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N-colored generalized Frobenius partitions: generalized Kolitsch identities

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Abstract. Let $N \geq 1$ be squarefree with $(N,6) = 1$. Let $c\phi_N(n)$ denote the number of N-colored generalized Frobenius partitions of n introduced by Andrews in 1984, and $P(n)$ denote the number of partitions of n. We prove

$$c\phi_N(n) = \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) + b(n),$$

where $C(z) := (q;q)_\infty^N \sum_{n=1}^\infty b(n)q^n$ is a cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$. This extends and strengthens earlier results of Kolitsch and Chan–Wang–Yan treating the case when N is a prime. As an immediate application, we obtain an asymptotic formula for $c\phi_N(n)$ in terms of the classical partition function $P(n)$.

1 Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, and $\mathbb{H}$ denote the sets of positive integers, non-negative integers, integers, rational numbers, complex numbers, and upper half plane of complex numbers, respectively. Throughout the paper, we denote $q = e^{2\pi iz}$, where $z \in \mathbb{H}$.

In 1984, Andrews [1] introduced the function $c\phi_N(n)$ counting the number of N-colored generalized Frobenius partitions of n with $N \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The generating function of $c\phi_N(n)$ is denoted by

$$C\Phi_N(q) := \sum_{n=0}^\infty c\phi_N(n)q^n.$$

Andrews [1] determined $C\Phi_N(q)$ in terms of a theta function divided by an infinite product, as follows. Let

$$\theta_N(x) := \sum_{i=1}^N x_i^2 + \sum_{1\leq i<j\leq N} x_i x_j.$$

be a quadratic form in N variables, and

$$f_{\theta_N}(z) := \sum_{x\in\mathbb{Z}^N} q^{\theta_N(x)},$$

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be the associated theta function. Then, by [1, Theorem 5.2], we have

$$C\Phi_N(z) = \frac{f_{\theta_{N-1}}(z)}{(q;q)_\infty^N},$$

where

$$(q;q)_\infty = \prod_{n\geq 1}(1-q^n).$$

There has been a plethora of research concerning the congruence properties of $c\phi_N(n)$; we leave the discussion of this topic and related results to [3] and its references. In this paper, we investigate relations between $c\phi_N(n)$ and $P(n)$, where $P(n)$ denotes the number of partitions of n. We define $P(0)=1$ and $P(a)=0$ when $a\notin\mathbb{N}_0$. From the description of $c\phi_N(n)$ (see [1, 3]), or from the formula for $C\Phi_N(z)$ and the product formula for the partition generating function, we clearly have

$$c\phi_1(n) = P(n).$$

In [7, 8], Kolitsch has shown rather surprising relationships between these two types of partitions which are stated below.

Theorem 1.1 (Kolitsch [8]) *For all $n\in\mathbb{N}_0$, we have*

$$c\phi_5(n) = 5P(5n-1) + P(n/5), \tag{1.1}$$

$$c\phi_7(n) = 7P(7n-2) + P(n/7), \tag{1.2}$$

and

$$c\phi_{11}(n) = 11P(11n-5) + P(n/11). \tag{1.3}$$

The proof of these beautiful identities relies on q-series identities from [5, equations (2.2) and (3.1)] that relate the generating function of t-cores to theta series.

Very recently in [3], Chan et al. have discovered the following more general relationships between $c\phi_p(n)$ and $P(n)$. Below, noting that the Dedekind eta function is defined by $\eta(z) = q^{1/24}(q;q)_\infty$, we restate the main aspects of their Theorem 4.1.

Theorem 1.2 (Chan–Wang–Yan [3]) *For all $n\in\mathbb{N}_0$, we have*

$$c\phi_{13}(n) = 13P(13n-7) + P(n/13) + a(n), \tag{1.4}$$

where $q\dfrac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2} = \sum_{n=1}^{\infty} a(n)q^n$. *When $p\geq 17$ is a prime, then we have*

$$\sum_{n=0}^{\infty}\left(c\phi_p(n) - p\cdot P\left(pn - \frac{p^2-1}{24}\right) - P(n/p)\right)q^n = \frac{h_p(z) + 2p^{(p-11)/2}(\eta(pz)/\eta(z))^{p-11}}{(q^p;q^p)_\infty},$$

where $h_p(z)$ is a modular function on $\Gamma_0(p)$ with a zero at ∞ and a pole of order $(p+1)(p-13)/24$ at 0. Additionally, the function $h_p(z)(\eta(z)\eta(pz))^{p-13}$ is a holomorphic modular form of weight $p-13$ with a zero of order $(p-1)(p-11)/24$ at ∞ and $h_p(z)$ is congruent to p times a cusp form on $\Gamma_0(1)$ of weight $p-1$ modulo p^2.

These results rely on some delicate residue calculations and properties of modular functions. *The goal of this paper* is to extend the above results of Kolitsch and Chan–Wang–Yan to give relations between $c\phi_N(n)$ and $P(n)$, where N is a squarefree integer that is coprime to 6. The method we use is quite different than that of [3] or [8]. We describe our method after stating our main theorem.

We fix $\chi_a(b)$ to be the Kronecker symbol $\left(\frac{(-1)^{(a-1)/2}a}{b}\right)_K$. Whenever a is a squarefree odd integer, $\chi_a(b)$ is a primitive Dirichlet character modulo a. The space of modular forms of weight k for the modular subgroup $\Gamma_0(N)$ with multiplier system χ_N is denoted by $M_k(\Gamma_0(N),\chi_N)$, and its subspace of cusp forms is denoted by $S_k(\Gamma_0(N),\chi_N)$.

Theorem 1.3 (Main Theorem) *Let N be a squarefree positive integer with* $\gcd(N,6)=1$.

(i) *Then for all $n\in\mathbb{N}_0$, we have*

$$c\phi_N(n)=\sum_{d|N}N/d\cdot P\left(\frac{N}{d^2}n-\frac{N^2-d^2}{24d^2}\right)+b(n), \tag{1.5}$$

where

$$C(z):=(q;q)_\infty^N\sum_{n=1}^{\infty}b(n)q^n$$

is a cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$.

(ii) *We have $C(z)=0$ if and only if $N=5,7,$ or 11.*

(iii) *If $N\neq 5,7,$ or 11, then there is no $M\geq 0$ such that $b(n)=0$ for all $n>M$.*

Theorem 1.3 is the result of a chain of modular identities. We first discover an identity that relates the theta function $f_{\theta_{N-1}}(z)$ to Eisenstein series. Then we find another identity that relates these Eisenstein series to the partition function $P(n)$ using an intimate relationship between eta quotients and Eisenstein series. This relationship between eta quotients and Eisenstein series is not valid unless $\frac{N^2-d^2}{24d}\in\mathbb{N}_0$ for $d\mid N$, see Theorem 5.1. Therefore, we put the restriction $\gcd(N,6)=1$. These modular identities are determined using [2, Theorem 1.1]. Finally, we combine these identities to obtain Theorem 1.3.

In contrast with [3, Theorem 4.1 (c)], when $N=p$ a prime greater than 13, our theorem gives slightly more information about $h_p(z)$. As a result of our Theorem 1.3 we obtain that

$$h_p(z)+2p^{(p-11)/2}(\eta(pz)/\eta(z))^{p-11}$$

is simply a cusp form in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$. Therefore, it is evident that $h_p(z)$ is congruent to a cusp form modulo p^2.

On the other hand, when $N=p$ a prime greater than 3, our Theorem 1.3 leads to the equation

$$c\phi_p(n)=p\cdot P\left(pn-\frac{p^2-1}{24}\right)+P(n/p)+b(n),$$

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where

$$C(z) := (q;q)_\infty^p \sum_{n=1}^{\infty} b(n)q^n$$

is a cusp form in $S_{(p-1)/2}(\Gamma_0(p), \chi_p)$. Therefore, (1.1)–(1.4) can easily be deduced from our Theorem 1.3. Using Sturm's Theorem, one observes that in the cases $N = 5, 7$, and 11, we have $C(z) = 0$, which leads to (1.1)–(1.3).

As an application of Theorem 1.3, we establish the following asymptotic formula for $c\phi_N(n)$ in terms of linear combinations of partition functions.

Theorem 1.4 *Let N be a squarefree positive integer with $(N, 6) = 1$. We have*

$$c\phi_N(n) \sim \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2 - d^2}{24d^2}\right)$$

as $n \to \infty$.

The organization of the paper is as follows. In Section 2, we introduce further notation and prove an important theorem concerning the modular forms in $M_k(\Gamma_0(N), \chi_N)$, see Theorem 2.1. In Section 3, we compute the constant terms of $f_{\theta_{N-1}}(z)$ at the cusps $1/c$ where $c \mid N$. This requires computing some Gauss sums related to the quadratic form θ_{N-1}. These Gauss sum computations could be of independent interest to an audience with particular interest in the subject. In Section 4, we compute the constant terms of the eta quotient $\frac{\eta^N((N/d)z)}{\eta(dz)}$ at the cusps $1/c$ where $c \mid N$. In Section 5, we use Theorem 2.1 and the calculations of Sections 3 and 4 to give $f_{\theta_{N-1}}(z)$ and $\frac{\eta^N((N/d)z)}{\eta(dz)}$ in terms of Eisenstein series. We then use the relationship between $\frac{\eta^N((N/d)z)}{\eta(dz)}$ and the partition function to prove an identity relating Eisenstein series and the partition function. Then we combine these identities to prove Theorem 1.3. In Section 6, we show that the error term $b(n)$ is much smaller than

$$\sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2 - d^2}{24d^2}\right)$$

by combining estimates involving coefficients of various q-series and this proves Theorem 1.4.

2 Notation and preliminaries

In this section, we introduce further notation and prove a theorem on a certain space of modular forms, see Theorem 2.1. This theorem is the backbone of the paper. We start with some notation.

Recall that $\chi_a(b)$ denotes the Kronecker symbol $\left(\frac{(-1)^{(a-1)/2}a}{b}\right)_K$. Let $k \in \mathbb{N}$. The generalized sum of divisors function associated with χ_d and $\chi_{N/d}$ is defined by

$$\sigma_{k-1}(\chi_{N/d}, \chi_d; n) := \sum_{1 \le t|n} \chi_{N/d}(n/t)\chi_d(t)t^{k-1}.$$

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Let B_{k,χ_N} denote the kth generalized Bernoulli number associated with χ_N defined by the series

$$\sum_{k=0}^{\infty} \frac{B_{k,\chi_N}}{k!} t^k = \sum_{a=1}^{N} \frac{\chi_N(a) t e^{at}}{e^{Nt} - 1}.$$

Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}_0$ be coprime. For an $f(z) \in M_k(\Gamma_0(N), \chi)$ we denote the constant term of $f(z)$ in the Fourier expansion of $f(z)$ at the cusp a/c by

$$[f]_{a/c} = \lim_{z \to i\infty} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right),$$

where $b, d \in \mathbb{Z}$ are such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. The value of $[f]_{a/c}$ does not depend on the choice of b, d. Throughout the paper we denote

$$\varepsilon_c = \begin{cases} 1 & \text{if } c \equiv 1 (\text{mod } 4), \\ i & \text{if } c \equiv 3 (\text{mod } 4). \end{cases}$$

We are now ready to state and prove the following statement.

Theorem 2.1 *Let N be a squarefree integer with* $\gcd(N, 6) = 1$. *Let* $f(z) \in M_{(N-1)/2}(\Gamma_0(N), \chi_N)$. *Then we have*

$$f(z) = [f]_{1/N} + \sum_{d|N} \frac{[f]_{1/d}}{A(d, N)} \cdot \frac{(1 - N)(N/d)^{(N-2)/2}}{B_{(N-1)/2,\chi_N}} \sum_{n \geq 1} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n) q^n$$
$$+ C(z),$$

where $C(z)$ is some cusp form in $S_{(N-1)/2}(\Gamma_0(N), \chi_N)$ and

$$A(d, N) = (-1)^{\frac{(d+1)(N/d-1)}{4}} \varepsilon_{N/d} = \begin{cases} 1 & \text{if } d \equiv 1 (\text{mod } 4) \text{ and } N \equiv 1 (\text{mod } 4), \\ i & \text{if } d \equiv 3 (\text{mod } 4) \text{ and } N \equiv 1 (\text{mod } 4), \\ -i & \text{if } d \equiv 1 (\text{mod } 4) \text{ and } N \equiv 3 (\text{mod } 4), \\ 1 & \text{if } d \equiv 3 (\text{mod } 4) \text{ and } N \equiv 3 (\text{mod } 4). \end{cases}$$

Proof This theorem is a direct application of [2, Theorem 1.1]. The specialized version of the set of tuples of characters defined in [2] and given below simplifies to

$$\begin{aligned} \mathcal{E}((N-1)/2, N, \chi_N) :=& \{(\varepsilon, \psi) \in D(L, \mathbb{C}) \times D(M, \mathbb{C}) : \varepsilon, \psi \text{ primitive}, \\ & \varepsilon(-1)\psi(-1) = (-1)^{(N-1)/2},\ \varepsilon\psi = \chi_N \text{ and } LM \mid N\} \\ =& \{(\chi_{N/d}, \chi_d) : d \mid N\}. \end{aligned}$$

Therefore, using [2, Theorem 1.1], we obtain

$$\begin{aligned} f(z) =& \sum_{d|N} \chi_{N/d}(-1)[f]_{1/d} \\ & \times \left(\chi_{N/d}(0) + \frac{W(\chi_d)}{W(\chi_N)} \frac{(1-N)(N/d)^{(N-1)/2}}{B_{(N-1)/2,\chi_N}} \sum_{n \geq 1} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n) q^n\right) \\ & + C(z), \end{aligned} \tag{2.1}$$

for some $C(z)$ in $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$, where the Gauss sum $W(\chi_d)$ is defined by

$$W(\chi_d) := \sum_{a=1}^{d} \chi_d(a) e^{2\pi i a/d}.$$

On the other hand, since N, d are squarefree and odd we have

$$\chi_N = \prod_{p|N} \chi_p, \text{ and } \chi_d = \prod_{p|d} \chi_p.$$

Additionally, we have

$$W(\chi_p) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ i\sqrt{p} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

By the multiplicative properties of the Gauss sums $W(\chi_N)$ for p an odd prime divisor of N we have

$$W(\chi_N) = (-1)^{(p-1)(N/p-1)/4} W(\chi_p) W(\chi_{N/p}),$$

see [9, Lemma 3.1.2]. Using this iteratively, we deduce that

$$W(\chi_N) = \varepsilon_N \sqrt{N} = \begin{cases} \sqrt{N} & \text{if } N \equiv 1 \pmod 4, \\ i\sqrt{N} & \text{if } N \equiv 3 \pmod 4. \end{cases}$$

Putting this in (2.1), we obtain the desired result. ■

In order to get necessary modular identities from Theorem 2.1, we need to compute $[f_{\theta_{N-1}}]_{1/d}$ and $\left[\frac{\eta^N((N/d)z)}{\eta(dz)}\right]_{1/d}$ for each $d \mid N$. Computation of $\left[\frac{\eta^N((N/d)z)}{\eta(dz)}\right]_{1/d}$ can be done using [6, Proposition 2.1]. This is carried out in Section 4. By [10, (10.2)] (see [2, (1.9)] for a refined version), we have

$$[f_{\theta_{N-1}}]_{1/d} = \left(\frac{-i}{d}\right)^{(N-1)/2} \frac{G_{N-1}(1,d)}{\sqrt{N}}, \tag{2.2}$$

where the quadratic Gauss sum $G_N(a,c)$ for $N, a, c \in \mathbb{N}$ is defined by

$$G_N(a,c) := \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^N} e^{2\pi i a \theta_N(x)/c}.$$

Therefore, to calculate $[f_{\theta_{N-1}}]_{1/d}$, we need to calculate $G_{N-1}(1,d)$, which is carried out in the next section.

3 Gauss sums and constant terms of $f_{\theta_{N-1}}(z)$

Let N be an odd squarefree positive integer. In this section, we compute $G_{N-1}(a,d)$ for all $d \mid N$ and $a \in \mathbb{N}$ with $\gcd(a,d) = 1$. Then when $\gcd(N,6) = 1$, we use our computations together with [10, (10.2)] to obtain the constant term $[f_{\theta_{N-1}}(z)]_{1/d}$ of $f_{\theta_{N-1}}(z)$ in its Fourier series expansion at $1/d$, see Theorem 3.7. In this section, for a set A and an N-tuple $x \in A^N$, we use the notation $x = (x_1, \dots, x_N)$, i.e., x_i denotes the ith coordinate of the tuple x. We first prove a multiplicativity result concerning $G_N(a,c)$.

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Lemma 3.1 *Let $N \in \mathbb{N}$. Let $\alpha, \beta, \gamma \in \mathbb{N}$ be mutually coprime. Then we have*

$$G_N(\gamma, \alpha\beta) = G_N(\beta\gamma, \alpha)G_N(\alpha\gamma, \beta).$$

Proof The map $\mathbb{Z}/\alpha\mathbb{Z} \times \mathbb{Z}/\beta\mathbb{Z} \to \mathbb{Z}/\alpha\beta\mathbb{Z}$ given by $(x, y) \mapsto z = \beta x + \alpha y$ is bijective. Therefore, each $z \in (\mathbb{Z}/\alpha\beta\mathbb{Z})^N$ can be expressed as $z = \beta x + \alpha y$ for a unique $x \in (\mathbb{Z}/\alpha\mathbb{Z})^N$, $y \in (\mathbb{Z}/\beta\mathbb{Z})^N$. From

$$z_i = \beta \cdot x_i + \alpha \cdot y_i,$$

we have

$$z_i^2 \equiv (\alpha + \beta)(\beta \cdot x_i^2 + \alpha \cdot y_i^2) (\text{mod } a\beta), \tag{3.1}$$

$$z_i \cdot z_j \equiv (a + \beta)(\beta \cdot x_i x_j + \alpha \cdot y_i y_j) (\text{mod } \alpha\beta). \tag{3.2}$$

Using (3.1) and (3.2), we have

$$\begin{aligned} \sum_{i=1}^{N} z_i^2 + \sum_{\substack{i,j=1,\\ i<j}}^{N} z_i z_j &\equiv (\alpha + \beta)\left(\sum_{i=1}^{N}(\beta \cdot x_i^2 + \alpha \cdot y_i^2) + \sum_{\substack{i,j=1,\\ i<j}}^{N} (\beta \cdot x_i x_j + \alpha \cdot y_i y_j)\right) \\ &\equiv (\alpha + \beta)\left(\beta \sum_{i=1}^{N} x_i^2 + \beta \sum_{\substack{i,j=1,\\ i<j}}^{N} x_i x_j + \alpha \sum_{i=1}^{N} y_i^2 + \alpha \sum_{\substack{i,j=1,\\ i<j}}^{N} y_i y_j\right). \end{aligned}$$

Therefore, using the notation $e(x) := e^{2\pi i x}$, we have

$$\begin{aligned} G_N(\gamma, \alpha\beta) &= \sum_{z \in (\mathbb{Z}/\alpha\beta\mathbb{Z})^N} e\left(\gamma \frac{\theta_N(z)}{\alpha\beta}\right) = \sum_{z \in (\mathbb{Z}/\alpha\beta\mathbb{Z})^N} e\left(\gamma \frac{\sum z_i^2 + \sum z_i z_j}{\alpha\beta}\right) \\ &= \sum_{x \in (\mathbb{Z}/\alpha\mathbb{Z})^N} e\left(\beta\gamma \frac{\sum x_i^2 + \sum x_i x_j}{\alpha}\right) \sum_{y \in (\mathbb{Z}/\beta\mathbb{Z})^N} e\left(\alpha\gamma \frac{\sum y_i^2 + \sum y_i y_j}{\beta}\right) \\ &= \sum_{x \in (\mathbb{Z}/\alpha\mathbb{Z})^N} e\left(\beta\gamma \frac{\theta_N(x)}{\alpha}\right) \sum_{y \in (\mathbb{Z}/\beta\mathbb{Z})^N} e\left(\alpha\gamma \frac{\theta_N(y)}{\beta}\right) \\ &= G_N(\beta\gamma, \alpha)G_N(\alpha\gamma, \beta). \end{aligned}$$ ∎

For an odd prime p, in order to relate the relevant quadratic Gauss sums (over $\mathbb{Z}/p\mathbb{Z}$) in N variables to quadratic Gauss sums in $N-1$ or $N-2$ variables, we need the function $\mathcal{C}_p : (\mathbb{Z}/p\mathbb{Z})\backslash\{1 \text{ mod } p\} \to \mathbb{Z}/p\mathbb{Z}$ given by $\mathcal{C}_p(R) := \frac{1}{4(1-R)}$. When p is clear from the context, we can omit it from the subscript and use $\mathcal{C}(R)$ instead.

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Lemma 3.2 *Let p be an odd prime. Let N, R and a be positive integers such that* $\gcd(a,p)=1$. *We have*

$$\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^N} e\left(a\frac{\theta_N(x)}{p} - a\frac{Rx_N^2}{p}\right)$$
$$= \begin{cases} p & \text{if } R\equiv 1 (\text{mod } p), \text{ and } N\le 2,\\ p\displaystyle\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-2}} e\left(a\frac{\theta_{N-2}(x)}{p}\right) = p\cdot G_{N-2}(a,p) & \text{if } R\equiv 1(\text{mod } p), \text{ and } N>2,\\ \varepsilon_p\sqrt{p}\left(\dfrac{a(1-R)}{p}\right)_K \displaystyle\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}} e\left(a\frac{\theta_{N-1}(x)}{p} - a\frac{\mathcal{C}(R)x_{N-1}^2}{p}\right) & \text{if } R\not\equiv 1(\text{mod } p). \end{cases}$$

Proof The following easily proved identities are used throughout the proof:

$$\theta_N(x_1,\dots,x_N) = \theta_{N-1}(x_1,\dots,x_{N-1}) + x_N\sum_{j=1}^N x_j,$$

$$\sum_{x\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{Ax^2+Bx+C}{p}\right) = \sum_{y\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{Ay^2}{p}\right)e\left(\frac{C-(4A)^{-1}B^2}{p}\right)$$
$$= e\left(\frac{C-(4A)^{-1}B^2}{p}\right)\left(\frac{A}{p}\right)_K\varepsilon_p\sqrt{p}$$

for $A,B,C\in\mathbb{Z}/p\mathbb{Z}$ with $A\neq 0$ by the change of variables $y = x+(2A)^{-1}B$.
The case $R\equiv 1(\text{mod } p)$ and $N=1$ is obvious:

$$\sum_{x_1=0}^{p-1} e\left(a\frac{\theta_1(x_1)-x_1^2}{p}\right) = \sum_{x_1=0}^{p-1} 1 = p.$$

When $R\equiv 1(\text{mod } p)$ and $N=2$, we have

$$\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^2} e\left(a\frac{\theta_2(x_1,x_2)-x_1^2}{p}\right) = \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^2} e\left(\frac{ax_2^2+ax_1x_2}{p}\right)$$
$$= \sum_{y_1,y_2\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{y_1y_2}{p}\right)$$
$$= p,$$

where in the last second line, we make the change of variables $y_1 = ax_2$ and $y_2 = x_2 + x_1$ and the last line follows from orthogonality.

Next, we prove the case $R\equiv 1(\text{mod } p)$ and $N>2$. We have

$$\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^N} e\left(a\frac{\theta_N(x)}{p} - a\frac{Rx_N^2}{p}\right)$$
$$= \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^N} e\left(a\frac{\theta_N(x)}{p} - a\frac{x_N^2}{p}\right)$$

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$$= \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^N} e\left(a\frac{\theta_{N-1}(x_1,\ldots,x_{N-1}) + x_N\sum_{j=1}^{N-1}x_j}{p}\right)$$

$$= \sum_{A=0}^{p-1} \sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}\\ \sum x_i\equiv A \bmod p}} e\left(a\frac{\theta_{N-1}(x)}{p}\right) \sum_{x_N=0}^{p-1} e\left(\frac{(aA)x_N}{p}\right). \tag{3.3}$$

Now, we observe that if $A\not\equiv 0 \bmod p$ then $\sum_{x_N\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{(aA)x_N}{p}\right) = 0$. Therefore, the right hand side (RHS) of (3.3) is

$$p\sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}\\ \sum x_i=0}} e\left(a\frac{\theta_{N-1}(x)}{p}\right). \tag{3.4}$$

We use $-\sum_{j=1}^{N-2}x_j = x_{N-1}$ to eliminate x_{N-1} so that the above expression is

$$p\sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}\\ \sum x_i\equiv 0}} e\left(a\frac{\theta_{N-2}(x_1,\ldots,x_{N-2})}{p}\right) = p\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-2}} e\left(a\frac{\theta_{N-2}(x)}{p}\right).$$

Finally, we prove the case $R\not\equiv 1 (\mathrm{mod}\ c)$. We have

$$\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^N} e\left(a\frac{\theta_N(x)}{p} - a\frac{Rx_N^2}{p}\right)$$

$$= \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^N} e\left(a\frac{\theta_{N-1}(x_1,\ldots,x_{N-1}) + (1-R)x_N^2 + x_N\sum_{j=1}^{N-1}x_j}{p}\right)$$

$$= \sum_{A=0}^{p-1} \sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}\\ \sum x_i\equiv A}} e\left(a\frac{\theta_{N-1}(x)}{p}\right) \sum_{x_N\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{a(1-R)x_N^2 + (aA)x_N}{p}\right). \tag{3.5}$$

Now in (3.5), we use

$$\sum_{x_N\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{a(1-R)x_N^2 + (aA)x_N}{p}\right) = \varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K e\left(-\frac{\mathcal{C}(R)aA^2}{p}\right)$$

so that the RHS of (3.5) becomes

$$\varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K \sum_{A=0}^{p-1} \sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}\\ \sum x_i\equiv A}} e\left(a\frac{\theta_{N-1}(x) - \mathcal{C}(R)A^2}{p}\right)$$

$$= \varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K \sum_{A=0}^{p-1} \sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}\\ \sum x_i\equiv A}} e\left(a\frac{\theta_{N-2}(x_1,\ldots,x_{N-2}) + x_{N-1}\sum_{j=1}^{N-1}x_j - \mathcal{C}(R)A^2}{p}\right).$$

(3.6)

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We employ $x_{N-1} = A - \sum_{j=1}^{N-2} x_j$ in (3.6) so that its RHS is

$$\varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K \sum_{A=0}^{p-1}\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-2}} e\left(a\frac{\theta_{N-2}(x) + A(A-\sum_{j=1}^{N-2}x_j) - \mathcal{C}(R)A^2}{p}\right)$$

$$= \varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K \sum_{A=0}^{p-1}\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-2}} e\left(a\frac{\theta_{N-2}(x) + A^2 - A\sum_{j=1}^{N-2}x_j - \mathcal{C}(R)A^2}{p}\right). \tag{3.7}$$

Then we replace A by $-A$ in (3.7) to obtain

$$\varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K \sum_{A=0}^{p-1}\sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-2}} e\left(a\frac{\theta_{N-2}(x) + A^2 + A\sum_{j=1}^{N-2}x_j - \mathcal{C}(R)A^2}{p}\right)$$

$$= \varepsilon_p\sqrt{p}\left(\frac{a(1-R)}{p}\right)_K \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-1}} e\left(a\frac{\theta_{N-1}(x) - \mathcal{C}(R)x_{N-1}^2}{p}\right), \tag{3.8}$$

where $x \in (\mathbb{Z}/p\mathbb{Z})^{N-1}$ in the last sum has the form $x = (x_1, \ldots, x_{N-1}, A)$ for an arbitrary $(x_1, \ldots, x_{N-1}) \in (\mathbb{Z}/p\mathbb{Z})^{N-1}$ and $A \in \mathbb{Z}/p\mathbb{Z}$. ∎

We want to show that sufficiently many iterations of Lemma 3.2 will relate $G_{N_1}(a,c)$ to $G_{N_2}(a,c)$ where $N_1 > N_2$. For any positive integer t, let $\mathcal{C}^t$ denote the tth fold iterate of $\mathcal{C}$. The value of $\mathcal{C}^t(R)$ is well defined when none of the $R, \mathcal{C}(R), \ldots, \mathcal{C}^{t-1}(R)$ is $1 \bmod p$. When $t = 0$, we let $\mathcal{C}^t$ be the identity function on $\mathbb{Z}/p\mathbb{Z}\backslash\{1 \bmod p\}$. The next lemma describes the orbit of $0 \bmod p$ under $\mathcal{C}$.

Lemma 3.3 *Let p be an odd prime. We have the following:*

(i) $\mathcal{C}^t((p+1)/2) = (p+1)/2$ *for every* $t \in \mathbb{N}$.
(ii) $\{\mathcal{C}^t(0) : \quad t = 0, 1\ldots, p-2\} = \{0, 1\ldots, (p-1)/2, (p+3)/2, \ldots, p-1\} \bmod p$ *with* $\mathcal{C}^{p-2}(0) = 1 \bmod p$.

Proof Part (a) follows from the fact that $\mathcal{C}((p+1)/2) = (p+1)/2$. For part (b), one can prove by induction on t the formula:

$$\mathcal{C}^t(0) = \frac{t}{2t+2} \bmod p \text{ for } 0 \le t \le p-2.$$ ∎

Proposition 3.4 *Let p be an odd prime, $N \in \mathbb{N}$ be such that $N \ge p-1$ and $a \in \mathbb{N}$ are coprime to p. Then we have*

$$G_N(a,p) = \begin{cases} i^{(p-p^2)/2}\cdot\left(\frac{a}{p}\right)_K p^{p/2} & \text{if } N = p-1, \text{ or } p, \\ i^{(p-p^2)/2}\cdot\left(\frac{a}{p}\right)_K p^{p/2} G_{N-p}(a,p) & \text{if } N > p. \end{cases}$$

Proof By Lemma 3.3, we have $\mathcal{C}^t(0) \not\equiv 1 (\bmod\ p)$ for $0 \le t \le p-3$. Therefore, we apply Lemma 3.2 repeatedly for $p-2$ many times and obtain

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$$G_N(a,p) = \left(\varepsilon_p\sqrt{p}\left(\frac{a}{p}\right)_K\right)^{p-2}\prod_{t=1}^{p-2}\left(\frac{1-\mathcal{C}^{t-1}(0)}{p}\right)_K \times \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-(p-2)}} e\left(a\frac{\theta_{N-(p-2)}(x)}{p} - a\frac{\mathcal{C}^{p-2}(0)x^2_{N-(p-2)}}{p}\right)$$

$$= \left(\varepsilon_p\sqrt{p}\left(\frac{a}{p}\right)_K\right)^{p-2}\prod_{t=1}^{p-2}\left(\frac{1-\mathcal{C}^{t-1}(0)}{p}\right)_K \times \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-(p-2)}} e\left(a\frac{\theta_{N-(p-2)}(x)}{p} - a\frac{x^2_{N-(p-2)}}{p}\right), \tag{3.9}$$

where in the second step, we use $\mathcal{C}^{p-2}(0) \equiv 1 (\bmod\ p)$ that comes from Lemma 3.3. When $N > p$, we apply Lemma 3.2 to (3.9) to obtain

$$G_N(a,p) = \left(\varepsilon_p\sqrt{p}\left(\frac{a}{p}\right)_K\right)^{p-2}\prod_{t=1}^{p-2}\left(\frac{1-\mathcal{C}^{t-1}(0)}{p}\right)_K \cdot p \cdot \sum_{x\in(\mathbb{Z}/p\mathbb{Z})^{N-p}} e\left(a\frac{\theta_{N-p}(x)}{p}\right)$$
$$= (\varepsilon_p)^{p-2}\left(\frac{a}{p}\right)_K\prod_{t=0}^{p-2}\left(\frac{1-\mathcal{C}^{t-1}(0)}{p}\right)_K \cdot p^{p/2}\cdot G_{N-p}(a,p).$$

Finally, the desired result follows by employing the elementary identities

$$\varepsilon_p = i^{(1-p)/2}\left(\frac{(p+1)/2}{p}\right)_K \tag{3.10}$$

and

$$\prod_{t=1}^{p-2}\left(\frac{1-\mathcal{C}^{t-1}(0)}{p}\right)_K = (-1)^{(p-1)/2}\left(\frac{(p+1)/2}{p}\right)_K. \tag{3.11}$$

When $N = p-1$, or p, by similar arguments, we obtain

$$G_N(a,p) = \left(\varepsilon_p\sqrt{p}\left(\frac{a}{p}\right)_K\right)^{p-2}\prod_{t=1}^{p-2}\left(\frac{1-\mathcal{C}^{t-1}(0)}{p}\right)_K \cdot p.$$

The desired result in this case follows similarly by employing (3.10) and (3.11). ∎

Proposition 3.5 *Let $N > 1$ be an odd positive squarefree integer and let p be a prime divisor of N. If $\gcd(a,p) = 1$, then we have*

$$G_{N-1}(a,p) = i^{(N-Np)/2}\cdot\left(\frac{a}{p}\right)_K p^{N/2}.$$

Proof We apply Proposition 3.4 to $G_{N-1}(a,p)$ for $N/p - 1$ many times and obtain

$$G_{N-1}(a,p) = \left(i^{(p-p^2)/2}\cdot\left(\frac{a}{p}\right)_K p^{p/2}\right)^{N/p} = i^{(N-Np)/2}\cdot\left(\frac{a}{p}\right)_K p^{N/2}.$$ ∎

Theorem 3.6 *Let N be an odd positive squarefree integer, let d be a divisor of N, and let $a \in \mathbb{Z}$ with* $\gcd(a,d) = 1$. *Then we have*

$$G_{N-1}(a,d) = \left(\frac{a}{d}\right)_K \cdot i^{(N-Nd)/2} \cdot d^{N/2}.$$

Proof First, we compute $G_{N-1}(1,d)$. By Lemma 3.1 and Proposition 3.5, we have

$$\begin{aligned} G_{N-1}(1,d) &= \prod_{p|d} G_{N-1}(d/p,p) = \prod_{p|d} i^{(N-Np)/2} \cdot \left(\frac{d/p}{p}\right)_K p^{N/2} \\ &= d^{N/2} \prod_{p|d} i^{(N-Np)/2} \cdot \left(\frac{d/p}{p}\right)_K. \end{aligned}$$

We let

$$B(d,N) := \frac{\prod_{p|d} i^{(N-Np)/2} \cdot \left(\frac{d/p}{p}\right)_K}{i^{(N-Nd)/2}}.$$

Now, let p_1 be an odd prime such that $p_1 \nmid N$. Then for all $d \mid N$, we have

$$B(d,Np_1) = \frac{\prod_{p|d} i^{(Np_1-Np_1p)/2} \cdot \left(\frac{d/p}{p}\right)_K}{i^{(Np_1-Np_1d)/2}} = (B(d,N))^{p_1}, \tag{3.12}$$

and

$$\begin{aligned} B(dp_1,Np_1) &= \frac{\prod_{p|dp_1} i^{(Np_1-Np_1p)/2} \cdot \left(\frac{dp_2/p}{p}\right)_K}{i^{(Np_1-Np_1d)/2}} \\ &= \frac{i^{(Np_1-Np_1^2)/2} \cdot \left(\frac{d}{p_1}\right)_K \prod_{p|d}\left(\frac{p_1}{p}\right)_K \prod_{p|d} i^{(Np_1-Np_1p)/2} \cdot \left(\frac{d/p}{p}\right)_K}{i^{(Np_1-Ndp_1^2)/2}} \\ &= \frac{(-1)^{(p_1-1)(d-1)/4} \prod_{p|d} i^{(Np_1-Np_1p)/2} \cdot \left(\frac{d/p}{p}\right)_K}{i^{(Np_1^2-Ndp_1^2)/2}} \\ &= \frac{(-1)^{(p_1-1)(d-1)/4} (B(d,N))^{p_1}}{i^{(Np_1^2-Ndp_1^2-Np_1+Ndp_1)/2}} \\ &= \frac{(-1)^{(p_1-1)(d-1)/4} (B(d,N))^{p_1}}{(i^{(p_1-1)(1-d)/2})^{Np_1}} \\ &= (B(d,N))^{p_1}. \end{aligned} \tag{3.13}$$

Clearly $B(1,1) = 1$. Therefore, by (3.12) and (3.13), we have $B(d,N) = 1$ and this proves

$$G_{N-1}(1,d) = i^{(N-Nd)/2} \cdot d^{N/2}.$$

We now compute $G_{N-1}(a,d)$ when $\gcd(a,d)=1$. For $d\in\mathbb{N}$, let $\zeta_d := \exp(2\pi i/d)$. Let σ be the automorphism of $\mathbb{Q}(\zeta_d)$ such that $\sigma(\zeta_d)=\zeta_d^a$. This yields

$$G_{N-1}(a,d)=\sigma(G_{N-1}(1,d)). \tag{3.14}$$

Let k be the number of prime divisors of d that are congruent to 1 mod 4. From

$$\prod_{p|d}(\varepsilon_p\sqrt{p})^N = i^{Nk}d^{N/2}$$

and the fact that k and $(1-d)/2$ have the same parity, we have

$$G_{N-1}(1,d)=\pm\prod_{p|d}(\varepsilon_p\sqrt{p})^N. \tag{3.15}$$

For each prime $p\mid d$, the field $\mathbb{Q}(\varepsilon_p\sqrt{p})$ is the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$ which is also the fixed field of the quadratic residues in $(\mathbb{Z}/p\mathbb{Z})^*\cong \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Therefore the restriction of σ on $\mathbb{Q}(\varepsilon_p\sqrt{p})$ maps

$$\varepsilon_p\sqrt{p}\mapsto\left(\frac{a}{p}\right)_K\varepsilon_p\sqrt{p}.$$

Together with (3.14) and (3.15), we have

$$G_{N-1}(a,d)=\prod_{p|d}\left(\frac{a}{p}\right)_K G_{N-1}(1,d)=\left(\frac{a}{d}\right)_K G_{N-1}(1,d)$$

and this finishes the proof. ∎

Theorem 3.7 *Let N be a positive squarefree integer such that* $\gcd(N,6)=1$ *and d be a divisor of N. Then we have*

$$[f_{\theta_{N-1}}(z)]_{1/d} = i^{(1-Nd)/2}\cdot\sqrt{d/N}.$$

Proof We put the result of Theorem 3.6 in (2.2) to obtain the desired result. ∎

4 Constant terms of $\frac{\eta^N((N/d)z)}{\eta(dz)}$

Throughout this section, we let N be a positive squarefree integer such that $\gcd(N,6)=1$. We denote by $V_{1/c}\left(\frac{\eta^N((N/d)z)}{\eta(dz)}\right)$ the order of vanishing of the eta quotient $\frac{\eta^N((N/d)z)}{\eta(dz)}$ at the cusp $1/c$. We first show that $\frac{\eta^N((N/d)z)}{\eta(dz)}$ vanishes at all $1/c$ except when $c=d$.

Lemma 4.1 *We have* $V_{1/c}\left(\frac{\eta^N((N/d)z)}{\eta(dz)}\right)=0$ *if* $c=d$ *and* $V_{1/c}\left(\frac{\eta^N((N/d)z)}{\eta(dz)}\right)>0$ *otherwise.*

Proof By [4, Proposition 5.9.3] (with cusp width N/c), we have

$$V_{1/c}\left(\frac{\eta^N((N/d)z)}{\eta(dz)}\right)=\frac{N}{24c}\left(\frac{d^2\gcd(N/d,c)^2-\gcd(d,c)^2}{d}\right).$$

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Since N is squarefree $\gcd(N/d, d) = 1$, we have

$$V_{1/d}\left(\frac{\eta^N((N/d)z)}{\eta(dz)}\right) = \frac{N}{24d}\left(\frac{d^2\gcd(N/d,d)^2 - \gcd(d,d)^2}{d}\right) = 0.$$

If $c \neq d$, then we have $d > \gcd(d,c)$ and clearly $\gcd(N/d,c) \geq 1$, therefore, $d^2\gcd(N/d,c)^2 > \gcd(d,c)^2$. Hence, we have

$$V_{1/c}\left(\frac{\eta^N((N/d)z)}{\eta(dz)}\right) = \frac{N}{24c}\left(\frac{d^2\gcd(N/d,c)^2 - \gcd(d,c)^2}{d}\right) > 0.$$ ■

Now, we compute $\left[\frac{\eta^N((N/d)z)}{\eta(dz)}\right]_{1/c}$ for all $c \mid N$. Recall that we use the notation $e(x) = e^{2\pi i x}$.

Lemma 4.2 *Let $c \mid N$. Then we have*

$$\left[\frac{\eta^N((N/d)z)}{\eta(dz)}\right]_{1/c} = \begin{cases} \left(\frac{N/d}{d}\right)_K \cdot \left(\frac{d}{N}\right)^{N/2} \cdot i^{\frac{1-Nd}{2}} & \text{if } c = d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof The case where $c \neq d$ is a direct result of Lemma 4.1. Now, we prove the case when $c = d$. Let $L_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $L_2 = \begin{bmatrix} N/d & v \\ d & w \end{bmatrix} \in SL_2(\mathbb{Z})$. Then by [6, Proposition 2.1] we have

$$\left[\frac{\eta^N((N/d)z)}{\eta(dz)}\right]_{1/d} = \frac{v^N(L_2)e(-dv/24)}{v(L_1)}\left(\frac{d}{N}\right)^{N/2},$$

where

$$v(L_1) = e\left(\frac{-1}{24}\right),$$
$$v(L_2) = \left(\frac{w}{d}\right)_K e\left(\frac{1}{24}((N/d + w)d - vw(d^2 - 1) - 3d)\right).$$

Then we have

$$\begin{aligned} \frac{v^N(L_2)e(-dv/24)}{v(L_1)} &= \left(\frac{w}{d}\right)_K^N e\left(\frac{1}{24}(N(N/d + w)d - Nvw(d^2 - 1) - 3Nd + 1 - dv)\right) \\ &= \left(\frac{w}{d}\right)_K^N e\left(\frac{1}{24}(vd - 3Nd + 3 - dv)\right) \\ &= \left(\frac{N/d}{d}\right)_K e\left(\frac{1}{8}(1 - Nd)\right), \end{aligned}$$

where in the first step, we use $d^2 - 1 \equiv 0 \pmod{24}$, $N^2 \equiv 1 \pmod{24}$ and $Ndw \equiv 1 + dv \pmod{24}$, in the last step we use N is an odd integer, and $w \cdot N/d \equiv 1 \pmod{d}$. ■

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5 Relations among $\frac{\eta^N((N/d)z)}{\eta(dz)}$, Eisenstein series, $P(n)$, and $f_{\theta_{N-1}}(z)$

The end goal of this section is to prove Theorem 1.3. We first prove a relationship between $\frac{\eta^N((N/d)z)}{\eta(dz)}$ and Eisenstein series, see Theorem 5.1. Next, we prove a relationship between Eisenstein series and the partition function, see Theorem 5.2. To do this, we uncover a relationship between $\frac{\eta^N((N/d)z)}{\eta(dz)}$ and the partition function using arithmetic properties of Eisenstein series. We then prove another identity relating $f_{\theta_{N-1}}(z)$ to Eisenstein series, see Theorem 5.3. Finally, we show that Theorem 1.3 is a result of combination of these relations.

Now, we state and prove the relationship between $\frac{\eta^N((N/d)z)}{\eta(dz)}$ and Eisenstein series.

Theorem 5.1 *Let N be a positive squarefree integer such that* $\gcd(N,6)=1$. *Then we have*

$$\frac{\eta^N((N/d)z)}{\eta(dz)} = \chi_{N/d}(0) + \left(\frac{N/d}{d}\right)_K C(d,N)\cdot\frac{d}{N}\cdot\frac{(1-N)}{B_{(N-1)/2,\chi_N}}\cdot\sum_{n\geq 1}\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n + C_1(z),$$

where $C_1(z)\in S_{(N-1)/2}(\Gamma_0(N),\chi_N)$ *and*

$$C(d,N) := \frac{i^{(1-Nd)/2}}{A(d,N)} = \left(\frac{-8}{N}\right)_K\left(\frac{8}{d}\right)_K\left(\frac{-4}{d}\right)_K^{(N-1)/2}.$$

Proof Let N be a positive squarefree integer such that $\gcd(N,6)=1$. Then by [4, Propositions 5.9.2 and 5.9.3] and Lemma 4.1, we have

$$\frac{\eta^N((N/d)z)}{\eta(dz)} \in M_{(N-1)/2}(\Gamma_0(N),\chi_N).$$

Now the desired result follows by combining Theorem 2.1 and Lemma 4.2. ∎

Next we state and prove a relationship between $P(n)$ and Eisenstein series.

Theorem 5.2 *Let N be a positive squarefree integer such that* $\gcd(N,6)=1$. *Then we have*

$$\chi_{N/d}(0) + C(d,N)\cdot(N/d)^{(N-3)/2}\frac{(1-N)}{B_{(N-1)/2,\chi_N}}\cdot\sum_{n\geq1}\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)q^n$$
$$= N/d\cdot(q;q)_\infty^N\cdot\sum_{n\geq0}P\left(\frac{N}{d^2}n-\frac{N^2-d^2}{24d^2}\right)q^n + C_2(z),$$

where $C_2(z)$ *is some cusp form in* $S_{(N-1)/2}(\Gamma_0(N),\chi_N)$.

Proof For $m\in\mathbb{N}$, we define the operator $U(m)$ by

$$U(m)\Big|\sum_{n\geq0}a_nq^n = \sum_{n\geq0}a_{nm}q^n.$$

Then we have

$$U(N/d)\Big|\frac{\eta^N((N/d)z)}{\eta(dz)} = (q;q)_\infty^N\sum_{n\geq0}P\left(\frac{N}{d^2}n-\frac{N^2-d^2}{24d^2}\right)q^n. \tag{5.1}$$

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On the other hand, we observe that

$$\sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n \cdot N/d) = \chi_d(N/d)(N/d)^{(N-3)/2}\sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n).$$

Therefore, we have

$$U(N/d)\Big|\left(\chi_{N/d}(0) + C(d,N)\left(\frac{N/d}{d}\right)_K \cdot \frac{d}{N} \cdot \frac{(1-N)}{B_{(N-1)/2,\chi_N}} \cdot \sum_{n\geq 1} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n)q^n\right)$$
$$= \chi_{N/d}(0) + C(d,N)\left(\frac{N/d}{d}\right)_K \chi_d(N/d)(N/d)^{(N-3)/2}\frac{(1-N)d/N}{B_{(N-1)/2,\chi_N}} \cdot \sum_{n\geq 1} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n)q^n. \tag{5.2}$$

Finally the result follows from combining (5.1), (5.2), Theorem 5.1, and the elementary equation

$$\left(\frac{N/d}{d}\right)_K \chi_d(N/d) = 1,$$

and the property of modular forms that if $C_1(z) \in S_{(N-1)/2}(\Gamma_0(N), \chi_N)$ then

$$C_2(z) := U(N/d)\Big|C_1(z) \in S_{(N-1)/2}(\Gamma_0(N), \chi_N). \qquad \blacksquare$$

Theorem 5.3 *Let N be a positive squarefree integer such that* $\gcd(N,6) = 1$. *We have*

$$f_{\theta_{N-1}}(z) = 1 + \sum_{d|N} C(d,N)(N/d)^{(N-3)/2}\frac{(1-N)}{B_{(N-1)/2,\chi_N}} \cdot \sum_{n\geq 1} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n)q^n$$
$$+ C_3(z),$$

where $C_3(z)$ is some cusp form in $S_{(N-1)/2}(\Gamma_0(N), \chi_N)$.

Proof By [3, Theorem 2.1], we have $f_{\theta_{N-1}}(z) \in M_{(N-1)/2}(\Gamma_0(N), \chi_N)$. Therefore, the result follows from combining Theorems 2.1 and 3.7. $\blacksquare$

Now, we give the proof of Theorem 1.3.

Proof We start by proving part (i). By combining Theorems 5.2 and 5.3, we obtain

$$f_{\theta_{N-1}}(z) = (q;q)_\infty^N \sum_{d|N} \frac{N}{d} \sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)q^n + C(z) \tag{5.3}$$

for some $C(z) \in S_{(N-1)/2}(\Gamma_0(N), \chi_N)$. We divide both sides of (5.3) by $(q;q)_\infty^N$ to obtain

$$\sum_{n\geq 0} c\phi_N(n)q^n = \sum_{n\geq 0}\left(\sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)\right)q^n + \frac{C(z)}{(q;q)_\infty^N}. \tag{5.4}$$

(1.5) follows by comparing coefficients of q^n in (5.4).

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Now, we prove part (ii) of Theorem 1.3. When $N \geq 29$ a squarefree positive integer coprime to 6 and $d < N$ a divisor of N then $\frac{N}{d^2} - \frac{N^2-d^2}{24d^2} \leq 0$. Therefore by (1.5) and $c\phi_N(1) = N^2$, we have

$$b(1) = c\phi_N(1) - \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2} - \frac{N^2-d^2}{24d^2}\right)$$
$$= c\phi_N(1) - P\left(\frac{1}{N}\right) = N^2 \neq 0.$$

Hence, when $N \geq 29$ is a squarefree positive integer coprime to 6, we have $C(z) \neq 0$. Similarly when $N = 13, 17, 19$, or $N = 23$ by (1.5), we have

$$b(1) = c\phi_N(1) - N \cdot P\left(N - \frac{N^2-1}{24}\right) - P\left(\frac{1}{N}\right) = \begin{cases} 26 \neq 0 & \text{if } N = 13, \\ 170 \neq 0 & \text{if } N = 17, \\ 266 \neq 0 & \text{if } N = 19, \\ 506 \neq 0 & \text{if } N = 23. \end{cases}$$

This shows that $C(z) \neq 0$ when $N = 13, 17, 19$, or $N = 23$. Therefore by (1.1)–(1.3), we have $C(z) = 0$ if and only if $N = 5, 7$, or 11.

Finally we prove part (iii) of the theorem. We prove it by contradiction. Assume that there is an $M \geq 0$ such that $b(n) = 0$ for all $n > M$, then we would have

$$\sum_{n=1}^{M} b_n q^n = \frac{C(z)}{(q;q)_\infty^N}.$$

The RHS of this equation is a meromorphic modular function and the left hand side is an exponential sum. This is possible only if $\frac{C(z)}{(q;q)_\infty^N} = 0$, which is shown to be false unless $N = 5, 7$, or 11 in the proof of part (ii) of the theorem. ∎

6 Proof of Theorem 1.4

Throughout this section, let N be a positive integer such that $\gcd(N, 6) = 1$. We will use the Vinogradov symbols and various asymptotic notations in estimates involving functions in n where $n \in \mathbb{N}$ is large. The implicit constants in these estimates might depend on N but they are independent of n. Let

$$\mathcal{U}(n) := \frac{1-N}{B_{(N-1)/2,\chi_N}} \sum_{d|N} C(d,N)(N/d)^{(N-3)/2} \sigma_{(N-3)/2}(\chi_{N/d}, \chi_d; n).$$

We start by investigating the size of $\mathcal{U}(n)$.

Lemma 6.1 *We have $\mathcal{U}(n) > 0$ for every $n \in \mathbb{N}$ and*

$$\mathcal{U}(n) \gg n^{(N-3)/2} \quad \text{if } N > 5,$$
$$\mathcal{U}(n) \gg n/\log\log n \quad \text{if } N = 5.$$

Proof Let $n = \prod_{p|n} p^{e_p}$ be the prime factorization of n and write $k = (N-3)/2$. Then we have

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$$\sigma_k(\chi_{N/d},\chi_d;n)=\prod_{p|n}\frac{(\chi_d(p)p^k)^{e_p+1}-\chi_{N/d}(p)^{e_p+1}}{\chi_d(p)p^k-\chi_{N/d}(p)}$$
$$=\prod_{\substack{p|n\\p|d}}\chi_{N/d}(p^{e_p})\prod_{\substack{p|n\\p|N/d}}\chi_d(p^{e_p})p^{ke_p}\prod_{\substack{p|n\\p\nmid N}}\frac{(\chi_d(p)p^k)^{e_p+1}-\chi_{N/d}(p)^{e_p+1}}{\chi_d(p)p^k-\chi_{N/d}(p)}$$
$$=\prod_{\substack{p|n\\p|d}}\chi_{N/d}(p^{e_p})\prod_{\substack{p|n\\p\nmid d}}\chi_d(p^{e_p})\prod_{\substack{p|n\\p|N/d}}p^{ke_p}\prod_{\substack{p|n\\p\nmid N}}\frac{(p^k)^{e_p+1}-\chi_N(p)^{e_p+1}}{p^k-\chi_N(p)}.$$

Therefore, by elementary manipulations, we obtain

$$\sum_{d|N}C(d,N)(N/d)^{(N-3)/2}\cdot\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)$$
$$=\sum_{d|N}C(d,N)(N/d)^{(N-3)/2}\cdot\prod_{\substack{p|n\\p|d}}\chi_{N/d}(p^{e_p})\prod_{\substack{p|n\\p\nmid d}}\chi_d(p^{e_p})\prod_{\substack{p|n\\p|N/d}}p^{ke_p}$$
$$\times\prod_{\substack{p|n\\p\nmid N}}\frac{(p^k)^{e_p+1}-\chi_N(p)^{e_p+1}}{p^k-\chi_N(p)}$$
$$=\left(\frac{-8}{N}\right)_K N^{(N-3)/2}\prod_{\substack{p|n\\p|N}}p^{ke_p}\prod_{\substack{p|n\\p\nmid N}}\frac{(p^k)^{e_p+1}-\chi_N(p)^{e_p+1}}{p^k-\chi_N(p)}$$
$$\times\sum_{d|N}\left(\frac{-8}{N}\right)_K C(d,N)(1/d)^{(N-3)/2}\cdot\prod_{\substack{p|n\\p\nmid d}}\chi_d(p^{e_p})\prod_{\substack{p|n\\p|d}}\frac{\chi_{N/d}(p^{e_p})}{p^{ke_p}}.$$

On the other hand,

$$\left(\frac{-8}{N}\right)_K C(d,N)(1/d)^{(N-3)/2}\cdot\prod_{\substack{p|n\\p\nmid d}}\chi_d(p^{e_p})\prod_{\substack{p|n\\p|d}}\frac{\chi_{N/d}(p^{e_p})}{p^{ke_p}}$$

is a multiplicative function of $d\mid N$. Therefore, we have

$$\frac{1-N}{B_{(N-1)/2,\chi_N}}\sum_{d|N}C(d,N)(N/d)^{(N-3)/2}\cdot\sigma_{(N-3)/2}(\chi_{N/d},\chi_d;n)$$
$$=\left(\frac{-8}{N}\right)_K\frac{1-N}{B_{(N-1)/2,\chi_N}}N^{(N-3)/2}\prod_{\substack{p|n\\p|N}}p^{ke_p}\prod_{\substack{p|n\\p\nmid N}}\frac{(p^k)^{e_p+1}-\chi_N(p)^{e_p+1}}{p^k-\chi_N(p)}$$
$$\times\prod_{\substack{s|N\\s\text{ prime}}}\left(1+\left(\frac{-8}{N}\right)_K C(s,N)(1/s)^{(N-3)/2}\cdot\prod_{\substack{p|n\\p\nmid s}}\chi_s(p^{e_p})\prod_{\substack{p|n\\p|s}}\frac{\chi_{N/s}(p^{e_p})}{p^{ke_p}}\right)$$

$$= \left(\frac{-8}{N}\right)_K \frac{1-N}{B_{(N-1)/2,\chi_N}} N^{(N-3)/2} n^k \prod_{\substack{p|n \\ p\nmid N}} \frac{1-\chi_N(p)^{e_p+1}/(p^k)^{e_p+1}}{1-\chi_N(p)/p^k}$$

$$\times \prod_{\substack{s|N \\ s \text{ prime}}} \left(1 + \left(\frac{-8}{N}\right)_K C(s,N)(1/s)^{(N-3)/2} \cdot \prod_{\substack{p|n \\ p\nmid s}} \chi_s(p^{e_p}) \prod_{\substack{p|n \\ p|s}} \frac{\chi_{N/s}(p^{e_p})}{p^{ke_p}} \right).$$

The product over primes $s \mid N$ is at least

$$\prod_{s|N} \left(1 - \frac{1}{s}\right)$$

while the first product

$$\prod_{\substack{p|n \\ p\nmid N}} \frac{1-\chi_N(p)^{e_p+1}/(p^k)^{e_p+1}}{1-\chi_N(p)/p^k} = \prod_{\substack{p|n \\ p\nmid N}} \left(1 + \frac{\chi_N(p)p^{-k} - \chi_N(p)^{e_p+1}p^{-k(e_p+1)}}{1-\chi_N(p)p^{-k}}\right)$$

$$\geq \prod_{\substack{p|n \\ p\geq 3}} \left(1 - \frac{p^{-k}+p^{-2k}}{1-p^{-k}}\right).$$

When $N > 5$ and hence $k > 1$, we have

$$\prod_{\substack{p|n \\ p\geq 3}} \left(1 - \frac{p^{-k}+p^{-2k}}{1-p^{-k}}\right) > \prod_{p\geq 3} \left(1 - \frac{p^{-k}+p^{-2k}}{1-p^{-k}}\right)$$

which converges to a positive number. When $N = 5$, we have

$$\prod_{\substack{p|n \\ p\geq 3}} \left(1 - \frac{p^{-1}+p^{-2}}{1-p^{-1}}\right) \gg \prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{\varphi(n)}{n} \gg \frac{1}{\log\log n}.$$

It remains to show $\left(\frac{-8}{N}\right)_K \frac{1-N}{B_{(N-1)/2,\chi_N}} N^{(N-3)/2} > 0$. We have the relation between Dirichlet L-functions and Bernoulli numbers [9, Theorem 3.3.4]:

$$B_{(N-1)/2,\chi_N} = \frac{2k!N^k}{(-1)^{(N-3)/2}(2\pi i)^{(N-1)/2}W(\chi_N)} L((N-1)/2,\chi_N)$$

$$= \frac{1}{(-1)^{(N-3)/2} i^{(N-1)/2} \varepsilon_N} \frac{2k!N^{k-1/2}L((N-1)/2,\chi_N)}{(2\pi)^{(N-1)/2}}.$$

We have $\frac{2k!N^{k-1/2}L((N-1)/2,\chi_N)}{(2\pi)^{(N-1)/2}} > 0$ and it is easy to check $\frac{1}{(-1)^{(N-3)/2} i^{(N-1)/2} \varepsilon_N} = -\left(\frac{-8}{N}\right)_K$. This completes the proof. ∎

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For each non-negative integer r, we define $\mathcal{V}_r(n)$ for $n \geq 0$ by

$$\sum_{n\geq 0} \mathcal{V}_r(n)q^n = \frac{1}{(q;q)_\infty^r} = \left(\sum_{n\geq 0} P(n)q^n\right)^r = \sum_{n\geq 0} \sum_{\substack{x\in\mathbb{N}_0^r \\ \sum x_i = n}} \prod_{i=1}^r P(x_i)q^n.$$

We have

Proposition 6.2 *For $r \geq 1$,*

(i) $\displaystyle\lim_{n\to\infty} \frac{\mathcal{V}_r(n)}{\mathcal{V}_r(n-1)} = 1.$

(ii) $\displaystyle\lim_{n\to\infty} \frac{\mathcal{V}_{r-1}(n)}{\mathcal{V}_r(n)} = 0.$

Proof We use induction on r. When $r = 1$, both (i) and (ii) hold since $\mathcal{V}_1(n) = P(n)$ and $\mathcal{V}_0(n) = 0$ for $n > 0$. Consider $r \geq 2$ and assume that both (i) and (ii) hold for $r - 1$.

First, we prove part (ii) for r. We have

$$\mathcal{V}_r(n) = \mathcal{V}_{r-1}(n) + \mathcal{V}_{r-1}(n-1)P(1) + \mathcal{V}_{r-1}(n-2)P(2) + \cdots + \mathcal{V}_{r-1}(0)P(n).$$

By part (i) for $r - 1$, for each fixed positive integer k, we have

$$\lim_{n\to\infty} \frac{\mathcal{V}_{r-1}(n)}{\mathcal{V}_{r-1}(n-k)} = 1.$$

Therefore, part (ii) for r holds.

Finally, we prove part (i) for r. It suffices to show that for any given $\varepsilon > 0$, we have

$$\frac{\mathcal{V}_r(n)}{\mathcal{V}_r(n-1)} < 1 + \varepsilon \text{ for all sufficiently large } n.$$

Fix k such that $P(m)/P(m-1) < 1 + \varepsilon/2$ for every $m \geq k$. Let

$$S = \{x \in \mathbb{N}_0^r : \textstyle\sum x_i = n \text{ and } x_1 \geq k\},$$

$$S' = \{x \in \mathbb{N}_0^r : \textstyle\sum x_i = n \text{ and } x_1 < k\}.$$

For each $x = (x_1, \dots, x_r) \in S$, put $y = (y_1, \dots, y_r)$ with $y_1 = x_1 - 1$ and $y_i = x_i$ for $i \geq 2$. Then we have

$$\prod_{i=1}^r P(x_i) \Big/ \prod_{i=1}^r P(y_i) = P(x_1)/P(x_1 - 1) < 1 + \varepsilon/2,$$

which implies

$$\left(\sum_{x\in S} \prod_{i=1}^r P(x_i)\right) \Big/ \mathcal{V}_r(n-1) < 1 + \varepsilon/2. \tag{6.1}$$

On the other hand, we have

$$\sum_{x\in S'} \prod_{i=1}^r P(x_i) = \sum_{j=0}^{k-1} P(j)\mathcal{V}_{r-1}(n-j).$$

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And since each $\mathcal{V}_{r-1}(n-j)/\mathcal{V}_r(n) \to 0$ as $n \to \infty$, we have

$$\left(\sum_{x\in S'}\prod_{i=1}^{r} P(x_i)\right) \Big/ \mathcal{V}_r(n-1) < \varepsilon/2$$

for all sufficiently large n. Combining this with (6.1), we finish the proof that $\mathcal{V}_r(n)/\mathcal{V}_r(n-1) < 1+\varepsilon$ for all sufficiently large n. ∎

Now, we prove Theorem 1.4.

Proof When $N = 5, 7$, or 11 from (5.3) and Sturm's theorem, we have $c\phi_N(n) = \sum_{d|N} N/d \cdot P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) (\neq 0)$. Therefore, the statement for $N = 5, 7$, or 11 follows immediately. From now on assume $N > 11$. By Theorem 5.3, we have

$$f_{\theta_{N-1}}(z) - 1 - \sum_{n\geq 1}\mathcal{U}(n)q^n \in S_{(N-1)/2}(\Gamma_0(N), \chi_N).$$

Thus, by [4, Theorem 9.2.1.(a)], we have

$$f_{\theta_{N-1}}(z) - 1 - \sum_{n\geq 1}\mathcal{U}(n)q^n = \sum_{n\geq 1} O(n^{(N-1)/4})q^n.$$

On the other hand, by Theorem 5.2, we have

$$(q;q)_\infty^N \sum_{d|N}(N/d)\sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)q^n - 1 - \sum_{n\geq 1}\mathcal{U}(n)q^n \in S_{(N-1)/2}(\Gamma_0(N), \chi_N).$$

Hence, by [4, Theorem 9.2.1.(a)], we have

$$(q;q)_\infty^N \sum_{d|N}(N/d)\sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right)q^n - 1 - \sum_{n\geq 1}\mathcal{U}(n)q^n$$
$$= \sum_{n\geq 1} O(n^{(N-1)/4})q^n.$$

Now we let $\mathcal{V}(n) := \mathcal{V}_N(n)$ so that

$$\frac{1}{(q;q)_\infty^N} = \sum_{n\geq 0}\mathcal{V}(n)q^n.$$

With this notation and the earlier arguments, we obtain

$$c\phi_N(n) - \sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}() = O\left(\sum_{\ell+m=n}\mathcal{V}(m)\ell^{(N-1)/4}\right), \tag{6.2}$$

and

$$\sum_{d|N}(N/d)\sum_{n\geq 0} P\left(\frac{N}{d^2}n - \frac{N^2-d^2}{24d^2}\right) - \sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell)$$
$$= O\left(\sum_{\ell+m=n}\mathcal{V}(m)\ell^{(N-1)/4}\right). \tag{6.3}$$

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From (6.2) and (6.3), we have

$$\lim_{n\to\infty} \frac{c\phi_N(n)}{\sum_{d|N}(N/d)P\left(\frac{N}{d^2}n-\frac{N^2-d^2}{24d^2}\right)} = \lim_{n\to\infty} \frac{\sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell)+O\left(\sum_{\ell+m=n}\mathcal{V}(m)\ell^{(N-1)/4}\right)}{\sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell)+O\left(\sum_{\ell+m=n}\mathcal{V}(m)\ell^{(N-1)/4}\right)}.$$

To obtain the desired result, we prove

$$\sum_{\ell+m=n}\mathcal{V}(m)\ell^{(N-1)/4} = o\left(\sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell)\right) \text{ as } n\to\infty. \tag{6.4}$$

Let $\varepsilon>0$. Since $N>11$, we have that $\mathcal{U}(\ell)\gg\ell^{(N-3)/2}$ dominates $\ell^{(N-1)/4}$ when ℓ is large. Choose $L_\varepsilon>0$ such that for every $\ell\ge L_\varepsilon$, we have $\ell^{(N-1)/4}<\varepsilon\mathcal{U}(\ell)$. This yields:

$$\sum_{\ell+m=n,\ell\ge L_\varepsilon}\mathcal{V}(m)\ell^{(N-1)/4} < \varepsilon\sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell) \tag{6.5}$$

Choose a positive integer L'_ε such that

$$\mathcal{U}(\ell) > \varepsilon^{-1}L_\varepsilon^{(N+3)/4} \quad \text{for every } \ell\ge L'_\varepsilon. \tag{6.6}$$

We now consider $\ell<L_\varepsilon$. We have

$$\mathcal{V}(n-\ell)\ell^{(N-1)/4} \le \mathcal{V}(n-\ell)L_\varepsilon^{(N-1)/4} \le \frac{\varepsilon}{L_\varepsilon}\mathcal{V}(n-\ell)\mathcal{U}(L'_\varepsilon\ell).$$

Proposition 6.2 implies that $\mathcal{V}(n-\ell)<2\mathcal{V}(n-L'_\varepsilon\ell)$ for every $\ell<L_\varepsilon$ and for every sufficiently large n. This yields

$$\mathcal{V}(n-\ell)\ell^{(N-1)/4} \le \frac{2\varepsilon}{L_\varepsilon}\mathcal{V}(n-L'_\varepsilon\ell)\mathcal{U}(L'_\varepsilon\ell)$$

and hence

$$\sum_{\ell+m=n,\ell<L_\varepsilon}\mathcal{V}(m)\ell^{(N-1)/4} < 2\varepsilon\sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell) \tag{6.7}$$

for all sufficiently large n. From (6.5) and (6.7), we have

$$\sum_{\ell+m=n}\mathcal{V}(m)\ell^{(N-1)/4} < 3\varepsilon\sum_{\ell+m=n}\mathcal{V}(m)\mathcal{U}(\ell)$$

for all sufficiently large n and this finishes the proof. ■

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