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A CONTRIBUTION TO THE THEORY OF METRIZATION

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In a paper on the same subject [**28**] and another coming out at the same time [**27**], Nagata gave his celebrated Double (treble, really) Sequence Theorem, with which he deduced easily and thus brought together the *basic* metrization theorems, i.e. theorems in which the conditions for metrizability are given as the availability of bases or subbases of certain descriptions. The power of that theorem is demonstrated by the ease with which everything follows from it, and it must be that the theorem comes close to the heart of the matter for it to have that power; although, no doubt, from the other theorems the Double Sequence Theorem can also be deduced to different extents and with various degrees of difficulty: the theorem itself has in fact been proved [**27**; **28**; **29**] via the theorem of Alexandroff-Urysohn [**1**] with the help of A. H. Stone [**35**] and Michael's results [**21**; **22**; **23**] on paracompactness, and it has been demonstrated that it is an easy consequence of Frink's [**9**; **19**], etc. But there is not a description of bases, which this Nagata Structure, as the base described in the Double Sequence Theorem is called, and the Nagata-Smirnov Base [**6**; **26**; **34**] and others simultaneously fit. Such a description would certainly come even closer to the matter, providing another (perhaps better) view of the metric landscape. We propose the result in § 2 (largely formulated early in 1975 but unpublished [**14**]) which is implicit in Hung [**13**]. There we have a baselike *object*, generalizing the Nagata-Smirnov Base, the description of which Nagata Structure can be seen to fit, with a tilt of the head perhaps—but no tinkering whatsoever is necessary. Thus Theorem 2.1 generalizes (slightly) the Double Sequence Theorem, while truly unifying Nagata with Nagata-Smirnov and others, allowing at the same time other formulations.

One notion used here for our purpose is that of a (well ordered) family of *disjoint pairs* and their *separation of points*, which is explained in § 1. In § 2 we present the main Theorem and go on immediately to show how many *basic* theorems are its straightforward corollaries, deferring its two proofs to § 3 and § 4.

1. Preliminaries. A *disjoint pair* $\mathscr{B}$ is a collection of two disjoint sets. We say a disjoint pair $\mathscr{B} = \{B, \tilde{B}\}$ *separates* a point x and a point y if $x \in B$ and $y \in \tilde{B}$ or $x \in \tilde{B}$ and $y \in B$.

The axiom of choice is assumed. Ordinal numbers are denoted by α, β, ξ, ζ, η. An ordinal coincides with the set of all smaller ordinals, i.e., $\xi < \zeta$ is equivalent

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to $\xi \in \zeta$. Nevertheless we make the notational distinction between the first ordinal 0 and the empty set $\emptyset$. A cardinal number is an initial ordinal. The first infinite cardinal is ω. An ordinal ξ is a *non-limit ordinal* if $\xi = \zeta \cup \{\zeta\}$ for some ordinal ζ; otherwise ξ is a *limit ordinal*. Let ξ be a limit ordinal. The *cofinality* of ξ, denoted by cf (ξ), is the least ordinal ζ from which there is a function f into ξ such that

a) f is *order-preserving* (i.e., $f(\eta) \leqq f(\eta')$ for $\eta < \eta' < \zeta$), and

b) f is *unbounded* (i.e., $\sup_{\eta<\zeta} f(\eta) = \xi$). (See e.g. [**8**]).

A family of disjoint pairs $\{\mathscr{B}_\xi\}_{\xi<\alpha}$ *separates* the sets A_1 and A_2 if for every $x \in A_1$ and every $y \in A_2$ there is some $\zeta < \alpha$ such that $\mathscr{B}_\zeta$ separates x and y.

2. Main theorem.

We state our theorem as follows.

THEOREM 2.1. *A topological space X is metrizable if and only if it is T_0 and there exists on it a family of disjoint pairs, $\{\mathscr{A}_\xi\}_{\xi<\alpha}$, for some α confinal with ω,* i.e., cf $(\alpha) = \omega$, *with the following property. For each $x \in X$ and each $\beta < \alpha$, the set*

$$\bigcap \{\sim \tilde{B} : x \in B, \{B, \tilde{B}\} = \mathscr{A}_\xi, \xi < \beta\}$$

is a neighbourhood of x (it is sufficient that these sets form a *weak base* of the topology (cf. [**5**, **12**; **20**])), *and*

(i) *for each $x \in X$ and each open U containing x, there exist a neighbourhood V of x and an ordinal $\zeta < \alpha$ such that V and $\sim U$ are separated by the family $\{\mathscr{A}_\xi\}_{\xi<\zeta}$; or*

(ii) *every compact set K is separated from every disjoint closed set C by a family $\{\mathscr{A}_\xi\}_{\xi<\eta}$ for some $\eta < \alpha$, depending on K and C* (cf. Michael [**24**] and O'Meara [**33**] on pseudobases); *or*

(iii) *for each $x \in X$ and each open U containing x, there exist $\eta, \zeta < \alpha$ such that*

$$x \in B \subset \sim \tilde{B} \subset C \subset \sim \tilde{C} \subset U$$

where $\{B, \tilde{B}\} = \mathscr{A}_\eta$ $\{C, \tilde{C}\} = \mathscr{A}_\zeta$ (cf. Harley and Faulkner [**10**]).

One proof of this theorem parallels exactly that in Hung [**13**], which we shall give briefly in § 3. Another proof is given in § 4. We now, in order to acquaint our readers with the meaning of the conditions in the theorem, give an example of a baselike object of the descriptions in Theorem 2.1 in a metric space. For all $n \in \mathbf{N}$, let $\mathscr{A}^n$ be the family of *all* disjoint pairs of sets distant at least $1/n$ apart. The family

$$\mathscr{A} \equiv \bigcup \{\mathscr{A}^n : n \in \mathbf{N}\}$$

can clearly be well ordered into one that fits the descriptions given in the theorem. At the same time, σ-locally finite bases, bases described by Nagata, Hung [**7**; **13**; **19**] are of those descriptions. The difference between our theorem here and Nagata's in [**7**; **13**; **19**] may formally look small, but it represents a movement into the crux of the matter, as seen below. While the success of the

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σ-discrete bases and Nagata and Smirnov bases and even the Nagata [**7**; **13**; **19**] bases are due to the severe and not so severe limitation on the ability of the parts of the bases to *accumulate*; in our case, we let them accumulate to the best of their ability only keeping them at an arm's (shortening) length, as it were. To demonstrate that we may be moving into the crux of the matter, we note the following obvious corollary to our theoren, alternative (i), which is Nagata's celebrated Double (treble really) Sequence Theorem.

COROLLARY 2.2 (Nagata). *A topological space X is metrizable if and only if it is T_0 and has two (countable) sequences, $S_n(x)$ and $U_n(x)$, of neighbourhoods about every point $x \in X$ satisfying the following. At each $x \in X$,*

(i) *the family $\{U_n(x) : n \in \mathbf{N}\}$ is a fundamental system of neighbourhoods at x* (These neighbourhoods and fundamental systems requirements at every point can evidently be weakened to anything that satisfies (i) in Theorem 2.1.)*;*

(ii) *for every $n \in \mathbf{N}$, there is one (open) neighbourhood $\Omega(x)$ of x such that*

$$\Omega(x) \cap S_n(y) = \emptyset \quad \text{if } x \notin U_n(y);$$

(iii) *for every $n \in \mathbf{N}$,*

$$\Omega'(x) \subset \bigcap \{U_n(y) : x \in S_n(y)\}$$

for some (open) neighbourhood $\Omega'(x)$ of x.

Corollary 2.2 is obvious if one sees in the two neighbourhoods $S_n(x)$ and $U_n(x)$ the disjoint pairs $\{S_n(x), \sim U_n(x)\}$.

We may further note the following equally obvious corollary to our theorem, again alternative (i), which is (a strengthened version of) Frink's [**9**; **19**].

COROLLARY 2.3. *A topological space X is metrizable if and only if it is T_0 and has neighbourhoods $\{U_n(x) : n \in \mathbf{N}\}$ at every $x \in X$ with the following property: For each $x \in X$ and each open U containing x, there exist a neighbourhood V of x and an $m \in \mathbf{N}$ such that*

i) $U_m(y) \cap V = \emptyset$ *if* $y \notin U$, *and*

ii) $U_m(y) \subseteq U$ *if* $y \in V$.

Corollary 2.3 is obvious if one notes that $\{\sim U, V\}$ is a disjoint pair. Readers may note that Corollary 2.3 strengthens simultaneously Arhangel'skiĭ-Stone [**2**; **5**; **36**] and Alexandroff-Urysohn [**1**]. It also strengthens the *usual* Double Sequence Theorem, although not Corollary 2.2 above in which the original conditions have already been relaxed somewhat. This corollary and the one above point up the redundancies in the Double Sequence Theorem.

The next corollary, representing an alternative to Nagata's formulation above and a substantial improvement on an earlier unifying result [**15**], itself generalizing Arhangel'skiĭ-Stone [**2**; **5**; **36**] and Alexandroff-Urysohn [**1**], follows also from alternative (i) of Theorem 2.1.

COROLLARY 2.4. *A topological space X is metrizable if and only if it is T_0 and has at every point $p \in X$ a set of neighbourhoods $\{U_n(p)\}_{n \in \mathbf{N}}$ satisfying the following:*

(i) *for every $n \in \mathbf{N}$, there is one neighbourhood Ω for each point x such that $y \notin \Omega$ if $x \notin U_n(y)$; and*

(ii) *for every open neighbourhood W of x, there exist a neighbourhood V and an $m \in \mathbf{N}$ such that either* a) $V \cap U_m(y) = \emptyset$ *if $y \notin W$, or* b) $W \supset U_m(z)$, *if $z \in V$.*

Condition (ii) can evidently be weakened to the following:

(ii)′ *for every open neighbourhood W of x, there exist a neighbourhood V and an $m \in \mathbf{N}$ such that for each $y \notin W$ and $z \in V$, either $y \notin U_m(z)$ or $z \notin U_m(y)$.*

Alternative a) is Heath's characterization of Nagata Spaces [**11**]. (Every $U_n(p)$ may be looked upon as a disjoint pair, $\{\{p\}, \sim U_n(p)\}$.)

The same corollary with alternative a) may be reformulated as follows.

COROLLARY 2.4a. *A topological space x is metrizable if and only if it is T_0 and has at every point $p \in X$ a countable local base $\{U_n(p)\}_{n \in \mathbf{N}}$ satisfying the following: Given any (countable) sequence $\{x_i\}$,* (i) *if x is a cluster point, then for each $n \in \mathbf{N}$, $x \in U_n(x_i)$ for infinitely many i; on the other hand* (ii) *if x is not a cluster point, then there is at least one $m \in \mathbf{N}$ such that not only $x \notin U_m(x_i)$ for any i, but no points in some neighbourhood V belong to $U_m(x_i)$ for any i.*

Nagami's Theorem [**30**; **31**] as stated below, and therefore Morita's [**25**], are also obvious corollaries to Theorem 2.1, alternative (i).

COROLLARY 2.5 (Nagami). *A topological space X is metrizable if and only if it is T_0 and has a (countable) sequence $\mathscr{F}_n$ of closure preserving closed covers with the following property: For each $x \in X$ and each open U containing x, there exists such an $n \in \mathbf{N}$ that* St $(x, \mathscr{F}_n) \subset U$.

(Corollary 2.5 is obvious if one observes that

$$\{\sim \text{St}\,(x, \mathscr{F}_n), \sim \bigcup \{F : x \notin F \in \mathscr{F}_n\}\}$$

is a disjoint pair.)

Readers may have noticed that Corollaries 2.2-2.5 are all corollaries to Theorem 2.1 with alternative (i). Clearly, if alternatives (ii) or (iii) are invoked instead in each instance, we would have parallels to the above every time. Of these parallels, we name but two in the following and remark that clearly a parallel to 2.4 would improve on Theorem 1.1 of [**15**], itself generalizing Arhangel'skiĭ-Jones [**3**; **4**; **16**; **17**].

COROLLARY 2.6. *A topological space x is metrizable if and only if it is T_0 and has two (countable) sequences $S_n(K)$ and $U_n(K)$ of sets about (i.e. containing) every compact K satisfying the following:*

(i) *for every* $K \subset X$ *contained in an open set* U, *there exists an* $n \in \mathbf{N}$ *such that*

$$K \subset S_n(K) \subset U_n(K) \subset U \quad \text{at each } x \in X;$$

(ii) *for every* $n \in \mathbf{N}$, *there is one (open) neighbourhood* $\Omega(x)$ *of* x *such that*

$$\Omega(x) \cap S_n(K) = \emptyset \quad \text{if } x \notin U_n(K); \text{ and}$$

(iii) *for every* $n \in \mathbf{N}$,

$$\Omega'(x) \subset \cap \{U_n(K) : x \in S_n(K)\}$$

for some (open) neighbourhood $\Omega'(x)$ *of* x.

COROLLARY 2.7. *A topological space* X *is metrizable if and only if it is* T_0 *and has two (countable) sequences,* $S_n(x)$ *and* $U_n(x)$, *of sets above every point* $x \in X$ *satisfying the following. At each* $x \in X$,

(i) *for every* $x \in X$ *contained in an open set* U, *there exist* $m, n \in \mathbf{N}$ *such that*

$$x \in S_n(x) \subset U_n(x) \subset S_m(x) \subset U_m(x) \subset U;$$

(ii) *for every* $n \in \mathbf{N}$, *there is one (open) neighbourhood* $\Omega(x)$ *of* x *such that*

$$\Omega(x) \cap S_n(y) = \emptyset \quad \text{if } x \notin U_n(y); \text{ and}$$

(iii) *for every* $n \in \mathbf{N}$,

$$\Omega'(x) \subset \cap \{U_n(y) : x \in S_n(y)\}$$

for some (open) neighbourhood $\Omega'(x)$ *of* x.

3. Proof of main theorem. Given any space X and any family of disjoint pairs separating points, $\{\mathscr{A}_\xi\}_{\xi<\alpha}$, where the cofinality cf (α) of α is ω, the first infinite ordinal. For any countable sequence $\{\beta_i\}_{i<\omega}$ of ordinals cofinal with α, we can define a non-negative real valued function ρ on $X \times X$ as follows. For all $x, y \in X$, $x \neq y$, we can define $\rho(x, y)$ such that $1/\rho(x, y)$ equals the smallest non-zero i for which the family $\{\mathscr{A}_\xi\}_{\xi<\beta_i}$ separates x, y; which is always possible. For all $x \in X$, $\rho(x, x)$ is defined to be 0. Such a ρ is obviously a *symmetric*. (Cf. [**5**; **10**; **13**; **14**; **15**; **18**; **32**; **37**]. Briefly, a *symmetric* is that which if it also satisfies the usual triangle inequality is also a metric. A *symmetric space* is a space the topology of which consists of those (and only those) sets that contain a ball of some radius around every one of their members. Such a topology is said to be *induced* by the symmetric onto the space.) Since, for all $x \in X$, $j \in \mathbf{N}$, the set

$$\cap \{\sim \tilde{B} : x \in B, \{B, \tilde{B}\} = \mathscr{A}_\xi, \xi < \beta_j\}$$

is $N(x, 1/j) \equiv \{x \in X, \rho(x, y) < 1/j\}$; this symmetric induces a topology not finer than that of X, which is at the same time guaranteed not to be less fine by the facts (i), (ii) or (iii) in 2.1. X is therefore a symmetric space. The symmetric ρ is also guaranteed coherent (A symmetric ρ (on a symmetric space) is *coherent* if, for any compact set K and any disjoint closed set C, $\rho(K, C) > 0$;

according to Martin [**18**].) by the facts (i), (ii) or (iii) in 2.1. X is metrizable by Niemytzki-Wilson [**32**; **37**; **5**; **18**].

4. Another proof of the main theorem. A space X satisfying Theorem 2.1 by way of alternative (i) has, for every open cover of it, obviously a σ-cushioned open refinement and has therefore also a σ-discrete open refinement according to Michael [**23**]. While it is clear that condition (i) is a special case of (ii) and (iii) is a special case of (i), condition (ii) can be seen to be eqivalent to (i) with an argument similar to what F. B. Jones used to strengthen Moore's Metrization Theorem in [**16**; **17**]. Thus the conclusion above is good for all three alternatives.

We can also prove with another argument similar to F. B. Jones' that if, for every $x \in X$ and every $\beta < \alpha$, we write $F_{\beta,x}$ for the set

$$\bigcap \{\sim \tilde{B} : x \in B, \{B, \tilde{B}\} = \mathscr{A}_{\xi}, \xi < \beta\},$$

write $G_{\beta,x}$ for the interior of $F_{\beta,x'}$ write for every $\beta < \alpha$, $\mathscr{G}_{\beta}$ for the collection $\{G_{\beta,x} : x \in X\}$, and name any arbitrary increasing sequence $\{\beta_i\}_{i<\omega}$ of ordinals cofinal with α; then at every $x \in X$, the family of stars $\{\text{St}\ (x, \mathscr{G}_{\beta_i}) : i < \omega\}$ forms a local base.

If we now apply our first observation to the family of (open) covers $\{\mathscr{G}_{\beta_i} : i < \omega\}$, we will have a σ-discrete base. Our space X being clearly normal, it is almost obvious how a (compatible) metric can be constructed [**6**].

Since the argument we refer to in the second paragraph above is all-important, we set it out in full here. If we write, for every $\beta < \alpha, \mathscr{F}_{\beta}$ for the collection $\{F_{\beta,x} : x \in X\}$, it clearly suffices to argue that the family of stars

$$\{\text{St}\ (x\mathscr{F}_{\beta_i}) : i < \omega\}$$

forms a local base. Suppose such is not the case. Then there exist such an $x \in X$ and such an open set U containing x that St $(x, \mathscr{F}_{\beta_i}) \not\subset U$ for any $i < \omega$, i.e. that there exists a sequence $\{y_i\}_{i<\omega}$, with $y_i \in F_{\beta_i,x}$ and $F_{\beta_i,y_i} \not\subset U$. Clearly, $y_i \rightarrow x$ and for some N, the set $\{x, y_N, y_{N+1}, \ldots\}$ is compact and disjoint from $\sim U$ but not separated from it by any family $\{\mathscr{A}_{\xi}\}_{\xi<\eta}$, $\eta < \alpha$; which contradicts (ii) and proves our assertion.

It is interesting to compare this proof with that of Nagata's Theorem in [**29**].

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