

THE DRAZIN INVERSES OF PRODUCTS AND DIFFERENCES OF PROJECTIONS IN A C^* -ALGEBRA

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Abstract

For two given projections p and q in a C^* -algebra, we investigate how to express the Drazin inverses of the product pq and the difference $p - q$, and give applications. As a special case, we obtain the results of [C. Y. Deng, ‘The Drazin inverses of products and differences of orthogonal projections’, *J. Math. Anal. Appl.* **335** (2007) 64–71], with considerably simpler proofs.

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1. Introduction

Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on \mathcal{H} . Let \mathcal{A} be a unital C^* -algebra with unit e . An element $a \in \mathcal{A}$ is said to be *normal* (*self-adjoint*) if $aa^* = a^*a$ ($a = a^*$) (see [17]). An element $p \in \mathcal{A}$ is said to be a *projection* if $p^2 = p = p^*$. We use the notation

$$\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^*\}, \quad \mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}.$$

If $a \in \mathcal{A}$, then $\sigma(a)$ and $\text{acc } \sigma(a)$ denote the spectrum and the set of all accumulation points of $\sigma(a)$, respectively; a is *quasipolar* if $0 \notin \text{acc } \sigma(a)$, and *polar* if it is quasipolar and 0 is at most a pole of the resolvent $R(\lambda; a) = (\lambda e - a)^{-1}$ of a . In particular, a is *simply polar* if 0 is at most a simple pole of the resolvent $R(\lambda; a)$.

In 1958, Drazin (see [8]) introduced a different kind of a generalized inverse in associative rings and semigroups, one which does not have the reflexivity property, but commutes with the element. Let a, b be elements of a semigroup. An element b is the *Drazin inverse* of a , written $b = a^D$, if

$$ab = ba, \quad b = b^2a, \quad a^k = ba^{k+1}, \quad (1)$$

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for some nonnegative integer k . It is well known that if a is Drazin invertible, then the Drazin inverse is unique.

In 1996, Koliha (see [12]) introduced the generalized Drazin inverse in a ring. An element a of a unital ring R (with unit e') is *quasinilpotent* if $e' - xa \in \text{Inv}(R)$ for every x commuting with a , where $\text{Inv}(R)$ is the set of all invertible elements of R . Let $a \in R$. Then an element $b \in R$ is the *generalized Drazin inverse* of a if

$$ab = ba, \quad b = b^2a, \quad a - ba^2 \text{ is quasinilpotent.} \quad (2)$$

It is shown that a is quasipolar if and only if a is generalized Drazin invertible (see [12]). An element $a \in \mathcal{A}$ is *Moore–Penrose invertible* if there exists $x \in \mathcal{A}$ such that

$$axa = a, \quad xax = x, \quad (ax)^* = ax \quad (xa)^* = xa. \quad (3)$$

There is at most one element x satisfying Equation (3); if $a \in \mathcal{A}$ is Moore–Penrose invertible, then the unique solution of Equation (3) is called the Moore–Penrose inverse of a and is denoted by a^+ . In recent years, a number of researchers have given much attention to the Drazin inverse and Moore–Penrose inverse in C^* -algebras (see [9, 10, 12–15]). In particular, Koliha (see [13]) found that $a \in \mathcal{A}$ is Moore–Penrose invertible if and only if a^*a (or aa^*) is Moore–Penrose invertible if and only if a^*a (or aa^*) is quasipolar if and only if a^*a (or aa^*) is simply polar. Then a^*a (or aa^*) is generalized Drazin invertible if and only if a^*a (or aa^*) is Drazin invertible.

In this note, the Drazin inverses of products and differences of projections in a C^* -algebra are considered. The problem of the Drazin invertibility of the sum was first considered by Drazin in his celebrated paper [8]. Some additive perturbation results for Drazin inverses are given in [1, 4, 7, 11]. In particular, the Drazin and Moore–Penrose inverses of products and differences of orthogonal projections on a Hilbert space are obtained in [2, 6].

The aim of this note is to present some formulae for the Drazin inverses of products and differences of projections in a C^* -algebra. As a special case, we recover theorems and corollaries of [6] which consider similar questions on a Hilbert space. The techniques used by Deng in [6] are based on complicated space decompositions and on operator matrix representations which require pages of calculations. In contrast, the methods of the present paper, based on algebraic and spectral techniques in C^* -algebras, lead to a significant simplification of arguments.

2. Main results

To prove the main results, we shall begin with some lemmas.

If A and B are $n \times n$ complex matrices, Cline's formula (see [3]) is

$$(AB)^D = A[(BA)^D]^2B.$$

For a, b in an associative semigroup, if ba is Drazin invertible, then it is easy to verify that the element $c = a((ba)^D)^2b$ (see [5]) satisfies the definition of the Drazin inverse

of ab , so ab is Drazin invertible. Similarly, if ab is Drazin invertible, then ba is Drazin invertible. Therefore, we have the following lemma.

LEMMA 1. *Let $a, b \in \mathcal{A}$. Then ab is Drazin invertible if and only if ba is Drazin invertible.*

LEMMA 2 (see [17, Remark 1.2.1]). *Let $a, b \in \mathcal{A}$. Then $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.*

REMARK. From Lemma 2, it is clear that ab is generalized Drazin invertible if and only if ba is generalized Drazin invertible.

LEMMA 3. *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then $e - p - q$ is Drazin invertible if and only if pq is Drazin invertible.*

PROOF. For $p, q \in \mathcal{P}(\mathcal{H})$,

$$(\lambda - 1 + p)(\lambda - p - q)(\lambda - 1 + q) = \lambda((\lambda - 1)^2 - pq) \quad (4)$$

[16, Equation 2.5]. It is easy to verify that the equation holds for $p, q \in \mathcal{P}(\mathcal{A})$. Then for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, it is clear that $\lambda \in \sigma(p + q)$ if and only if $(\lambda - 1)^2 \in \sigma(pq)$.

By Lemma 1, pq is Drazin invertible if and only if pqp is Drazin invertible. It follows from [13, Theorem 2.4] that pqp is Drazin invertible if and only if $0 \notin \text{acc } \sigma(pqp)$. From Lemma 2 and Equation (4), we know that $0 \notin \text{acc } \sigma(pqp)$ if and only if $0 \notin \text{acc } \sigma(pq)$ if and only if $1 \notin \text{acc } \sigma(e - p - q)$ if and only if $e - p - q$ is Drazin invertible. \square

In the following, some equivalent conditions for pq to be Drazin invertible are given. As a corollary, we obtain [6, Theorem 2.1].

PROPOSITION 4. *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:*

- (a) pq is Drazin invertible;
- (b) qp is Drazin invertible;
- (c) $(e - p)(e - q)$ is Drazin invertible;
- (d) $(e - q)(e - p)$ is Drazin invertible;
- (e) pq is quasipolar (or generalized Drazin invertible);
- (f) pqp is Drazin invertible.

PROOF. (b) if and only if (c). It follows from Lemma 3 that pq is Drazin invertible if and only if $e - p - q$ is Drazin invertible if and only if $e - (e - p) - (e - q)$ is Drazin invertible if and only if $(e - p)(e - q)$ is Drazin invertible. The proof of the rest follows from Lemmas 1 and 2. \square

For $P, Q \in \mathcal{P}(\mathcal{H})$, [6, Theorem 2.4] has shown the equivalences of the following Proposition 5, by space decompositions and operator block techniques. The proof of [6, Theorem 2.4] is long and complicated. However, our proof of Proposition 5 is simple and algebraic.

PROPOSITION 5. *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:*

- (a) $p - q$ is Drazin invertible;
- (b) $p + q$ is Drazin invertible;
- (c) $p - qp$ is Drazin invertible;
- (d) $q - qp$ is Drazin invertible.

PROOF. For $p, q \in \mathcal{P}(\mathcal{H})$,

$$(\lambda - 1 + p)(\lambda - p + q)(\lambda - 1 + q) = \lambda(\lambda^2 - 1 + pq) \quad (5)$$

[16, Equation 2.4]. It is also easy to verify that the equation holds for $p, q \in \mathcal{P}(\mathcal{A})$. From Equations (4) and (5), it is easy to see that $0 \in \text{acc } \sigma(p - q)$ if and only if $0 \in \text{acc } \sigma(p + q)$. Thus $p - q$ is Drazin invertible if and only if $p + q$ is Drazin invertible.

It follows from Lemma 3 that $q(e - p)$ is Drazin invertible if and only if $p - q$ is Drazin invertible if and only if $p(e - q)$ is Drazin invertible. \square

Let $a, b \in \mathcal{A}$ be Drazin invertible. It has been shown that if $ab = ba = 0$, then $(a + b)^D = a^D + b^D$ (see [8, 12]). This leads to the following lemma.

LEMMA 6. *Let $p \in \mathcal{P}(\mathcal{A})$ and $a \in \mathcal{A}$ be self-adjoint and Drazin invertible. If $pa(e - p) = 0$ then $pa^D(e - p) = 0$.*

PROOF. Since $pa(e - p) = 0$, then $(e - p)ap = 0$, so $a = pap + (e - p)a(e - p)$, hence $\sigma(pap) \setminus \{0\} \subseteq \sigma(a) \setminus \{0\}$. It follows from $0 \notin \text{acc } \sigma(a)$ that $0 \notin \text{acc } \sigma(pap)$, so pap is Drazin invertible. By a similar argument, $(e - p)a(e - p)$ is Drazin invertible. Thus $a^D = (pap)^D + [(e - p)a(e - p)]^D$. Since $(e - p)pap = 0$ and $(pap)^D = (pap)[(pap)^D]^2$, then $(e - p)(pap)^D = 0$, so $p(pap)^D = (pap)^D$. Similarly, $(pap)^D p = (pap)^D$ and

$$[(e - p)a(e - p)]^D = (e - p)[(e - p)a(e - p)]^D = [(e - p)a(e - p)]^D(e - p).$$

Thus $a^D = (pap)^D p + (e - p)[(e - p)a(e - p)]^D$, so $pa^D(e - p) = 0$. \square

In the following, we present some formulae for $(pq)^D$, where $p, q \in \mathcal{P}(\mathcal{A})$. The results of [6, Theorem 2.2(1) and (2)] follow as special cases of (a) and (e). The result of (c) is an improvement of [6, Theorem 2.2(3)] which obtains $(PQP)^D PQ = (PQ)^D PQ$ for $P, Q \in \mathcal{P}(\mathcal{H})$.

PROPOSITION 7. *Let $p, q \in \mathcal{P}(\mathcal{A})$. If pq is Drazin invertible, then the following statements hold:*

- (a) $(pq)^D = qp$ if and only if $pq = qp$;
- (b) $p(pq)^D = (pq)^D$;
- (c) $(pqp)^D = (pq)^D p$;

- (d) $p[(e - q)(e - p)]^D = -(pq)^D(e - p);$
- (e) $(pq)^D = (pqp)^D - p[(e - q)(e - p)]^D.$

PROOF.

(a) Sufficiency is clear.

Necessity. If $(pq)^D = qp$, then $qp(pq)qp = qp$, so $(qp)^2 = qp$. Hence $pq = (pq)^2$, so $(pq)^D = pq$, then $pq = qp$, by the uniqueness of the Drazin inverse.

(b) Since $(e - p)pq = 0$ and $(pq)^D = (pq)(pq)^D(pq)^D$, then $(e - p)(pq)^D = 0$. Hence $p(pq)^D = (pq)^D$.

(c) Using Cline’s formula $(ab)^D = a[(ba)^D]^2b$, and letting $a = pq$ and $b = p$, then $(pqp)^D = pq[(pq)^D]^2p = (pq)^Dp$.

(d) By Proposition 4, $(e - q)(e - p)$ is Drazin invertible. It follows from (b) that $p[(e - q)(e - p)]^D = p[(e - q)(e - p)]^D(e - p)$ and $-(pq)^D(e - p) = -p(pq)^D(e - p)$, then

$$\begin{aligned} p[(e - q)(e - p)]^D + (pq)^D(e - p) &= p\{[(e - q)(e - p)]^D + (pq)^D\}(e - p) \\ &= p[(e - q)(e - p) + pq]^D(e - p) \\ &= p[e - (p - q)^2]^D(e - p). \end{aligned}$$

It is easy to see that $p[e - (p - q)^2](e - p) = 0$. Using Lemma 6, we have $p[e - (p - q)^2]^D(e - p) = 0$.

(e) From (c) and (d), (e) is clear. □

For $P, Q \in \mathcal{P}(\mathcal{H})$, the sufficiency of the following statements is given in [6, Remark]. By simpler algebraic and spectral techniques, we obtain the equivalence of the following statements, for $p, q \in \mathcal{P}(\mathcal{A})$.

THEOREM 8. *Let $p, q \in \mathcal{P}(\mathcal{A})$. If $p - q$ is Drazin invertible, then the following statements hold:*

- (a) $(p - q)^D = p - q$ if and only if $pq = qp$;
- (b) $(p + q)^D = p + q$ if and only if $pq = qp = 0$;
- (c) $(p + q)^D = p + q - \frac{3}{2}pq$ if and only if $pq = qp$;
- (d) $(p + q)^D = -\frac{1}{2}p + q$ if and only if $pq = p$;
- (e) $(p + q)^D = p - \frac{1}{2}q$ if and only if $pq = q$.

PROOF. (a) If $pq = qp$, then obviously $(p - q)^3 = p - q$, so it is easy to verify that $(p - q)^D = p - q$.

Conversely, if $(p - q)^D = p - q$, then $(p - q)^3 = p - q$, so $pqp = qpq$. Hence $(pq)(pq)^* = pqp = qpq = (pq)^*(pq)$, so pq is normal. Since $\sigma(pq) \subseteq [0, 1]$, then by the Gelfand representation theorem [17, Theorem 2.1.10], pq is self-adjoint. Hence $pq = qp$.

(b) Sufficiency is clear.

Necessity. If $(p + q)^D = (p + q)$, then $(p + q)^3 = (p + q)$. By a direct calculation, $pqp + qpq = -2(pq + qp)$, so $pq + qp \leq 0$. Then $p(pq + qp)p \leq 0$, hence $pqp \leq 0$, so $pqp = 0$. Then $pq = 0$, so $pq = qp = 0$.

(c) If $pq = qp$, then it is easy to verify that $(p + q)^D = p + q - (3/2)pq$.
 If $(p + q)^D = p + q - (3/2)pq$, then

$$(p + q)(p + q - \frac{3}{2}pq) = (p + q - \frac{3}{2}pq)(p + q)$$

and, by a direct calculation, $pqp = qpq$. It follows from the proof of (a) that $pq = qp$.

(d) Sufficiency is clear, by (c).

If $(p + q)^D = -\frac{1}{2}p + q$, then

$$(p + q)(-\frac{1}{2}p + q) = (-\frac{1}{2}p + q)(p + q)$$

and, by a direct calculation, $pq = qp$. Since $p + q$ is self-adjoint, then $(p + q)^D = -\frac{1}{2}p + q$ implies that

$$(p + q)(-\frac{1}{2}p + q)(p + q) = p + q.$$

It follows that $2pqp + qp + pq - qpq = 3p$, then $pqp = p$, since $pq = qp$. Hence $pq = pqp = p$. The proof of (e) is similar to that of (d). □

Let $p, q \in \mathcal{A}$ be projections. We denote by $S_{(m,p)}$ and $S_{(n,q)}$ an m -factor product of p and q with first factor p and an n -factor product of p and q with first factor q ; in both cases p and q occur alternately. The result of [6, Corollary 2.3] follows as a special case of the following result.

COROLLARY 9. *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:*

- (a) $S_{(m,p)}^D = S_{(n,q)}^D$ for some $m, n \geq 1$;
- (b) $pq = qp$;
- (c) $S_{(m,p)}^D = S_{(n,q)}^D$ for every $m, n \geq 1$.

PROOF. (a) implies (b). Without loss of generality, we assume that $m \leq n$. Obviously, $S_{(m,p)}^D = S_{(n,q)}^D$ implies that $[(pq)^D]^m = [(qp)^D]^n$. Since $p(pq)^D = (pq)^D = (pq)^D q$, we have

$$[(pq)^D]^m = (pq)^D p(pq)^D \dots p(pq)^D = [(qp)^D]^n,$$

so

$$[(pq)^D]^m p = (pq)^D p(pq)^D \dots p(pq)^D p = [(qp)^D]^n p = [(qp)^D]^n.$$

It follows from $(pq)^D p = (pqp)^D$ that $[(pq)^D]^m = [(pqp)^D]^m$, so $[(pq)^D]^m$ is self-adjoint, then $[(pq)^D]^m = [(qp)^D]^m$. Similarly,

$$[(pqp)^D]^m = [(pq)^D]^m = [(qp)^D]^m = [(qpq)^D]^m,$$

then $(pqp)^D = (qpq)^D$. Therefore,

$$pqp = [(pqp)^D]^D = [(qpq)^D]^D = qpq.$$

As in the proof of Theorem 8(a), $pq = qp$.

(b) implies (c) and (c) implies (a) are clear. □

In the following, we present two formulae for $(p - q)^D$ and $(p + q)^D$ where $p, q \in \mathcal{P}(\mathcal{A})$. As a corollary of formula (a), we obtain [6, Theorem 2.5(3)]. Formula (b) is an improvement of [6, Theorem 2.5(4)] which leads to

$$P[(P + Q)^D - (P - Q)^D](P - Q)^2 = 0,$$

for $P, Q \in \mathcal{P}(\mathcal{H})$.

THEOREM 10. *Let $p, q \in \mathcal{P}(\mathcal{A})$. If $p - q$ is Drazin invertible, then the following statements hold:*

- (a) $(p - q)^D = (p - q)^2[(p - qp)^D - (q - qp)^D]$;
 (b) $p[(p + q)^D - (p - q)^D](p - q) = 0$.

PROOF. (a) By Proposition 5, $p - qp$ and $q - qp$ are Drazin invertible. It is clear that

$$\begin{aligned} & (p - q)^2[(p - qp)^D - (q - qp)^D] \\ &= (p - q)(p - pq + pq - q)[(p - qp)^D - (q - qp)^D] \\ &= (p - q)[p(e - q) - (e - p)q]\{[(e - q)p]^D - [q(e - p)]^D\}. \end{aligned}$$

It follows from Proposition 7(b) that $p(e - q)[q(e - p)]^D = 0$ and $(e - p)q[(e - q)p]^D = 0$, hence

$$\begin{aligned} (p - q)^2[(p - qp)^D - (q - qp)^D] &= (p - q)\{p(e - q)[(e - q)p]^D \\ &\quad + (e - p)q[q(e - p)]^D\}. \end{aligned}$$

Using Proposition 7(c) again,

$$\begin{aligned} & (p - q)^2[(p - qp)^D - (q - qp)^D] \\ &= (p - q)\{[p(e - q)p]^D + [(e - p)q(e - p)]^D\} \\ &= (p - q)\{[p(e - q)p + (e - p)q(e - p)]^D\} \\ &= (p - q)[(p - q)^2]^D \\ &= (p - q)(p - q)^D(p - q)^D \\ &= (p - q)^D. \end{aligned}$$

(b) To show this, we need the following lemmas.

LEMMA 11. *Let $p, q, w \in \mathcal{P}(\mathcal{A})$. If $(p + q)w = 0$, then $pqw = 0$.*

PROOF. $(p + q)w = 0$ implies that $pwp + qwp = 0$ and $(e - p)qw = 0$. Thus $qw = pqw$ and qwp is self-adjoint. Since

$$\sigma(qwp) \setminus \{0\} = \sigma(pqw) \setminus \{0\} = \sigma(qw) \setminus \{0\} = \sigma(qwq) \setminus \{0\} \subseteq [0, 1],$$

then $\sigma(qwp) \subseteq [0, 1]$, so $qwp \geq 0$. Hence $pwp = 0$, then $pw = 0$, so $qw = 0$. It follows that $pqw = 0$.

LEMMA 12 (see [13]). *Let $a \in \mathcal{A}$ be self-adjoint. Then:*

- (a) *a is Moore–Penrose invertible if and only if a is Drazin invertible;*
- (b) *if a is Drazin invertible, then $a^+ = a^D$, and $a^2 a^D = a$.*

The following identity was first obtained by Wedin (see [18]) for matrices and by Harte and Mbekhta (see [10, Theorem 5]) for C^* -algebras.

LEMMA 13. *Let $a, b \in \mathcal{A}$ be Moore–Penrose invertible. Then*

$$b^+ - a^+ = -b^+(b - a)a^+ + (e - b^+b)(b^* - a^*)(a^+)^*a^+ \\ + b^+(b^+)^*(b^* - a^*)(e - aa^+).$$

PROOF OF (b) OF THEOREM 10. By Proposition 5, $p + q$ is Drazin invertible. For convenience, let $b = p + q$ and $a = p - q$. Then by Lemmas 12 and 13,

$$b^D - a^D = b^+ - a^+ \\ = -b^D(2q)a^D + (e - b^D b)(2q)(a^D)^2 + (b^D)^2(2q)(e - aa^D),$$

so

$$(b^D - a^D)a = -b^D(2q)a^D a + (e - b^D b)(2q)a^D,$$

and hence

$$p[(p + q)^D - (p - q)^D](p - q) \\ = p[2q(p - q)^D - 2(p + q)^D q(p - q)^D(p - q) \\ - 2(p + q)^D(p + q)q(p - q)^D].$$

Since

$$(p + q)(qp + pq) = (qp + pq)(p + q),$$

it follows from [12, Theorem 4.4] that

$$(p + q)^D(qp + pq) = (qp + pq)(p + q)^D.$$

Therefore,

$$p[(p + q)^D - (p - q)^D](p - q) \\ = 2p\{q(p - q)^D - (p + q)^D[q(p - q) + (p + q)q](p - q)^D\} \\ = 2p[q(p - q)^D - (p + q)^D(qp + pq)(p - q)^D] \\ = 2[pq(p - q)^D - p(qp + pq)(p + q)^D(p - q)^D] \\ = 2[pq(p - q)^D - p(qp + pq)(p + q)^D(p - q)^D] \\ = 2[pq(p - q)^D - pq(p + q)(p + q)^D(p - q)^D] \\ = 2[pq\{e - (p + q)(p + q)^D\}(p - q)^D].$$

Note that

$$e - (p + q)(p + q)^D \in \mathcal{P}(\mathcal{A})$$

and

$$(p + q)[e - (p + q)(p + q)^D] = 0;$$

then, by Lemma 11,

$$pq[e - (p + q)(p + q)^D] = 0,$$

so

$$p[(p + q)^D - (p - q)^D](p - q) = 0. \quad \square$$

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