Note on Relativistic Mechanics

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§ 1. *Introduction.*

The object of this note is to derive a form of Poisson's equation from general relativistic mechanics, without assuming the field to be either static or "weak." The problem is essentially a "local" problem, all observations being made by one observer; this observer determines the apparent gravitational field in his vicinity by observing the motions of free (isolated) particles. Defining gravitational mass by means of Poisson's equation, we find the relation between the densities of gravitational and inertial mass relative to any observer. We also find what may be called the non-rotating frame of reference belonging to any observer.

§ 2. *The frame of reference.*

Consider an observer A whose world-line in space-time V_4 is the curve C. An event E in A's history corresponds to a point P of C, and A's instantaneous space at this event is the region of any 3-space, orthogonal to C at P, which is in the neighbourhood of P, distances from P being such that their squares may be neglected. The observer A can refer events occurring in his vicinity to a cartesian frame of reference with himself at the origin. The axes are determined by a triad of unit vectors h^i_σ, $\sigma = 1, 2, 3$, at each point of C, these vectors being mutually orthogonal and orthogonal to C. If now P' is a point near C, and if P is that point of C such that PP' is orthogonal to C at P, the vector PP' can be written $z^\sigma h^i_\sigma$, where h^i_σ are the vectors of the triad at P. It follows from the general theory[1] that z^1, z^2, z^3 so defined are the cartesian coordinates assigned to the event $E'(P')$ by A at the event $E(P)$. Further, if s is the arcual distance of P measured along C from some fixed point P_0, and if A's proper time t

[1] The line-element of V_4 has the significance that the element of distance in an instantaneous space is the element of length measured by a rigid scale, and the element of distance along a world-line is the element of proper-time multiplied by the velocity of light.

is measured from the event $E_0(P_0)$, the time of occurrence of the event $E'(P')$ according to A is s/c, where c is the velocity of light.

A curve C' near C can now be specified by the z's as functions of s, the z's for a particular s being the coordinates of the point P' of C' corresponding to the point $P(s)$ of C. Hence, if C' is the world-line of a particle, the motion of this particle as observed by A is given by the equations

$$z^\sigma = z^\sigma(s), \qquad s = ct. \tag{1}$$

For convenience, we shall consider the cartesian axes defined by Fermi transport along C, this being the natural generalisation to any curve of Levi-Civita parallel transport along a geodesic. Fermi transport conserves the angle between any two vectors orthogonal to C, and a vector initially orthogonal to C retains this property when transported along the curve. It is in fact the most simple transport possessing these properties[1], and it is therefore not surprising that, as we shall see later, the frame of reference so defined has particular significance. The vectors h^i_σ are now solutions of the equations

$$\frac{d\lambda^i}{ds} + (\Gamma^i_{jk}\,h^k + g_{jk}\,\eta^k\,h^i)\,\lambda^j = 0 \tag{2}$$

where $h^i(s)$ is the unit vector tangent to C at $P(s)$ and η^i is the curvature vector at P, *i.e.*

$$\eta^i = \frac{dh^i}{ds} + \Gamma^i_{jk}\,h^j\,h^k, \tag{3}$$

the components of the fundamental tensor g_{jk} and the Christoffel symbols Γ^i_{jk} being evaluated at P. The vectors h^i_σ are completely determined at points of C by these transport equations and by an arbitrary triad $(h^i_\sigma)_0$ at some point P_0 of C; it can easily be verified that if the initial vectors satisfy the required conditions of orthogonality, the vectors at each other point of C also satisfy these conditions. Any other frame of reference could now be obtained by rotating the axes defined above.

§ 3. *The gravitational field.*

The observer A can explore the gravitational field in his vicinity at the event $E(P(s))$ by measuring the accelerations of free (isolated) particles at time $t = s/c$. It has been deduced[2] from the equations of

[1] See A. G. Walker, "Relative Coordinates," *Proc. Roy. Soc. Edinburgh*, 52 (1932), 346.

[2] Eddington, *The Mathematical Theory of Relativity* (1930), § 56.

conservation $T^{ij}{}_{,j} = 0$ satisfied by the energy tensor that the world-line of an isolated particle is a geodesic. We shall therefore consider the geodesics, issuing from the point (z) relative to $P\,(s)$, which are such that dz/ds is small of the order z, these being the world-lines of particles, near A at the event $E\,(P)$, whose velocities relative to A are small compared with the velocity of light.

It has been shown[1] that in the system of coordinates defined above, the equations of geodesics can be written

$$\frac{d^2 z^\sigma}{ds^2} = -\,\Gamma_{\sigma\nu}\, z^\nu + g_\sigma, \qquad (\sigma = 1,\, 2,\, 3) \tag{4}$$

where

$$\Gamma_{\sigma\nu} = \Gamma_{\nu\sigma} = R_{kijl}\, h^i\, h^j\, h^k_\sigma\, h^l_\nu, \tag{5}$$

$$g_\sigma = g_{ij}\, \eta^i\, h^j_\sigma. \tag{6}$$

The expressions on the right in (5) and (6) are evaluated at $P\,(s)$, so that $\Gamma_{\sigma\nu}$, g_σ are defined at points of C. In deriving equations (4), it is assumed that z and dz/ds are small. It can be shown however that the components g_σ and hence $d^2 z^\sigma/ds^2$ are not necessarily small for equations (4) to be valid at the point of the geodesic under consideration, *i.e.* it is not necessary to assume that the geodesic *remains* near C. Since $ds = cdt$, the components of acceleration derived by A are $c^2 d^2 z^\sigma/ds^2$. Hence at the event $E\,(P)$, the components of the apparent field of force are

$$F^\sigma = -\,c^2\, \Gamma_{\sigma\nu}\, z^\nu + c^2 g_\sigma. \tag{7}$$

We observe that the expressions for the F's do not involve the velocities dz/dt, whence, *according to any observer, the axes defined by Fermi transport along his world-line do not rotate relative to the matter in his neighbourhood.* This property justifies the use of the particular frame of reference defined above.

The components of force at the origin are $c^2 g_\sigma$, found by putting $z = 0$ in (7). The space-time vector giving the space-vector $c^2 g_\sigma$ is $c^2 \sum\limits_{\sigma=1}^{3} g_\sigma h^i_\sigma$, and from (6) since $g_{ij}\, h^i\, \eta^j = 0$ and $e\,(h_\sigma) = -\,1$, this vector is $-\,c^2 \eta^i$. Hence, *the apparent force at an event $E\,(P)$ in the history of an observer is given by the vector $-\,c^2 \eta^i$, where η^i is the curvature vector of the observer's world-line at P.* This result appears to be well known.[2]

[1] Walker, *loc. cit.* p. 351. The right-hand side of (4.3) should read $-v_r$ instead of v_r; we are now writing g_σ for v_σ. It must be remembered that in the V_4 we are considering, the indicators are $e\,(h) = g_{ij}\, h^i\, h^j = +\,1$, $e\,(h_\sigma) = g_{ij}\, h_\sigma{}^i\, h_\sigma{}^j = -\,1$.

[2] *Cf.* E. T. Whittaker, *Proc. Roy. Soc.*, **149** *A* (1935), 385.

§ 4. *Poisson's equation and gravitational mass.*

Since $\Gamma_{\sigma\nu} = \Gamma_{\nu\sigma}$, we see at once from (7) that $\mathbf{F} = -\operatorname{grad} V$, where

$$V = \tfrac{1}{2}c^2\, \Gamma_{\sigma\nu}\, z^\sigma z^\nu - c^2 g_\sigma\, z^\sigma. \tag{8}$$

Hence the apparent field of force in the neighbourhood of the observer at the event $E(P)$ can be derived from the potential V given by (8), the quantities $\Gamma_{\sigma\nu}$, g_σ being evaluated at P.

From (8) we find

$$\nabla^2 V = \sum_{\sigma=1}^{3} \frac{\partial^2 V}{(\partial z^\sigma)^2} = c^2 \sum_{\sigma=1}^{3} \Gamma_{\sigma\sigma}. \tag{9}$$

Now the vectors h^i, h^i_σ form an orthogonal quadruple, whence

$$\sum_{\sigma=1}^{3} h^k_\sigma h^l_\sigma = h^k h^l - g^{kl}.$$

Substituting in (5) and (9), we find

$$\nabla^2 V = -\, c^2 R_{ij}\, h^i h^j \tag{10}$$

where R_{ij} is the Ricci tensor $g^{kl} R_{kijl}$. Neglecting the cosmical constant, the field-equations expressing R_{ij} in terms of the energy tensor T_{ij} can be written

$$R_{ij} = -\, \kappa\,(T_{ij} - \tfrac{1}{2}T g_{ij}), \qquad T = g^{ij}\, T_{ij}, \tag{11}$$

where $\kappa = 8\pi\gamma/c^2$. Hence (10) becomes

$$\nabla^2 V = 4\pi\gamma\,(2T_{ij}\, h^i h^j - T). \tag{12}$$

It is of interest to note that a form of this expression arose in recent work on Gauss' Theorem.[1]

The energy tensor can be defined[2] as $T^{ij} = \Sigma\, \rho'_0 \lambda'^i \lambda'^j$, where ρ'_0 is the proper density of all particles whose world-lines are in the direction λ'^i at the point under consideration, and Σ denotes summation over all time-like directions λ'^i. It follows at once that

$$T = \rho_0, \quad T_{ij}\, h^i h^j = \rho, \tag{13}$$

where ρ_0 is the proper density at the event $E(P)$ and ρ is the relative density measured by the observer at this event. Thus finally we have

$$\nabla^2 V = 4\pi\gamma\,(2\rho - \rho_0). \tag{14}$$

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[1] E. T. Whittaker, *loc. cit.*; and H. S. Ruse, page 151 (5.7) of the present volume of *Proc. Edin. Math. Soc.*

[2] Eddington, *op. cit.*, § 53.

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Comparing (14) with Poisson's equation, we see that *the relative density of gravitational mass defined by Poisson's equation is*

$$\rho_g = 2\rho - \rho_0, \tag{15}$$

where ρ_0 *and* ρ *are the proper and relative densities of inertial mass.* It must be remembered that equation (14) applies only to events at the observer and not to all events in his instantaneous space. It is assumed that the observer measures distances directly and consequently the field of observation is so limited that $\nabla^2 V$ can be evaluated only at the observer himself. Thus ρ_g is the density of gravitational mass at the observer.

The relation (15) can be written otherwise, using the known relations between proper and relative inertial density and pressure. If v is the mean velocity of matter relative to the observer A at the event $E(P)$,

$$\rho_0 = \rho\left(1 - \frac{v^2}{c^2}\right) - \frac{3}{c^2}\, p_m,$$

where p_m is the mean relative pressure, *i.e.* $p_m = \frac{1}{3}(p_{xx} + p_{yy} + p_{zz})$. Hence

$$\rho_g = \rho\left(1 + \frac{v^2}{c^2}\right) + \frac{3}{c^2}\, p_m. \tag{16}$$

Also, if the system is isotropic, the proper hydrostatic pressure being p, we have

$$\rho = \rho_0 \frac{1}{1 - v^2/c^2} + \frac{3p}{c^2}\, \frac{1 + v^2/3c^2}{1 - v^2/c^2}.$$

Hence,

$$\rho_g = \rho_0 \frac{1 + v^2/c^2}{1 - v^2/c^2} + \frac{3p}{c^2}\, \frac{1 + v^2/3c^2}{1 - v^2/c^2}. \tag{17}$$

The relation (17) shows that the density of gravitational mass varies for different observers at the same event. It is generally assumed that the pressure can be neglected in comparison with the density. In this case, we see that *the gravitational density is equal to the inertial density when and only when the observer is at rest relative to the matter in his neighbourhood.*