q)$ is an open set containing (e,y), this gives a contradiction. It follows that $(x,y) = (e,e) \in S_{CLB} \subseteq S$.

Suppose $x = d_i$ for some $i > 1$. Then (x,y) is an element of $C\ell(S_D)$, $C\ell(S_{\text{even}})$, or $C\ell(S_{\text{odd}})$ where

$$S_D = \{(d_i,v) \in S_{NLB}\} ,$$

$$S_{\text{even}} = \{(u,v) \in S_{NLB}\colon u = a_{i-1,2j} \text{ or } a_{i,2j} \text{ for some } j \in \underset{\sim}{Z}\} , \text{ and}$$

$$S_{\text{odd}} = \{(u,v) \in S_{NLB}\colon u = a_{i-1,2j+1} \text{ or } a_{i,2j+1} \text{ for some } j \in \underset{\sim}{Z}\} .$$

In the first case, when $(x,y) \in C\ell(S_D)$, it follows that

$y \in C\ell(\{v: (d_i, v) \in S_{NLB}\})$ and this forces $y = d_{i-1}, d_i, d_{i+1}$, or, for some $j \in \mathbb{Z}$, $a_{i-1,2j+1}$ or $a_{i,2j+1}$.

Consider the second case when $(x,y) \in C\ell(S_{\text{even}})$. Then (x,y) is a limit point of the set of points $\{(a_{i-1,2j}, v) \in S_{NLB} : j \in \mathbb{Z}\}$. But if $(a_{i-1,2j}, v) \in S_{NLB}$, then $a_{i-1,2j} = a_{i-1,\xi(p)+8q}$ for some p and q such that $0 \leq q \leq p-1$. Whereupon, $v = a_{i-1,k}$ for some $k \geq \xi(p-1)$. It follows that $y = d_i$.

Finally, consider the third case when $(x,y) \in C\ell(S_{\text{odd}})$. Then the points (u,v) of S_{odd} may be partitioned into two sets according to whether $v = a_{i-1,2j+1}, a_{i,2j+1}, d_{i-1}, d_i$, or d_{i+1}, or whether $v = a_{i-1,2j}$ or $a_{i,2j}$ for some $j \in \mathbb{Z}$. It follows, since (x,y) is in the closure of one of these two sets, that $y = a_{i-1,2k+1}, a_{i,2k+1}, d_{i-1}, d_i$, or d_{i+1} for some $k \in \mathbb{Z}$ or that $y = d_i$, respectively.

In any event, $(x,y) \in S$ whenever $x = d_i$ for some $i > 1$.

The argument when $x = d_i'$ for some $i > 1$ is essentially the same, as are the arguments for $x = d_1$ or d_1'. By symmetry, the proof is complete. □

For $(P,\tau,P_0) \in \mathbb{T}_5$, for $\Psi(P,\tau,P_0) = (R,\rho,R_0,R_1)$. The above shows:

LEMMA 9. *For $(P,\tau,P_0) \in \mathbb{T}_5$, $\Psi(P,\tau,P_0) \in \mathbb{T}_K$.*

We now extend Ψ to morphisms. For $P, P' \in \mathbb{T}_5$ and a morphism $\psi: P \to P'$, let $\Psi(\psi): \Psi(P) \to \Psi(P')$ be defined as follows:

$$\Psi(\psi)(x) = \begin{cases} x & \text{for } x \in Q, \\ \psi(x) & \text{otherwise.} \end{cases}$$

Since ψ is a continuous order-preserving map that is the identity on P_0, it is routine to verify that $\Psi(\psi)$ is a continuous order-preserving map. Clearly, $\Psi(\psi)(R_0) \subseteq R_0'$. For $x, y \in P \backslash P_0$, $\{x,y\} \in R_1$ if and only if x and y have a common lower bound. Thus, since $\Psi(\psi)$

is order-preserving and is the identity on Q, $\Psi(\psi)(R_1) \subseteq R_1'$.

To summarise at this point:

LEMMA 10. $\Psi: T_5 \to \underset{\sim}{T}_K$ *is a well defined faithful functor.*

It remains to show that Ψ is almost full; that is, if $\phi:\Psi(P) \to \Psi(P')$ is a non-constant k-map, then $\phi = \Psi(\psi)$ for some morphism $\psi: P \to P'$. For the remainder of this section let ϕ denote such a map. Thus, $\phi:\Psi(P) \to \Psi(P')$ is a k-map such that $\phi(R) \not\subseteq R_0' = E$.

The main thrust is to establish that ϕ is the identity on Q.

LEMMA 11. $\phi(R \setminus E) \subseteq (R' \setminus E)$.

Proof. For any $i > 0$, A_i and E are order components of R . Thus if $\phi(A_i) \cap E \neq \emptyset$, then $\phi(A_i) = E$ and, since $d_i, d_{i+1} \in C\ell(A_i)$, $\phi(D) \cap E \neq \emptyset$. Similarly, if the image under ϕ of any one of the order components $B \cup B' \cup \bigcup(C_i : 0 \leq i < 5) \cup (P \setminus P_0)$ or A_i' for $i > 0$ has non-empty intersection with E , then $\phi(D \cup D') \cap E \neq \emptyset$. Consequently, if $\phi(R \setminus E) \not\subseteq (R' \setminus E)$, then $\phi(D \cup D') \cap E \neq \emptyset$.

Suppose, for some $i > 1$, $\phi(d_i) = e$. Then, since there exists no nontrivial path X in R such that $e \in C\ell(X)$, both $\phi(A_{i-1}) \cap E$ and $\phi(A_i) \cap E$ are non-void. It follows that $\phi(A_{i-1}) = \phi(A_i) = E$ and $\phi(d_{i-1}) = \phi(d_i) = e$. By an analogous argument, if $\phi(d_1) = e$, then $\phi(B \cup B' \cup \bigcup(C_i : 0 \leq i < 5) \cup (P \setminus P_0)) = \phi(A_1) = E$ and, consequently, $\phi(d_1')$ $= \phi(d_2) = e$. As similar statements hold whenever $\phi(D') \cap E \neq \emptyset$, $\phi(R \setminus E) \subseteq E$ whenever $\phi(D \cup D') \cap E \neq \emptyset$.

The above shows that if $\phi(R \setminus E) \not\subseteq (R' \setminus E)$, then $\phi(R \setminus E) \subseteq E$. But, by hypothesis, $\phi(R \setminus E) \not\subseteq E$. □

A two-element set was included in R_1 for every pair of maximal elements of an infinite fence in Q except for pairs of maximal elements at distance four: see condition (ii) in the definition of R_1 . The omitted pairs will guarantee that the restriction of ϕ to an infinite

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fence is either one-to-one or has a finite image: showing this is our first step in varifying that ϕ is the identity on Q. We will make these remarks precise in Lemmas 13 to 17. First we must check that the omitted pairs have not been included inadvertently.

LEMMA 12. *For* $j \in \mathbb{Z}$ *and* $0 < i$, *none of* $\{a_{i,2j+1}, a_{i,2(j\pm 2)+1}\}$, $\{a'_{i,2j+1}, a'_{i,2(j\pm 2)+1}\}$, $\{b_{2j+1}, b_{2(j\pm 2)+1}\}$, *and* $\{b'_{2j+1}, b'_{2(j\pm 2)+1}\}$ *belongs to* R_1.

Proof. Begin by noting that the distance between any pair of elements occurring in the statement of the lemma is four.

Suppose, in particular that, contrary to hypothesis, $\{a_{i,2j+1}, a_{i,2(j+2)+1}\} \in R_1$. Clearly, the pair in question does not occur in R_1 as a result of conditions (i), (ii), (iii), (iv), or, since the value of ξ is always even, (v). Only condition (vi) remains. Since both $2j+1$ and $2(j+2)+1$ are odd, the pair $\{a_{i,2j+1}, a_{i,2(j+2)+1}\}$ must be of the form $\{a_{i,r\pm 1}, a_{i,s\pm 1}\}$ for some $\{a_{i,r}, a_{i,s}\}$ in (v). That is, for some $p \in N_i$ and $q = 0, \ldots, p-2$, either

$$r = \xi(p)+8q \quad \text{and} \quad s = \xi(p)+8(q+1),$$

or

$$r = \xi(p) \quad \text{and} \quad s = \xi(p)+8(p-1).$$

Since in no instance does $|(r\pm 1) - (s\pm 1)| = 4$, the assumption is false and, consequently, $\{a_{i,2j+1}, a_{i,2(j+2)+1}\} \notin R_1$.

A similar argument in each of the remaining cases completes the proof. □

It is at this point that our attempt to clarify the definition of Q by means of a fairly liberal use of letters works to our disadvantage: the statements of succeeding lemmas are slightly cumbersome.

Lemmas 13, 14, 15, and 16 describe what happens in the event that ϕ identifies two or more elements of distance less than or equal to four in any of the fences $(A_i : 0 < i)$, $(A'_i : 0 < i)$, B, and B'. Although each lemma applies equally well to any one of these fences, we will only give complete statements for $(A_i : 0 < i)$. Each of the proofs relies on the fact that ϕ preserves both the R_1 relation and connectivity.

LEMMA 13. *If, for some* $i,p > 0$ *and* $j,q \in \mathbb{Z}$, $\phi(a_{i,2j+1}) = \phi(a_{i,2(j+1)+1}) = a_{p,2q+1}$, $a'_{p,2q+1}$, b_{2q+1} , *or* b'_{2q+1} , *then*

$$\phi(A_i) \subseteq \{a_{p,2q+1\pm r} : 0 \le r < 4\} ,$$

$$\phi(A_i) \subseteq \{a'_{p,2q+1\pm r} : 0 \le r < 4\} ,$$

$$\phi(A_i) \cap B \subseteq \{b_{2q+1\pm r} : 0 \le r < 4\} , \text{ or}$$

$$\phi(A_i) \cap B' \subseteq \{b'_{2q+1\pm r} : 0 \le r < 4\} , \text{ respectively.}$$

Analogous statements hold for A'_i, B , *and* B' .

Proof. Consider the case that $\phi(a_{i,2j+1}) = \phi(a_{i,2(j+1)+1}) = a_{p,2q+1}$. If $a_{p,2(q+2)+1} \in \phi(A_i)$, then $\phi(a_{i,2k+1}) = a_{p,2(q+2)+1}$ for some $k \in \mathbb{Z}$. However, for every $k \in \mathbb{Z}$, either $\{a_{i,2j+1}, a_{i,2k+1}\}$ or $\{a_{i,2(j+1)+1}, a_{i,2k+1}\}$ is in R_1 . This is impossible since ϕ preserves R_1 and

$$\phi(\{a_{i,2j+1}, a_{i,2k+1}\}) = \phi(\{a_{i,2(j+1)+1}, a_{i,2k+1}\})$$
$$= \{a_{p,2q+1}, a_{p,2(q+2)+1}\}$$

which, as shown in Lemma 12, is not in R'_1 . A similar argument shows that $\phi(a_{p,2(q-2)+1}) \notin \phi(A_i)$. Thus, by connectivity, $\phi(A_i) \subseteq \{a_{p,2q+1\pm r} : 0 \le r < 4\}$.

Clearly, $\phi(a_{i,2j+1}) = \phi(a_{i,2(j+1)+1}) = a'_{p,2q+1}$ is dispensed in like fashion, as are the remaining possibilities once it is noted that, for $0 \le k < 5$, neither $\{b_{2k+1}, c_{k,3}\}$ nor $\{b'_{2k+1}, c_{k,5}\}$ is in R'_1. □

Similar arguments prove the next two Lemmas.

LEMMA 14. *If, for some* $i,p > 0$ *and* $j,q \in \mathbb{Z}$, $\phi(a_{i,2j+1}) = \phi(a_{i,2(j+2)+1}) = a_{p,2q+1}$, $a'_{p,2q+1}$, b_{2q+1}, *or* b'_{2q+1}, *then*

$$\phi(A_i) \subseteq \{a_{p,2q+1\pm r} : 0 \le r < 4\} ,$$

$$\phi(A_i) \subseteq \{a'_{p,2q+1\pm r} : 0 \le r < 4\} ,$$

$$\phi(A_i) \cap B \subseteq \{b_{2q+1\pm r} : 0 \le r < 4\} , \text{ or}$$

$$\phi(A_i) \cap B' \subseteq \{b'_{2q+1\pm r} : 0 \le r < 4\} , \text{ respectively.}$$

Analogous statements hold for A'_i, B, *and* B'.

LEMMA 15. *If, for some* $i,p > 0$ *and* $j,q \in \mathbb{Z}$, $\phi(a_{i,2j+1}) = \phi(a_{i,2(j+1)+1}) = a_{p,2q}$, $a'_{p,2q}$, b_{2q}, *or* b'_{2q}, *then*

$$\phi(A_i) \subseteq \{a_{p,2q\pm r} : 0 \le r < 3\} ,$$

$$\phi(A_i) \subseteq \{a'_{p,2q\pm r} : 0 \le r < 3\} ,$$

$$\phi(A_i) \cap B \subseteq \{b_{2q\pm r} : 0 \le r < 3\} , \text{ or}$$

$$\phi(A_i) \cap B' \subseteq \{b'_{2q\pm r} : 0 \le r < 3\} , \text{ respectively.}$$

Analogous statements hold for A'_i, B, *and* B'.

Finally,

LEMMA 16. *If, for some* $i,p > 0$ *and* $j,q \in \mathbb{Z}$, $\phi(a_{i,2j+1}) = a_{p,2q}$, $a'_{p,2q}$, b_{2q}, *or* b'_{2q}, *then*

$$\phi(A_i) \subseteq \{a_{p,2q\pm r} : 0 \le r < 5\} ,$$

$$\phi(A_i) \subseteq \{a'_{p,2q\pm r} : 0 \le r < 5\} ,$$

$$\phi(A_i) \cap B \subseteq \{b_{2q\pm r} : 0 \le r < 5\} , \text{ or}$$

$$\phi(A_i) \cap B' \subseteq \{b'_{2q\pm r} : 0 \le r < 5\} , \text{ respectively.}$$

Analogous statements hold for A'_i, B, *and* B'.

Proof. Suppose $\phi(a_{i,2j+1}) = a_{p,2q}$. Clearly,

$$\phi(\{a_{i,2(j-1)+1}, a_{i,2(j+1)+1}\}) \subseteq \{a_{p,2q-1}, a_{p,2q}, a_{p,2q+1}\} .$$

In the event that either $\phi(a_{i,2(j-1)+1})$ or $\phi(a_{i,2(j+1)+1}) = a_{p,2q}$, Lemma 15 yields the required conclusion.

If $\phi(a_{i,2(j-1)+1}) = \phi(a_{i,2(j+1)+1}) \neq a_{p,2q}$, then Lemma 14 may be applied.

The only case that remains to be considered is when $\phi(\{a_{i,2(j-1)+1}, a_{i,2j+1}, a_{i,2(j+1)+1}\}) = \{a_{p,2q-1}, a_{p,2q}, a_{p,2q+1}\}$. Observe that, for any $k \in \underset{\sim}{Z}$, $a_{i,2k+1}$ is an element of two distinct members of R_1 that have non-empty intersection with

$$\{a_{i,2(j-1)+1}, a_{i,2j+1}, a_{i,2(j+1)+1}\} .$$

Since $a_{p,2(q-1)-1}$, $a_{p,2(q-1)}$, $a_{p,2(q+1)}$, and $a_{p,2(q+1)+1}$ are each elements of only one member of R_1' that has non-empty intersection with

$$\{a_{p,2q-1}, a_{p,2q}, a_{p,2q+1}\} ,$$

it follows in the remaining case that, for every $k \in \underset{\sim}{Z}$

$$\phi(a_{i,2k+1}) \in \{a_{p,2q-1}, a_{p,2q}, a_{p,2q+1}\} :$$

by connectivity, $\phi(A_i) \subseteq \{a_{p,2q\pm r} : 0 \leq r < 3\}$.

The other possibilities are treated analogously. □

Essentially, the above shows that, for an infinite fence in Q , either ϕ is one-to-one or it has a finite image. In the event that ϕ is one-to-one,the minimal points of the fence are sent by ϕ to the minimal points of some other fence. It is here that condition (v) enters. The minimal elements of any infinite fence in Q together with all the two-element subsets of it that are in R_1 form a graph. Thus, if the image is also in Q , the restriction of ϕ is then a compatible mapping from one graph to another. Condition (v) implies that each such graph has infinitely many isolated vertices and infinitely many disjoint cycles of distinct length. Consequently, ϕ is a graph isomorphism and, by necessity, the identity. The following lemma states this in a precise

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manner.

LEMMA 17. *If, for some* $i, p > 0$, $\phi(A_i) \cap A_p$, $\phi(A_i) \cap A'_p$, $\phi(A_i) \cap B$ *or* $\phi(A_i) \cap B'$ *is non-empty, then, for some* $q \in \mathbb{Z}$.

$$\phi(A_i) \subseteq \{a_{p,q\pm r}: 0 \leq r < 5\} ,$$

$$\phi(A_i) \subseteq \{a'_{p,q\pm r}: 0 \leq r < 5\} ,$$

$$\phi(A_i) \cap B \subseteq \{b_{q\pm r}: 0 \leq r < 5\} ,$$

$$\phi(A_i) \cap B' \subseteq \{b'_{q\pm r}: 0 \leq r < 5\} ,$$

respectively, or $\phi \restriction A_i$ *is the identity.*

Analogous statements hold for A'_i, B, *and* B'.

Proof. Suppose $\phi(A_i) \cap A_p \neq \emptyset$ and $\phi(A_i) \not\subseteq \{a_{p,q\pm r}: 0 \leq r < 5\}$ for any $q \in \mathbb{Z}$. By Lemma 16, $\phi(a_{i,2j+1}) \neq a_{p,2q}$ for any $j,q \in \mathbb{Z}$. Thus, $\phi(\{a_{i,2j+1}: j \in \mathbb{Z}\}) \subseteq \{a_{p,2q+1}: q \in \mathbb{Z}\}$ and, by Lemmas 13 and 14, ϕ is one-to-one on $\{a_{i,2j+1}: j \in \mathbb{Z}\}$. Thus, ϕ is one-to-one on A_i.

Since ϕ is one-to-one on $\{a_{i,2j}: j \in \mathbb{Z}\}$ and $\phi(\{a_{i,2j}: j \in \mathbb{Z}\}) \subseteq \{a_{p,2q}: q \in \mathbb{Z}\}$, it follows from condition (v) of the definition of R_1 that $p = i$ and ϕ is the identity.

The remaining possibilities are treated similarly. □

LEMMA 18. *The restriction of* ϕ *to each of* $(A_i: 0 < i)$, $(A'_i: 0 < i)$, B, B', D, D', *and* E *is the identity.*

Proof. Since $R_0 = R'_0 = E$, ϕ is the identity on E.

We claim that $\phi(D \cup D') \subseteq D \cup D'$. If, for some $i,p > 0$, $\phi(d_i) \in A_p$, then $\phi(A_i) \cap A_p \neq \emptyset$ and $\phi \restriction A_i$ is not one-to-one. By Lemma 17, $\phi(A_i)$ is a finite subset of A_p and, hence, $\phi(d_{i+1}) \in A_p$. By induction and continuity we deduce that $\phi(e) \in C\ell(A_p)$. This is impossible since $\phi(e) = e$. Thus, $\phi(D) \cap (\bigcup(A_i: 0 < i)) = \emptyset$. Similar arguments show that $\phi(d_i) \notin U(A_i^!: 0 < i)$, B, B', $U(C_i: 0 \le i < 5)$, or $P'\backslash P_0'$, that is that $\phi(D) \subseteq D \cup D'$. Clearly, as required, $\phi(D') \subseteq D \cup D'$ too.

It follows from the preceding claim that ϕ is the identity on each of $(A_i: 0 < i)$, $(A_i^!: 0 < i)$, B, and B' (otherwise, by Lemma 17, $\phi(D \cup D) \not\subseteq D \cup D'$). This being the case, ϕ is also the identity on $D \cup D'$. □

The following lemma completes the proof of Theorem 1.

LEMMA 19. *$\Psi: \underset{\sim}{T}_5 \to \underset{\sim}{T}_K$ is an almost full functor.*

Proof. By Lemma 18, ϕ is the identity on $(A_i: 0 < i)$, $(A_i^!: 0 < i)$, B, B', D, D', and E. In particular, for $0 \le i < 5$, $\phi(b_{2i+1}) = b_{2i+1}$ and $\phi(b_{2i+1}') = b_{2i+1}'$. Since C_i is the shortest path connecting b_{2i+1} and b_{2i+1}', ϕ is the identity on C_i. In particular, $\phi(c_{i,4}) = c_{i,4}$ for every $0 \le i < 5$. By hypothesis, for $(P,\tau,P_0) \in \underset{\sim}{T}_5$, if $x \in \mathrm{Max}(P\backslash P_0)$, then there are distinct i,j such that $x \ge c_{i,4}, c_{j,4}$. Thus,

$$\phi((P\backslash P_0) \cup \{c_{i,4}: 0 \le i < 5\}) \subseteq ((P\backslash P_0) \cup \{c_{i,4}: 0 \le i < 5\}).$$

If ψ denotes the restriction of ϕ to $(P\backslash P_0) \cup \{c_{i,4}: 0 \le i < 5\}$, then ψ is a morphism of $\underset{\sim}{T}_5$ such that $\phi = \Psi(\psi)$. □

4. MS-algebras.

The theorems of this paper and [3] give information about endomorphism monoids of de Morgan algebras and Kleene algebras. In [1], endomorphism monoids of Stone algebras, another well known variety generalising Boolean algebras, were discussed. These results enable us, without further effort, to describe the behaviour of endomorphism monoids of all the subvarieties of the variety $\underset{\sim}{MS}$ of *MS*-algebras. The variety $\underset{\sim}{MS}$ was introduced by Blyth and Varlet [7] and has been extensively studied since. An *MS*-<u>algebra</u> $(L;\vee,\wedge,\sim,0,1)$ is an algebra of type $(2,2,1,0,0)$ such that $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and $\sim$ is a dual $(0,1)$-lattice endomorphism for which $a = a\wedge\sim\sim a$, $\sim(a\wedge b) = \sim a\vee\sim b$, and

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$\sim 0 = 1$. MS-algebras generalise both de Morgan and Stone algebras. The complete lattice of subvarieties of $\underset{\sim}{MS}$ is described in Blyth and Varlet [8] (see Beazer [5]). This lattice is depicted in Figure 6; here $\underset{\sim}{S}$ denotes the variety of Stone algebras, determined by the equation $a \wedge \sim a = 0$

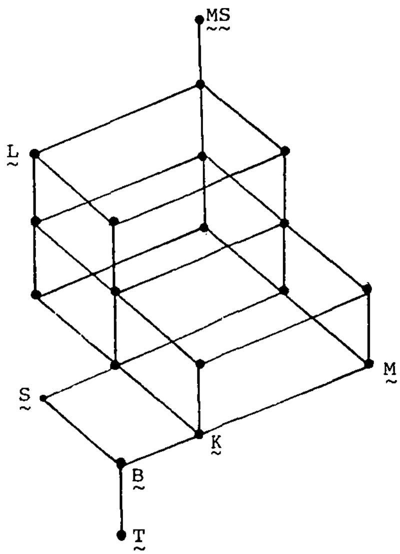

Figure 6

(see, for example, Balbes and Dwinger [4]), and $\underset{\sim}{L}$ denotes the subvariety determined by $a \wedge \sim a = (a \wedge \sim a) \wedge (\sim b \vee \sim\sim b)$.

In [1], it was shown that, for $L, L' \in \underset{\sim}{S}$, if $\mathrm{End}(L) \cong \mathrm{End}(L')$, then $L \cong L'$. (Recall that a similar statement holds for Boolean algebras, of which Stone algebras are a generalization.)

Now observe that, by definition, any variety that contains a universal (almost universal) variety is itself universal (almost universal). Further, observe that, as noted in [7], an MS-algebra is a de Morgan algebra if and only if it satisfies $\sim\sim x = x$. In particular, if L is an MS-algebra that is not a de Morgan algebra, then, for some $x \in L$, $\sim\sim x \neq x$. Since $\sim$ is a dual $(0,1)$-lattice endomorphism, $\sim\sim$ is a $(0,1)$-lattice

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endomorphism that preserves $\sim$. Hence, any *MS*-algebra that is not a de Morgan algebra has a non-trivial endomorphism. Thus, since every non-trivial Kleene algebra has a non-trivial endomorphism, the variety $\underset{\sim}{L}$ has no non-trivial rigid algebras. Consequently, $\underset{\sim}{L}$ is not universal.

By [1] (see also Magill [24], Maxson [25], and Schein [30]), Theorem 1 of [3], and Theorem 1 the following holds:

THEOREM 20. *Let* $\underset{\sim}{V}$ *be a variety of MS-algebras. One of the following holds:*

(i) if $\underset{\sim}{T} \subseteq \underset{\sim}{V} \subseteq \underset{\sim}{S}$, *then, for* $L, L' \in \underset{\sim}{V}$, $End(L) \cong End(L')$ *implies* $L \cong L'$;

(ii) if $\underset{\sim}{K} \subseteq \underset{\sim}{V} \subseteq \underset{\sim}{L}$, *then* $\underset{\sim}{V}$ *is almost universal and not universal;*

(iii) if $\underset{\sim}{M} \subseteq \underset{\sim}{V} \subseteq \underset{\sim}{MS}$, *then* $\underset{\sim}{V}$ *is universal.*

In conclusion, we remark that all the varieties we have considered are subvarieties of the variety $\underset{\sim}{O}$ of Ockham algebras (see Urquhart [31] and Goldberg [16], [17]). A more systematic study of universality and recoverability in varieties of Ockham algebras will be presented in a subsequent paper.

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