

Integral representation of hydraulic permeability

Chuan Bi 

Department of Psychiatry, University of Maryland School of Medicine,
Baltimore, MD 21201, USA (chuan.bi@som.umaryland.edu)

Miao-Jung Yvonne Ou  and Shangyou Zhang

Department of Mathematical Sciences, University of Delaware, Newark,
DE 19716, USA (mou@udel.edu, szhang@udel.edu)

(Received 4 August 2021; accepted 25 March 2022)

In this paper, we show that the permeability of a porous material (Tartar (1980)) and that of a bubbly fluid (Lipton and Avellaneda. *Proc. R. Soc. Edinburgh Sect. A: Math.* **114** (1990), 71–79) are limiting cases of the complexified version of the two-fluid models posed in Lipton and Avellaneda (*Proc. R. Soc. Edinburgh Sect. A: Math.* **114** (1990), 71–79). We assume the viscosity of the inclusion fluid is $z\mu_1$ and the viscosity of the hosting fluid is $\mu_1 \in \mathbb{R}^+$, $z \in \mathbb{C}$. The proof is carried out by the construction of solutions for large $|z|$ and small $|z|$ with an iteration process similar to the one used in Bruno and Leo (*Arch. Ration. Mech. Anal.* **121** (1993), 303–338) and Golden and Papanicolaou (*Commun. Math. Phys.* **90** (1983), 473–491) and the analytic continuation. Moreover, we also show that for a fixed microstructure, the permeabilities of these three cases share the same integral representation formula (3.17) with different values of contrast parameter $s := 1/(z - 1)$, as long as s is outside the interval $\left[-\frac{2E_2^2}{1+2E_2^2}, -\frac{1}{1+2E_1^2}\right]$, where the positive constants E_1 and E_2 are the extension constants that depend only on the geometry of the periodic pore space of the material.

Keywords: Hydraulic permeability; Stokes equations; composite materials; integral representation formula; Stieltjes class

2010 Mathematics subject classification: 35Q35, 35Q70

1. Introduction

Darcy's law, which was first proposed by Darcy in 1856 [19] based on experimental observation of water flowing through beds of sand, describes the relationship between the spontaneous flow discharge rate of steady state through a porous medium, the viscosity of the fluid and the pressure drop over a distance. Later, theoretical/mathematical derivations of Darcy's law were presented in many works, e.g. [5, 25, 28, 32, 36, 38], just to name a few.

In the setting of a periodic pore microstructure, as the period goes to zero, the convergence to the Darcy's law of the Stokes system with no-slip boundary condition posed on the boundary of the pore space was proved by Tartar using

the energy method [40]. Allaire implemented the two-scale convergence method introduced by Nguetseng [33] to derive the Darcy's law and show the convergence [3, 4]. Prior to the proof of Darcy's law in the 1980s, Brinkman [14] studied the viscous force exerted by a flowing flow on a dense swarm of particles by adding a diffusion term to the Darcy's law so as to take into account the transitional flow between boundaries. Brinkman's method was further studied in [30, 37, 39]. In the case of a porous material where the solid region is much smaller than the fluid part, Levy [26] and Sanchez-Palencia [38] proposed the same form of Darcy's law but with a different representation of the permeability tensor \mathbf{K} . Later on, Allaire [2] showed the continuity of the transition between the two forms of Darcy's laws by considering various ratios between the size of the solid inclusion and the size of the separation. Moreover, instead of considering the porous materials as a periodic structured material, Beliaev [9] considered the porous materials as a random and stochastically homogeneous material and deduced the same Darcy's law. Allaire [1] generalized the homogenization to handle the more realistic micro-geometries of the porous medium where both the solid part and the fluid part are connected. Furthermore, in terms of the fluid–solid interface conditions, a slip boundary condition is considered by Allaire in [1, 17]. In the case of the fluid flow through a porous medium subject to a time-harmonic pressure gradient, the permeability depends on the frequency and is referred to as the dynamic permeability. The theory of dynamic permeability is established [5, 8, 12, 22] and further developed by Ou [35].

The goal of this paper is to study how the permeability tensor derived from the homogenization approach for porous materials [40, 41] depends on the microstructure of the pore space. Details of this will be presented in § 1.1.

The main tool we use will be the integral representation formula (IRF) for composite materials. Composite materials are materials made from more than one constituent material with different physical or chemical properties. The effective properties of composites, such as elasticity, conductivity and permeability are of great interest in different application fields. Homogenization theory for composite materials has been extensively studied in [10, 31, 34]. Mathematically, for a two-component composite material, the microstructural information is carried into the analytical formulation of the effective properties of the composite. Bergman pioneered the study of analyticity of the effective dielectric constant [11], and in terms of integral representation of effective material properties, a rigorous basis of integral representation of the effective conductivity is established by Golden and Papanicolaou [21], the effective elastic constants by Kantor and Bergman [23], the effective diffusivity in convection-enhanced diffusion was derived by Avellaneda and Majda [6, 7]. Further enlargements of the domain of analyticity of the IRF of elasticity tensor to the case where one phase is a void or a hard inclusion is studied by Bruno and Leo [15, 16].

Unlike the problem set-up for calculating effective material properties such as effective conductivity, elasticity and diffusivity, where the physical property of interest is well defined both in the micro-scale and the macro(homogenized)-scale, the permeability of porous material is by definition an effective property and hence

it makes sense only in the macro-scale. To overcome this difficulty, we consider a porous material as the limit case of a two-fluid mixture.

Specifically, we will start with the two-fluid mixture problem studied in [29], where the effective property is called the *self-permeability*. We will derive the IRF for the self-permeability and show that the permeability for a porous material is equal to the limit of the self-permeability when the viscosity of one phase becomes infinite. Similar to the hard/soft inclusion case studied in [15, 16], we will extend the domain of analyticity of the IRF to ∞ and to 0 by an iterative process. As a result, the IRF derived here is valid for porous materials with a solid skeleton as well as for fluid-bubble mixtures. Hence it provides a theoretical connection between the permeability defined in [40] and the self-permeability for the bubbly fluid studied in [29] and any mixture in between these two limiting cases.

The paper is organized as follows. The permeability of a porous material is defined in § 1.1. Section 2 starts with the definition of the *self-permeability* \mathbf{K} of a two-fluid mixture and the corresponding cell problem, followed by an analysis of the cell problem and the construction of the solution in the vicinity of the two limiting cases of $z = \infty$ and $z = 0$. In § 3, the IRF of \mathbf{K} is obtained by applying the theory of matrix-valued Stieltjes functions. In this section, the spectral representation of \mathbf{K} is also derived. The relationships between the moments of the measure in the IRF and the geometry of the pore space are derived by comparing these two representations. Section 4 presents the numerical solutions of the cell problem of a special pore structure, which validate the theoretical results given in § 3.

Einstein summation convention is applied unless stated otherwise.

1.1. Definition of permeability from homogenization

Following the convention of homogenization, the space coordinates for the cell problem in the open unit cell $Q = (0, 1)^n$ for $n = 2, 3$, are denoted by $\mathbf{y} = (y_1, y_2, y_3)$. Let Ω be a smooth bounded open set and Q an open unit cube made of two open sets Q_1, Q_2 and the interface $\Gamma = \text{cl}(Q_1) \cap \text{cl}(Q_2)$ with $\text{cl}(A)$ being the closure of a set A . Moreover, \widetilde{Q}_i denotes the Q -periodic extension of $Q_i, i = 1, 2$. Following [1], we assume that (1) Q_1 and Q_2 have strictly positive measures in $\text{cl}(Q)$. (2) The set \widetilde{Q}_i is open with C^1 boundary and is locally located on one side of its boundary, $i = 1, 2$, and \widetilde{Q}_1 is connected. (3) Q_1 is connected with a Lipschitz boundary. In addition, we consider the case of inclusion, i.e. $Q_2 \cap \partial Q = \emptyset$.

Consider $\epsilon > 0$ much smaller than the size of Ω and ϵQ -periodically extend ϵQ_1 in the entire space. Ω_ϵ denotes the intersection of Ω and this ϵQ -periodically extended structure. In [40], the permeability is derived from the Stokes equation in Ω_ϵ , which reads: find $\mathbf{u}^\epsilon \in H_0^1(\Omega_\epsilon)^n$ and $p^\epsilon \in L^2(\Omega_\epsilon)/\mathbb{R}$ such that

$$\begin{cases} -\mu \Delta \mathbf{u}^\epsilon + \nabla p^\epsilon = \mathbf{f} & \text{in } \Omega_\epsilon. \\ \text{div } \mathbf{u}^\epsilon = 0 & \text{in } \Omega_\epsilon \end{cases} \tag{1.1}$$

where $\mathbf{f} \in L^2(\Omega)$ is independent of ϵ and the viscosity μ is a constant (μ is set to 1 in [40, 41]). See figure 1 for an example of the unit cube. Note that the superscript ϵ is used to signify that the solutions \mathbf{u}^ϵ and p^ϵ depend on ϵ . To be able to prove the convergence of $(\mathbf{u}^\epsilon, p^\epsilon)$ as $\epsilon \rightarrow 0$, it is necessary to extend these solutions from Ω_ϵ to Ω so they are defined in the same spatial domain. In [40, 41], \mathbf{u}^ϵ was extended by

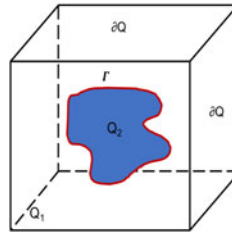


Figure 1. Sample illustration of a periodic cell.

zero and p^ϵ by a properly defined extension operator with their extensions denoted by $\hat{\mathbf{u}}^\epsilon$ and \hat{p}^ϵ , respectively. As $\epsilon \rightarrow 0$,

$$\frac{\hat{\mathbf{u}}^\epsilon}{\epsilon^2} \rightharpoonup \mathbf{U} \text{ weakly in } L^2(\Omega)^n, \operatorname{div} \mathbf{U} = 0, \mathbf{U} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \text{ and } \hat{p}^\epsilon \rightarrow p \text{ in } L^2(\Omega)/\mathbb{R}$$

and the limit functions satisfy the following Darcy’s law [40]

$$\mathbf{U} = \frac{\mathbf{K}^{(D)}}{\mu} (\mathbf{f} - \nabla p) \tag{1.2}$$

where the permeability tensor $\mathbf{K}^{(D)}$ is defined as

$$K_{ij}^{(D)} = \int_{Q_1} \mathbf{u}_D^j \cdot \mathbf{e}_i \, dy, \quad i, j = 1, \dots, n \tag{1.3}$$

with \mathbf{e}_i denoting the unit vector in the i -th direction and \mathbf{u}_D^j the unique solution of the following boundary value problem

$$\begin{cases} \mu \Delta_{\mathbf{y}} \mathbf{u}_D^j - \nabla_{\mathbf{y}} p^j = -\mathbf{e}_j & \text{in } Q_1 \\ \operatorname{div}_{\mathbf{y}} \mathbf{u}_D^j = 0 & \text{in } Q_1 \\ \mathbf{u}_D^j = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{1.4}$$

in the space $\mathring{H}(Q_1) := \{\mathbf{v} : \mathbf{v} \in H^1(Q_1)^n \mid \operatorname{div}_{\mathbf{y}} \mathbf{v} = 0, \mathbf{v}|_\Gamma = \mathbf{0}, Q\text{-periodic}\}$. Note that the superscript j of \mathbf{u} and p signifies the solutions corresponding to the force term \mathbf{e}_j .

Since μ is set to 1 in [40, 41], the permeability \mathbf{K} presented there is related to $\mathbf{K}^{(D)}$ by $\mathbf{K}^{(D)} = \mathbf{K}/\mu$. For future analysis, we will derive here the quadratic form representation of the permeability. We start by observing that for incompressible fluid, we have

$$\Delta \mathbf{u} = \operatorname{div}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$$

Therefore, (1.3) can be expressed as

$$K_{ij}^{(D)} = \int_{Q_1} \mathbf{u}_D^j \cdot (\nabla_{\mathbf{y}} p^i - \mu \Delta_{\mathbf{y}} \mathbf{u}_D^i) \, dy = \int_{Q_1} \mu \nabla_{\mathbf{y}} \mathbf{u}_D^j : (\nabla_{\mathbf{y}} \mathbf{u}_D^i + \nabla_{\mathbf{y}}^T \mathbf{u}_D^i) \, dy \tag{1.5}$$

after applying the divergence theorem, the periodicity of \mathbf{u} , p and the no-slip conditions on Γ . Here we have used the Frobenius inner product of matrices

$\mathbf{A} : \mathbf{B} = \sum_{i,j=1} A_{ij} B_{ij}$. In terms of the usual notion of the symmetric part and the antisymmetric part of vector field $\nabla \mathbf{u}$

$$e(\mathbf{u}) := \frac{1}{2} (\nabla \mathbf{u} + \nabla^T \mathbf{u}), \quad \tilde{e}(\mathbf{u}) := \frac{1}{2} (\nabla \mathbf{u} - \nabla^T \mathbf{u}), \tag{1.6}$$

the right-hand side of equation (1.5) becomes $\int_{Q_1} 2\mu(e(\mathbf{u}_D^j) + \tilde{e}(\mathbf{u}_D^j)) : e(\mathbf{u}_D^i) \, d\mathbf{y} = \int_{Q_1} 2\mu e(\mathbf{u}_D^j) : e(\mathbf{u}_D^i) \, d\mathbf{y}$ because the Frobenius product of a symmetric matrix and an antisymmetric matrix must be 0. Therefore we have the quadratic form of permeability tensor $\mathbf{K}^{(D)}$

$$K_{ij}^{(D)} = \int_{Q_1} 2\mu e(\mathbf{u}_D^j) : e(\mathbf{u}_D^i) \, d\mathbf{y}. \tag{1.7}$$

2. Approximation of flow in porous medium by a two-phase Stokes flow

In this section, we consider the system for porous materials (1.4) as one of the limiting cases of the two-fluid problem described below, which is the same as the one studied in [29] with the exception that the fluid viscosity here can be complex-valued. It is easy to check that the homogenization process in [29] stays valid after making small modifications to accommodate the complex valued viscosity described below.

Let Ω , Q and ϵ be the same as in § 1.1. Q_2 is still the inclusion in the periodic cell. Consider the ϵQ -periodic extension of ϵQ_1 (ϵQ_2) and denote by $\Omega_{1\epsilon}$ ($\Omega_{2\epsilon}$) its intersection with Ω . We note that $\Omega_{1\epsilon}$ (region of the hosting fluid) is the same as Ω_ϵ in the previous section. Suppose $\Omega_{1\epsilon}$ is occupied by fluid with viscosity $\mu_1 > 0$ and Ω_2 by fluid with viscosity $z\mu_1$ with $z \in \mathbb{C}$. The interface $\tilde{\Gamma} = \partial\Omega_{1\epsilon} \cap \partial\Omega_{2\epsilon}$ is such that $\Omega_{1\epsilon} \cup \tilde{\Gamma} \cup \Omega_{2\epsilon} = \Omega$. For the ease of notation, we define the stress tensor $\boldsymbol{\tau}(\mathbf{u}, \mu)$ of a fluid with viscosity μ , velocity field \mathbf{u} and pressure field p as

$$\boldsymbol{\tau}(\mathbf{u}, p, \mu) = 2\mu e(\mathbf{u}) - p\mathbf{I}, \quad \mathbf{I} \text{ is the identity matrix.} \tag{2.1}$$

Let χ_i be the characteristic function of $\Omega_{\epsilon i}$, $i = 1, 2$. Consider the viscosity function

$$\xi^\epsilon(\mathbf{x}; z) = (\chi_2(\mathbf{x})z\mu_1 + \chi_1(\mathbf{x})\mu_1), \quad z \in \mathbb{C}. \tag{2.2}$$

The two-fluid problem is given by the following Stokes system

$$\begin{cases} \operatorname{div} (2\xi^\epsilon(\mathbf{x}; z)e(\mathbf{u}^\epsilon)) - \nabla p^\epsilon = -\mathbf{f} & \text{in } \Omega \setminus \tilde{\Gamma} \\ \operatorname{div} \mathbf{u}^\epsilon = 0 & \text{in } \Omega \\ \mathbf{u}^\epsilon = \mathbf{0} & \text{on } \partial\Omega \\ \llbracket \mathbf{u}^\epsilon \rrbracket = 0, \mathbf{u}^\epsilon \cdot \mathbf{n} = 0 & \text{on } \tilde{\Gamma} \\ \llbracket \boldsymbol{\pi} \rrbracket \cdot \mathbf{n} = (\llbracket \boldsymbol{\pi} \cdot \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} \equiv \llbracket \boldsymbol{\pi} \cdot \mathbf{n} \rrbracket - \mathbf{n} \times \mathbf{n} \times \llbracket \boldsymbol{\pi} \cdot \mathbf{n} \rrbracket & \text{on } \tilde{\Gamma} \end{cases} \tag{2.3}$$

where $\boldsymbol{\pi} = \boldsymbol{\tau}(\mathbf{u}^\epsilon, p^\epsilon, \xi^\epsilon)$, \mathbf{f} is a square integrable momentum source independent of ϵ , $\llbracket \cdot \rrbracket$ the jump across the interface $\tilde{\Gamma}$ and \mathbf{n} is the outward unit normal of $\partial\Omega_{2\epsilon}$. The second jump condition in (2.3) means the traction can only jump in the normal direction. Also note that the superscript ϵ is used to signify that the solutions \mathbf{u}^ϵ and p^ϵ depend on ϵ .

It is shown in [29] that as $\epsilon \rightarrow 0$, \mathbf{u}^ϵ and the properly normalized p^ϵ , which is denoted by \hat{p}^ϵ , converge as follows

$$\frac{\mathbf{u}^\epsilon}{\epsilon^2} \rightarrow \mathbf{u}^0 \quad \text{weakly in } L^2(\Omega)^n, \quad \hat{p}^\epsilon \rightarrow P \quad \text{strongly in } L^2(\Omega)/\mathbb{R}$$

where \mathbf{u}^0 and P satisfy the homogenized system:

$$\begin{cases} \mathbf{u}^0 = -\mathbf{K}(\nabla P - \mathbf{f}) & \text{in } \Omega \\ \operatorname{div} \mathbf{u}^0 = 0 & \text{in } \Omega \end{cases} \tag{2.4}$$

where the components of \mathbf{K} , which is referred to as the *self-permeability* in [29], is defined as

$$K_{ij}(z) := \int_Q \mathbf{u}^j \cdot \mathbf{e}_i \, d\mathbf{y}, \quad i, j = 1, \dots, n \tag{2.5}$$

with \mathbf{u}^i being the unique solution to the cell problem posed in the function space $H(Q)$, which is defined in (2.7),

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\boldsymbol{\mu}(\mathbf{y}; z)e(\mathbf{u}^i) - p^i \mathbf{I}) + \mathbf{e}_i = \mathbf{0} & \text{in } Q_1 \cup Q_2 \\ \llbracket \boldsymbol{\pi} \rrbracket \cdot \mathbf{n} = (\llbracket \boldsymbol{\pi} \cdot \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} & \text{on } \Gamma \end{cases} \tag{2.6}$$

where $\boldsymbol{\mu}(\mathbf{y}; z) = \mu_1 \chi_1(\mathbf{y}) + z \mu_2 \chi_2(\mathbf{y})$ with χ_m being the characteristic functions of Q_m , $m = 1, 2$, and $\boldsymbol{\pi} = \boldsymbol{\tau}(\mathbf{u}^k, p^k, \boldsymbol{\mu})$, cf. (2.1). Note that the superscript i is used to signify that \mathbf{u}^i and p^i are solutions to the cell problem (2.6) with the force term $-\mathbf{e}_i$, $i = 1, \dots, n$.

2.1. Function spaces

Let $\mathcal{R}(Q_2)$ denote the space of rigid body displacements in Q_2 , i.e. $\mathbf{u} = \mathbf{A}\mathbf{y} + \mathbf{b}$ with constant skew-symmetric matrix \mathbf{A} and constant vector \mathbf{b} in Q_2 . We start with the space of admissible functions for the velocity

$$\begin{aligned} H(Q) := & \left\{ \mathbf{v} : \mathbf{v} \in H^1(Q_1 \cup Q_2)^n \mid \operatorname{div}_{\mathbf{y}} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ in } H^{-(1/2)}(\Gamma), \right. \\ & \left. \llbracket \mathbf{v} \rrbracket_\Gamma = \mathbf{0}, (\mathbf{v}, \eta)_{H^1(Q_2)} = 0, \forall \eta \in \mathcal{R}(Q_2), \mathbf{v} \text{ is } Q\text{-periodic} \right\} \end{aligned} \tag{2.7}$$

where \mathbf{n} is the outward unit normal of ∂Q_2 . $H(Q)$ is endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_Q = \int_Q 2\mu_1 e(\mathbf{u}) : \overline{e(\mathbf{v})} \, d\mathbf{y}. \tag{2.8}$$

The induced norm is denoted by $\|\mathbf{u}\|_Q^2 := (\mathbf{u}, \mathbf{u})_Q$. Note that we have $H(Q) \cap \mathcal{R}(Q) = \{\mathbf{0}\}$ because $\mathbf{A} = \mathbf{0}$ due to the Q -periodicity and $\mathbf{u} \cdot \mathbf{n} = 0$ implies $\mathbf{b} = \mathbf{0}$. We observe that if $\mathbf{u} \in H(Q)$ then $\mathbf{u} \in H^1(Q)^n$ by the following argument. Obviously, $\mathbf{u} \in L^2(Q)^n$. To prove $\partial u_i / \partial y_j \in L^2(Q)$ for $i, j = 1, 2, 3$, let ϕ be any C^∞ test

function compactly supported in Q and h be the i -th component u_i for any i . Then

$$\int_Q h \nabla \phi \, d\mathbf{y} = - \left(\int_{Q_1 \cap \text{Supp}(\phi)} \phi \nabla h \, d\mathbf{y} + \int_{Q_2 \cap \text{Supp}(\phi)} \phi \nabla h \, d\mathbf{y} \right)$$

here we used $[[h]] = \mathbf{0}$. Now we can define a candidate function \mathbf{g} such that

$$\mathbf{g}|_{Q_i} := \nabla h|_{Q_i}, \quad i = 1, 2 \tag{2.9}$$

then clearly $\mathbf{g} \in L^2(Q)^n$ and $\langle h, \nabla \phi \rangle = - \langle \mathbf{g}, \phi \rangle$, where $\langle \cdot \rangle$ denotes the usual L^2 inner product. Therefore $h \in H^1(Q)$ and hence $u_i \in H^1(Q)$, $i = 1, \dots, n$.

Next, we show that $\|\cdot\|_Q$ is equivalent to the usual H^1 norm, i.e., there exist constants B_1 and B_2 such that

$$B_1 \|\mathbf{u}\|_{H^1(Q)} \leq \|\mathbf{u}\|_Q \leq B_2 \|\mathbf{u}\|_{H^1(Q)} \tag{2.10}$$

Because $H^1(Q) \cap \mathcal{R}(Q) = \{0\}$, by theorem 2.5 in [34], there exists a Korn's constant C_1 such that

$$C_1 \|\mathbf{u}\|_{H^1(Q)} \leq \frac{1}{\sqrt{2\mu_1}} \|\mathbf{u}\|_Q \tag{2.11}$$

where C_1 depends only on Q . Therefore, we can take $B_1 = \sqrt{2\mu_1}C_1$. To emphasize the dependence on Q , we will write it as $B_1(Q)$. On the other hand, according to the orthogonal decomposition that $\nabla \mathbf{u} = e(\mathbf{u}) + \tilde{e}(\mathbf{u})$, see (1.6),

$$\|\mathbf{u}\|_{H^1(Q)}^2 \geq \|\nabla \mathbf{u}\|_{L^2(Q)}^2 = \|e(\mathbf{u})\|_{L^2(Q)}^2 + \|\tilde{e}(\mathbf{u})\|^2 \geq \|e(\mathbf{u})\|_{L^2(Q)}^2 = \frac{1}{2\mu_1} \|\mathbf{u}\|_Q^2$$

therefore $B_2 = \sqrt{2\mu_1}$. The reason for introducing the $H(Q)$ -norm is that the self-permeability in (2.5) can be represented in terms of the inner product. More specifically, using (2.5), (2.6) and the fact that $\bar{\mathbf{e}}_i = \mathbf{e}_i$, by a calculation similar to (1.5) and taking into account the interface condition $\mathbf{u} \cdot \mathbf{n} = 0$ and the jump conditions in (2.6), (2.5) can be expressed in the following form

$$K_{ij}(z) = \int_Q 2\mu(\mathbf{y}; \bar{z}) \overline{e(\mathbf{u}^i(z))} : e(\mathbf{u}^j(z)) \, d\mathbf{y} \tag{2.12}$$

and its conjugate transpose $\mathbf{K}^* := \overline{\mathbf{K}^T}$ is

$$(K^*)_{ij}(z) = \int_Q 2\mu(\mathbf{y}; z) e(\mathbf{u}^j(z)) : \overline{e(\mathbf{u}^i(z))} \, d\mathbf{y} \tag{2.13}$$

2.2. Weak solution of the cell problem (2.6)

The weak formulation of the cell problem (2.6) is

$$\int_{Q_1 \cup Q_2} 2\mu(\mathbf{y}; z)e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y} = \int_{Q_1 \cup Q_2} \mathbf{e}_k \cdot \overline{\mathbf{v}} \, d\mathbf{y}, \quad \forall \mathbf{v} \in H(Q) \tag{2.14}$$

From this, we see that the solutions satisfy the following symmetry

$$\mathbf{u}^k(\mathbf{y}; \bar{z}) = \overline{\mathbf{u}^k(\mathbf{y}; z)}$$

Define the sesquilinear form on $H(Q)$

$$a(\mathbf{u}, \mathbf{v}) = \int_{Q_1 \cup Q_2} 2\mu(\mathbf{y}; z)e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y} \tag{2.15}$$

It is clear that $a(\mathbf{u}, \mathbf{v})$ is bounded in $H(Q)$. To check the coercivity, assume $\mathbf{u}^k \neq 0$ and define the parameter

$$\lambda := \frac{\int_Q 2\mu_1 \chi_2 e(\mathbf{u}^k) : \overline{e(\mathbf{u}^k)} \, d\mathbf{y}}{\int_Q 2\mu_1 e(\mathbf{u}^k) : \overline{e(\mathbf{u}^k)} \, d\mathbf{y}} \tag{2.16}$$

then $0 \leq \lambda \leq 1$. We note that

$$\frac{a(\mathbf{u}^k, \mathbf{u}^k)}{\int_Q 2\mu_1 e(\mathbf{u}^k) : \overline{e(\mathbf{u}^k)} \, d\mathbf{y}} = \lambda z + (1 - \lambda) \cdot 1 \tag{2.17}$$

and hence as long as 0 is not on the line segment joining z and 1, there exist $\alpha(z) := \min_{0 \leq \lambda \leq 1} |\lambda z + 1 - \lambda| > 0$ such that

$$|a(\mathbf{u}^k, \mathbf{u}^k)| \geq \alpha(z) \int_Q 2\mu_1 e(\mathbf{u}^k) : \overline{e(\mathbf{u}^k)} \, d\mathbf{y} = \alpha \|\mathbf{u}^k\|_Q^2 \tag{2.18}$$

Therefore for $z \in \mathbb{C} \setminus \{\Re z \leq 0\}$, by the Lax–Milgram lemma [13, chapter 2], there exists a unique weak solution $\mathbf{u}^k \in H(Q)$ to the cell problem (2.6) and with the solution \mathbf{u}^k , we can construct $p^k \in L^2(Q)/\mathbb{C}$.

Since $\alpha(z)$ is a continuous function in z , the coercivity of the sesquilinear form can be applied to conclude that \mathbf{u}^k is analytic in z and its m -th derivative, $m \geq 1$, satisfies the following recursive equation

$$\begin{aligned} & \int_{Q_1 \cup Q_2} 2\mu(\mathbf{y}; z)e\left(\frac{d^m \mathbf{u}^k}{dz^m}\right) : \overline{e(\mathbf{v})} \, d\mathbf{y} \\ &= - \int_{Q_2} 2m\mu_1 e\left(\frac{d^{m-1} \mathbf{u}^k}{dz^{m-1}}\right) : \overline{e(\mathbf{v})} \, d\mathbf{y}, \quad \forall \mathbf{v} \in H(Q) \end{aligned} \tag{2.19}$$

As a result, $\mathbf{K}(z)$ is also analytic for $z \in \mathbb{C} \setminus \{\Re z \leq 0\}$. To relate the two-fluid problem with $\mathbf{K}^{(D)}$, we adapt the method used in [16] to study the behaviour of $\mathbf{K}(z)$ near $z = \infty$ in the following section.

2.3. Analyticity of the solution for large $|z|$

Let $w := \frac{1}{z}$ and consider Q -periodic solution in the series form near $w = 0$

$$\mathbf{u}_\infty(\mathbf{y}; \mathbf{e}, w) := \sum_{k=0}^\infty \mathbf{u}_k(\mathbf{y}; \mathbf{e})w^k \text{ and } p_\infty(\mathbf{y}; \mathbf{e}, w) := \sum_{k=0}^\infty p_k(\mathbf{y}; \mathbf{e})w^k \tag{2.20}$$

where the \mathbf{e} is an arbitrary constant unit vector. To set up the notation, we denote the restrictions of \mathbf{u}_k, p_k in Q_2 (inclusion) and Q_1 as $\mathbf{u}_k^{in}, p_k^{in}$ and $\mathbf{u}_k^{out}, p_k^{out}$ respectively and define

$$\mathbf{u}_\infty^{in}(\mathbf{y}; \mathbf{e}, w) := \sum_{k=0}^\infty \mathbf{u}_k^{in}(\mathbf{y}, \mathbf{e})w^k, \quad \mathbf{u}_\infty^{out}(\mathbf{y}; \mathbf{e}, w) := \sum_{k=0}^\infty \mathbf{u}_k^{out}(\mathbf{y}, \mathbf{e})w^k \tag{2.21}$$

By substituting (2.20) into (2.6) with the viscosity defined in (2.2), taking into account the additional two interface conditions $\mathbf{u} \cdot \mathbf{n} = 0$ and $[[\mathbf{u}]] = 0$, followed by equating terms of the same order with respect to w , we arrive in the following equations in Q_1 :

$$O(w^0): \quad \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_0^{out}) - p_0^{out} \mathbf{I}) = -\mathbf{e} \tag{2.22}$$

$$O(w^k): \quad \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) = \mathbf{0} \text{ for } k \geq 1 \tag{2.23}$$

and in Q_2 :

$$O(w^{-1}): \quad \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_0^{in})) = \mathbf{0} \tag{2.24}$$

$$O(w^0): \quad \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_1^{in}) - p_0^{in} \mathbf{I}) = -\mathbf{e} \tag{2.25}$$

$$O(w^k): \quad \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_{k+1}^{in}) - p_k^{in} \mathbf{I}) = \mathbf{0} \text{ for } k \geq 1 \tag{2.26}$$

and the following interface conditions on Γ

$$O(w^{-1}): \quad 2\mu_1 (e(\mathbf{u}_0^{in}) \cdot \mathbf{n})|_\Gamma = C(\mathbf{y})\mathbf{n} \text{ for some function } C(\mathbf{y}) \tag{2.27}$$

$$O(w^k), k \geq 0: \quad \left((2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}_{k+1}^{in}) - p_k^{in} \mathbf{I}) \right) \mathbf{n} \\ = \left\{ \left[(2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}_k^{in}) - p_k^{in} \mathbf{I}) \right] \mathbf{n} \right\} \cdot \mathbf{n}, \tag{2.28}$$

$$\mathbf{u}_k^{in} \cdot \mathbf{n} = \mathbf{u}_k^{out} \cdot \mathbf{n} = 0 \text{ and } \mathbf{u}_k^{in} = \mathbf{u}_k^{out} \tag{2.29}$$

We introduce the following spaces, $i = 1, 2$

$$H(Q_1) = \{ \mathbf{v} : \mathbf{v} \in H^1(Q_1)^n \mid \operatorname{div}_{\mathbf{y}} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma, Q\text{-periodic} \}$$

$$H(Q_2) = \{ \mathbf{v} : \mathbf{v} \in H^1(Q_2)^n \mid \operatorname{div}_{\mathbf{y}} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma, (\mathbf{v}, \mathcal{R}(Q_2))_{H^1(Q_2)} = 0, Q\text{-periodic} \}$$

$$\dot{H}(Q_i) = \{ \mathbf{v} : \mathbf{v} \in H^1(Q_i)^n \mid \operatorname{div}_{\mathbf{y}} \mathbf{v} = 0, \mathbf{v}|_\Gamma = \mathbf{0}, Q\text{-periodic} \} \subset H(Q_i),$$

$$L(Q_i)/\mathbb{C} = \left\{ p : p \in L^2(Q_i), \int_{Q_i} p(\mathbf{y}) \, d\mathbf{y} = 0, Q\text{-periodic}, \right\}$$

Note that $H(Q_1) \cap \mathcal{R}(Q_1) = \{ \mathbf{0} \}$ because $\partial Q \subset \partial Q_1$. For $H(Q_2)$, the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ implies $H(Q_2) \cap \mathcal{R}(Q_2) = \{ \mathbf{0} \}$ because of the extra condition $(\mathbf{v}, \mathcal{R}(Q_2))_{H^1(Q_2)} = 0$ [34]. Therefore, $H(Q_i)$ and $\dot{H}(Q_i)$ are equipped with inner

product $(\mathbf{u}, \mathbf{v})_{Q_i} = \int_{Q_i} 2\mu_1 e(\mathbf{u}) : \overline{e(\mathbf{v})} \, dy$ and Korn's inequalities are valid in $H(Q_i)$, $i = 1, 2$.

LEMMA 2.1. *Let Q_2 be a connected, open bounded set such that $\partial Q_2 \cap \partial Q = \emptyset$ and ∂Q_2 is in $C^{k,\sigma}$, $k, \sigma \geq 0$, $k + \sigma \geq 2$. For any vector field $\mathbf{u}^{in} \in H(Q_2)$, there exists a unique weak solution $\mathbf{u}^{out}(\mathbf{y}; \mathbf{f}^{out}) \in H(Q_1)$ that satisfies the following system*

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}^{out}) - p^{out} \mathbf{I}) = \mathbf{f}^{out} & \text{in } Q_1 \\ \mathbf{u}^{out} = \mathbf{u}^{in} & \text{on } \Gamma \end{cases} \tag{2.30}$$

where in our context, $\mathbf{f}^{out} = \mathbf{0}$ or $\mathbf{f}^{out} = -\mathbf{e}$, a constant unit vector. Moreover,

$$\|\mathbf{u}^{out}\|_{Q_1} \leq \frac{1}{B_1(Q)} \|\mathbf{f}^{out}\|_{L^2(Q_1)} + 2E_1 \|\mathbf{u}^{in}\|_{Q_2}. \tag{2.31}$$

where the positive constants $B_1(Q_1)$ is defined in (2.10) and $E_1 \geq 1$ depends only on Q_1 and Q_2 .

Proof. To handle the inhomogeneous boundary condition, we proceed as follows. By [24, corollary 3.2], there exists a bounded, divergence free extension $T(\mathbf{u}^{in})$ of \mathbf{u}^{in} to a small neighbourhood O of Q_2 and vanishes at $\partial O \subset Q_1$ such that

$$\|T(\mathbf{u}^{in})\|_Q \leq E_1 \|\mathbf{u}^{in}\|_{Q_2} \tag{2.32}$$

where $E_1 \geq 1$ depends only on Q_1 and Q_2 . Furthermore, the extension $T(\mathbf{u}^{in})$ on O can be extended periodically to \mathbb{R}^n [18] since $T(\mathbf{u}^{in})$ vanishes on ∂O and hence on ∂Q . We denote the restriction $T(\mathbf{u}^{in})|_{Q_1}$ as $\tilde{\mathbf{u}}^{out} \in H(Q_1)$ and $\hat{\mathbf{u}}^{out} := \mathbf{u}^{out} - \tilde{\mathbf{u}}^{out} \in \dot{H}(Q_1)$ and (2.30) becomes

$$\operatorname{div}_{\mathbf{y}} (2\mu_1 e(\hat{\mathbf{u}}^{out}) - p^{out} \mathbf{I}) = \mathbf{f}^{out} - \mu_1 \Delta \tilde{\mathbf{u}}^{out} \quad \text{in } Q_1 \tag{2.33}$$

Consider the variational formulation: find $\hat{\mathbf{u}}^{out} \in \dot{H}(Q_1)$ such that $\forall \Phi \in \dot{H}(Q_1)$,

$$\int_{Q_1} 2\mu_1 e(\hat{\mathbf{u}}^{out}) : \overline{e(\Phi)} \, dy = - \int_{Q_1} \mathbf{f}^{out} \cdot \overline{\Phi} \, dy - \int_{Q_1} 2\mu_1 e(\tilde{\mathbf{u}}^{out}) : \overline{e(\Phi)} \, dy, \tag{2.34}$$

The right-hand side of (2.34) can be bounded as follows

$$\left| \int_{Q_1} \mathbf{f}^{out} \cdot \overline{\Phi} \, dy + \int_{Q_1} 2\mu_1 e(\tilde{\mathbf{u}}^{out}) : \overline{e(\Phi)} \, dy \right| \leq \left(\frac{\|\mathbf{f}^{out}\|_{L^2(Q_1)}}{B_1(Q_1)} + \|\tilde{\mathbf{u}}^{out}\|_{Q_1} \right) \|\Phi\|_{Q_1}$$

The sesquilinear form $\int_{Q_1} 2\mu_1 e(\hat{\mathbf{u}}^{out}) : \overline{e(\Phi)} \, dy$ is clearly bounded and coercive with constant 1. Hence by the Lax–Milgram lemma there exists a unique weak solution $\hat{\mathbf{u}}^{out} \in \dot{H}(Q_1)$ to (2.33) such that $\|\hat{\mathbf{u}}^{out}\|_{Q_1} \leq ((1/B_1(Q_1)) \|\mathbf{f}^{out}\|_{L^2(Q_1)} + \|\tilde{\mathbf{u}}^{out}\|_{Q_1})$. In terms of $\hat{\mathbf{u}}^{out}$, \mathbf{u}^{out} can be expressed as $\mathbf{u}^{out} = \tilde{\mathbf{u}}^{out} + \hat{\mathbf{u}}^{out}$ and

satisfies the estimate

$$\|\mathbf{u}^{out}\|_{Q_1} \leq \|\mathring{\mathbf{u}}^{out}\|_{Q_1} + \|\tilde{\mathbf{u}}^{out}\|_{Q_1} \leq \frac{1}{B_1(Q_1)} \|\mathbf{f}^{out}\|_{L^2(Q_1)} + 2E_1 \|\mathbf{u}^{in}\|_{Q_2} \tag{2.35}$$

To show the solution \mathbf{u}^{out} is unique, suppose \mathbf{u}_1^{out} and \mathbf{u}_2^{out} both solve (2.30) then the difference $\mathbf{w}^{diff} = \mathbf{u}_1^{out} - \mathbf{u}_2^{out} \in \mathring{H}(Q_1)$ must satisfy

$$\int_{Q_1} 2\mu_1 e(\mathbf{w}^{diff}) : \overline{e(\Phi)} \, d\mathbf{y} = 0, \quad \forall \Phi \in \mathring{H}(Q_1)$$

Hence $\mathbf{w}^{diff} = 0$ in Q_1 because $\mathbf{w} \in \mathring{H}(Q_1)$. We note the two special cases:

$$\|\mathbf{u}^{out}\|_{Q_1} \leq 2E_1 \|\mathbf{u}^{in}\|_{Q_2} \text{ for } \mathbf{f}^{out} = \mathbf{0} \tag{2.36}$$

$$\|\mathbf{u}^{out}\|_{Q_1} \leq \frac{1}{B_1(Q_1)} \sqrt{|Q_1|} + 2E_1 \|\mathbf{u}^{in}\|_{Q_2} \text{ for } \mathbf{f}^{out} = -\mathbf{e}_j, \quad j = 1, \dots, n, \tag{2.37}$$

where $|Q_1|$ is the volume of Q_1 . □

LEMMA 2.2. *Let Q_2 satisfy the same assumptions as those in lemma 2.1. For any pair of $(\mathbf{u}^{out}, p^{out}) \in H(Q_1) \times L^2(Q_1)/\mathbb{C}$ that satisfies (2.30), there exists a unique vector $\mathbf{u}^{in}(\mathbf{y}; \mathbf{f}^{in}) \in H(Q_2)$ that satisfies the Stokes equation with continuity of tangential traction on Γ*

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}^{in}) - p^{in} \mathbf{I}) = \mathbf{f}^{in} & \text{in } Q_2, \\ \mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}^{out}) - p^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}^{in}) - p^{in} \mathbf{I})) \cdot \mathbf{n}] = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{2.38}$$

where in our context, $\mathbf{f}^{in} = \mathbf{0}$ or $\mathbf{f}^{in} = -\mathbf{e}$. Moreover,

$$\|\mathbf{u}^{in}\|_{Q_2} \leq \frac{1}{B_1(Q_2)} \|\mathbf{f}^{in}\|_{L^2(Q_2)} + \frac{E_1(Q_1)}{B_1(Q_1)} \|\mathbf{f}^{out}\|_{L^2(Q_1)} + E_1(Q_1) \|\mathbf{u}^{out}\|_{Q_1}$$

and $E_1(Q_1)$, $B_1(Q_1)$ and $B_1(Q_2)$ depend only on Q and Γ .

Proof. Take $\Phi \in H(Q_2)$, the variation formulation for the PDE is

$$\int_{Q_2} (\operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}^{in}) - p^{in} \mathbf{I})) \cdot \bar{\Phi} \, d\mathbf{y} = \int_{Q_2} \mathbf{f}^{in} \cdot \bar{\Phi} \, d\mathbf{y}.$$

For the ease of notation, let $\boldsymbol{\pi}^{in} := \boldsymbol{\tau}(\mathbf{u}^{in}, \mu_1, p^{in})$ and $\boldsymbol{\pi}^{out} := \boldsymbol{\tau}(\mathbf{u}^{out}, \mu_1, p^{out})$ where the stress function $\boldsymbol{\tau}$ is defined in (2.1). Applying integration by parts on the left-hand side, followed by an application of the divergence theorem leads to

$$-\int_{\Gamma} (\boldsymbol{\pi}^{in} \cdot \bar{\Phi}) \cdot \mathbf{n} \, dS - \int_{Q_2} 2\mu_1 e(\mathbf{u}^{in}) : \overline{e(\Phi)} \, d\mathbf{y} = \int_{Q_2} \mathbf{f}^{in} \cdot \bar{\Phi} \, d\mathbf{y} \tag{2.39}$$

Let \mathbf{t} denote the unit vector in the tangent plane such that $\mathbf{t} \cdot \mathbf{n} = 0$. The conditions on Γ in (2.38) imply

$$\begin{cases} \Phi \cdot \mathbf{n} = 0 \Rightarrow \Phi = d(\mathbf{y})\mathbf{t} \text{ for some function } d(\mathbf{y}), \\ \mathbf{n} \times \mathbf{n} \times [(\boldsymbol{\pi}^{out} - \boldsymbol{\pi}^{in}) \cdot \mathbf{n}] = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{out} - \boldsymbol{\pi}^{in}) \cdot \mathbf{n} \\ = C(\mathbf{y})\mathbf{n} \text{ for some function } C(\mathbf{y}) \end{cases}$$

With these observations and the fact that $\boldsymbol{\pi}^{in}$ is symmetric, the first term in (2.39) can be expressed as

$$\begin{aligned} - \int_{\Gamma} \bar{\Phi} \cdot \boldsymbol{\pi}^{in} \cdot \mathbf{n} \, dS &= \int_{\Gamma} \overline{d(\mathbf{y})}\mathbf{t} \cdot [(\boldsymbol{\pi}^{out} - \boldsymbol{\pi}^{in}) \cdot \mathbf{n}] - \int_{\Gamma} \overline{d(\mathbf{y})}\mathbf{t} \cdot (\boldsymbol{\pi}^{out} \cdot \mathbf{n}) \, dS \\ &= - \int_{\Gamma} \bar{\Phi} \cdot \boldsymbol{\pi}^{out} \cdot \mathbf{n} \, dS \end{aligned}$$

and hence the variational form (2.39) becomes for all $\Phi \in H(Q_2)$

$$- \int_{Q_2} 2\mu_1 e(\mathbf{u}^{in}) : \overline{e(\Phi)} \, d\mathbf{y} = \int_{Q_2} \mathbf{f}^{in} \cdot \bar{\Phi} \, d\mathbf{y} + \int_{\Gamma} \bar{\Phi} \cdot \boldsymbol{\pi}^{out} \cdot \mathbf{n} \, dS \tag{2.40}$$

To bound the right-hand side of (2.40), we first extend $\Phi \in H(Q_2)$ by the operator T described in (2.32)

$$\|T(\Phi)\|_Q \leq E_1 \|\Phi\|_{Q_2} \tag{2.41}$$

$T(\Phi)$ rapidly decays to zero in a small neighbourhood of Q_2 and stays 0 for the rest of Q_1 . The restriction of $T(\Phi)$ in Q_1 , denoted by Φ^{out} , has the following estimate

$$\|\Phi^{out}\|_{Q_1} \leq \|T(\Phi)\|_Q \leq E_1 \|\Phi\|_{Q_2} \tag{2.42}$$

Hence

$$\begin{aligned} \left| \int_{\Gamma} \bar{\Phi} \cdot \boldsymbol{\pi}^{out} \cdot \mathbf{n} \, dS \right| &= \left| \int_{\Gamma} \bar{\Phi}^{out} \cdot \boldsymbol{\pi}^{out} \cdot \mathbf{n} \, dS \right| \\ &= \left| \int_{Q_1} [\operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}^{out}) - p^{out}\mathbf{I})] \cdot \overline{\Phi^{out}} \, d\mathbf{y} \right. \\ &\quad \left. + \int_{Q_1} 2\mu_1 e(\mathbf{u}^{out}) : \overline{e(\Phi^{out})} \, d\mathbf{y} \right| \\ &\leq \left| \int_{Q_1} \mathbf{f}^{out} \cdot \overline{\Phi^{out}} \, d\mathbf{y} \right| + \left| \int_{Q_1} 2\mu_1 e(\mathbf{u}^{out}) : \overline{e(\Phi^{out})} \, d\mathbf{y} \right| \\ &\leq \frac{E_1}{B_1(Q_1)} \|\mathbf{f}^{out}\|_{L^2(Q_1)} \|\Phi\|_{Q_2} + E_1 \|\mathbf{u}^{out}\|_{Q_1} \|\Phi\|_{Q_2} \end{aligned}$$

Therefore the right-hand side of (2.40) is bounded by

$$\|\Phi\|_{Q_2} \left(\frac{E_1 \|\mathbf{f}^{in}\|_{L^2(Q_2)}}{B_1(Q_2)} + \frac{E_1 \|\mathbf{f}^{out}\|_{L^2(Q_1)}}{B_1(Q_1)} + E_1 \|\mathbf{u}^{out}\|_{Q_1} \right)$$

Finally, by the Lax–Milgram lemma a unique solution \mathbf{u}^{in} exists such that

$$\|\mathbf{u}^{in}\|_{Q_2} \leq \frac{E_1}{B_1(Q_2)} \|\mathbf{f}^{in}\|_{L^2(Q_2)} + \frac{E_1}{B_1(Q_1)} \|\mathbf{f}^{out}\|_{L^2(Q_1)} + E_1 \|\mathbf{u}^{out}\|_{Q_1} \quad (2.43)$$

□

The construction of the solution for z with large magnitude will be carried out using the following steps.

- (i) $O(w^{-1})$: Consider the system of (2.24) and (2.27) for $\mathbf{u}_0^{in}(\mathbf{y}; \mathbf{e}) \in H(Q_2)$.

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_0^{in})) = \mathbf{0} & \text{in } Q_2 \\ 2\mu_1 e(\mathbf{u}_0^{in}) \cdot \mathbf{n} = C(\mathbf{y})\mathbf{n} & \text{on } \Gamma \end{cases} \quad (2.44)$$

The variational formulation is

$$-\int_{\Gamma} \bar{\mathbf{v}} \cdot 2\mu_1 e(\mathbf{u}_0^{in}) \cdot \mathbf{n} - \int_{Q_2} 2\mu_1 e(\mathbf{u}_0^{in}) : \overline{e(\mathbf{v})} = 0 \quad \forall \mathbf{v} \in H(Q_2) \quad (2.45)$$

The first term vanishes because of the boundary conditions. Hence

$$\mathbf{u}_0^{in}(\mathbf{y}; \mathbf{e}) = \mathbf{0} \text{ in } Q_2 \text{ because } H(Q_2) \perp \mathcal{R}(Q_2).$$

- (ii) $O(w^0)$ in Q_1 : Solve the system of (2.29) and (2.22) for $\mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}) \in H(Q_1)$.

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_0^{out}) - p_0^{out} \mathbf{I}) = -\mathbf{e} & \text{in } Q_1 \\ \mathbf{u}_0^{out} = \mathbf{u}_0^{in} = \mathbf{0} & \text{on } \Gamma \end{cases} \quad (2.46)$$

An application of lemma 2.1 and (2.37) leads to the following result

$$\|\mathbf{u}_0^{out}\|_{Q_1} \leq \frac{\sqrt{|Q_1|}}{B_1(Q_1)} + 2E_1 \|\mathbf{u}_0^{in}\|_{Q_2} = \frac{\sqrt{|Q_1|}}{B_1(Q_1)} \quad (2.47)$$

- (iii) $O(w^0)$ in Q_2 : Consider the system of (2.25) and (2.28) for $\mathbf{u}_1^{in} \in H(Q_2)$:

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_1^{in}) - p_0^{in} \mathbf{I}) = -\mathbf{e} & \text{in } Q_2 \\ \mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}_0^{out}) - p_0^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}_1^{in}) - p_0^{in} \mathbf{I})) \cdot \mathbf{n}] = \mathbf{0} & \text{on } \Gamma \end{cases}$$

By applying lemma 2.2 and (2.43) with $\mathbf{f}^{out} = -\mathbf{e}$ and $\mathbf{f}^{in} = -\mathbf{e}$, we obtain

$$\|\mathbf{u}_1^{in}\|_{Q_2} \leq C_1 E_1, \quad C_1 := \frac{\sqrt{|Q_2|}}{B_1(Q_2)} + 2 \frac{\sqrt{|Q_1|}}{B_1(Q_1)} \quad (2.48)$$

- (iv) Induction step, $k \geq 1$: Given $\mathbf{u}_k^{in} \in H(Q_2)$ and $\mathbf{u}_{k-1}^{out} \in H(Q_1)$, find $\mathbf{u}_{k+1}^{in}(\mathbf{y}; \mathbf{0}) \in H(Q_2)$ and $\mathbf{u}_k^{out}(\mathbf{y}; \mathbf{0}) \in H(Q_1)$.

- (a) Applying lemma 2.1 with $\mathbf{f} = \mathbf{0}$, we conclude that for a given $\mathbf{u}_k^{in} \in H(Q_2)$, $k \geq 1$, there exists a unique $\mathbf{u}_k^{out} \in H(Q_1)$ that solves the system of (2.23) and (2.29) and assumes the estimate

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) = \mathbf{0} & \text{in } Q_1 \\ \mathbf{u}_k^{out} = \mathbf{u}_k^{in} & \text{on } \Gamma, \end{cases} \tag{2.49}$$

$$\|\mathbf{u}_k^{out}\|_{Q_1} \leq 2E_1 \|\mathbf{u}_k^{in}\|_{Q_2} \tag{2.50}$$

- (b) By applying lemma 2.2 with $\mathbf{f}^{in} = \mathbf{0} = \mathbf{f}^{out}$, we see that for any given $\mathbf{u}_k^{out} \in H(Q_1)$, $k \geq 1$ that satisfies (2.49), there exists a unique solution $\mathbf{u}_{k+1}^{in} \in H(Q_2)$ to the system of equations (2.26) and (2.28)

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_{k+1}^{in}) - p_k^{in} \mathbf{I}) = \mathbf{0} & \text{in } Q_2 \\ \mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}_{k+1}^{in}) - p_k^{in} \mathbf{I})) \mathbf{n}] = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{2.51}$$

Moreover,

$$\|\mathbf{u}_{k+1}^{in}\|_{Q_2} \leq E_1 \|\mathbf{u}_k^{out}\|_{Q_1} \tag{2.52}$$

Now we have found the coefficients $\mathbf{u}_n^{in}(\mathbf{y}; \mathbf{e})$ and $\mathbf{u}_n^{out}(\mathbf{y}; \mathbf{e})$ in (2.21) iteratively. We prove the convergence of the series in the following theorem by taking into account the fact that $\mathbf{u}_0^{in} = \mathbf{0}$.

THEOREM 2.3. *Define the partial sums*

$$\mathbf{S}_q^{in}(\mathbf{y}; \mathbf{e}, w) := \sum_{k=0}^q \mathbf{u}_{k+1}^{in}(\mathbf{y}, \mathbf{e}) w^{k+1}, \quad \mathbf{S}_q^{out}(\mathbf{y}; \mathbf{e}, w) := \sum_{k=0}^q \mathbf{u}_k^{out}(\mathbf{y}; \mathbf{e}) w^k.$$

Let $R \in (0, 1)$, in the disc $|w| \leq R/2E_1^2$, the series $\mathbf{S}_q^{in}(\mathbf{y}; \mathbf{e}, w)$ and $\mathbf{S}_q^{out}(\mathbf{y}; \mathbf{e}, w)$ converge uniformly to $\mathbf{u}_\infty^{in}(\mathbf{y}; \mathbf{e}, w) \in H(Q_2)$ and $\mathbf{u}_\infty^{out}(\mathbf{y}; \mathbf{e}, w) \in H(Q_1)$, respectively. Therefore, $\mathbf{u}_\infty(\mathbf{y}; \mathbf{e}, w) := \mathbf{u}_\infty^{in}(\mathbf{y}; \mathbf{e}, w)\chi_2 + \mathbf{u}_\infty^{out}(\mathbf{y}; \mathbf{e}, w)\chi_1 \in H(Q)$ solves the cell problem (2.6) and is analytic for $|w| < 1/2E_1^2$.

Proof. For each $q \in \mathbb{N}$, $\mathbf{S}_q^{in}(\mathbf{y}; \mathbf{e}, w)$ is a polynomial function of w and maps from \mathbb{C} to the Hilbert space $H(Q_2)$. Similarly, $\mathbf{S}_q^{out}(\mathbf{y}; \mathbf{e}, w)$ maps from \mathbb{C} to $H(Q_1)$. To show uniform convergence, we note that (2.50) and (2.52) imply there exists a positive constant E_1 that depends only on Q_1 and Q_2 such that $\|\mathbf{u}_{k+1}^{in}\|_{Q_2} \leq E_1 \|\mathbf{u}_k^{out}\|_{Q_1} \leq 2E_1^2 \|\mathbf{u}_k^{in}\|_{Q_2}$. Therefore,

$$\|\mathbf{u}_k^{in}\|_{Q_2} \leq (2E_1^2)^{k-1} \|\mathbf{u}_1^{in}\|_{Q_2}, \quad k \geq 1 \tag{2.53}$$

Let $m > q > N$, and define $r := 2E_1^2 |w|$. Then by (2.48) implies

$$\begin{aligned} \|\mathbf{S}_m^{in}(w) - \mathbf{S}_q^{in}(w)\|_{Q_2} &\leq \|\mathbf{u}_1^{in}\|_{Q_2} \left((2E_1^2)^q |w|^{q+1} + \dots + (2E_1^2)^{m-1} |w|^m \right) \\ &\leq \frac{r^{q+1} - r^{m+1}}{1 - r} \frac{\|\mathbf{u}_1^{in}\|_{Q_2}}{2E_1^2} \\ &\leq \frac{r^{q+1} - r^{m+1}}{1 - r} \left(\frac{C_1}{2E_1^2} \right), C_1 \text{ is defined in (2.48)}. \end{aligned}$$

Therefore, for $r \leq R < 1$, i.e. $|w| \leq R/2E_1^2$, where R is any fixed number in $(0, 1)$,

$$\|\mathbf{S}_m^{in}(w) - \mathbf{S}_q^{in}(w)\|_{Q_2} \leq \left(\frac{C_1}{2E_1^2} \right) \left(\frac{R^{N+1}}{1 - R} \right), \quad \forall m > q > N. \tag{2.54}$$

For $\mathbf{S}_q^{out}(\mathbf{y}; \mathbf{e}, w)$ we have $\|\mathbf{u}_k^{out}\|_{Q_1} \leq (2E_1^2)^{k-1} \|\mathbf{u}_1^{out}\|_{Q_1}$. By a similar procedure, for $m > q > N$ and $|w| \leq R/2E_1^2$ the following estimate is valid

$$\|\mathbf{S}_m^{out}(w) - \mathbf{S}_q^{out}(w)\|_{Q_1} \leq C_1 \left(\frac{R^{N+1}}{1 - R} \right)$$

Therefore, for every fixed w satisfying $|w| \leq R/2E_1^2$ for any $0 < R < 1$, $\mathbf{S}_q^{in}(\mathbf{y}; w)$ and $\mathbf{S}_q^{out}(\mathbf{y}; w)$ converge uniformly to $\mathbf{u}_\infty^{in}(\mathbf{y}; w) \in H(Q_2)$ and $\mathbf{u}_\infty^{out}(\mathbf{y}; w) \in H(Q_1)$, respectively. Since for each q , $\mathbf{S}_q^{in}(\mathbf{y}; w)$ and $\mathbf{S}_q^{out}(\mathbf{y}; w)$ are polynomials of w , hence analytic, the uniform convergence implies that the limit functions $\mathbf{u}_\infty^{in}(\mathbf{y}; w)$ and $\mathbf{u}_\infty^{out}(\mathbf{y}; w)$ are also analytic in $|w| < 1/2E_1^2$ with values in $H(Q_1)$ and $H(Q_2)$, respectively, by applying Morera’s theorem for Banach space valued analytic functions [27] to the uniformly converging sequences. By construction, the function $\mathbf{u}_\infty(\mathbf{y}; \mathbf{e}, w)$ defined in (2.20) solves the cell problem (2.6) for all w in the disc $\{w : |w| < 1/2E_1^2\} := B_0(1/2E_1^2)$. Moreover, the uniqueness of the solution implies that $\mathbf{u}_\infty(\mathbf{y}; \mathbf{e}_k, w) = \mathbf{u}^k(\mathbf{y}; \frac{1}{w})$ in $H(Q)$ for $w \in B_0(1/2E_1^2) \cap \{w \in \mathbb{C} \setminus (-\infty, 0]\}$. \square

The following theorem shows the relation between the two-fluid self-permeability \mathbf{K} in (2.5) and the Darcy permeability $\mathbf{K}^{(D)}$ in (1.5)

THEOREM 2.4. *In the case of large viscosity $|z| > 2E_1^2$ (or $|w| < 1/2E_1^2$), we have*

- (i) $\mathbf{u}_\infty^{in}(\mathbf{y}; \mathbf{e}_i, 0) = \mathbf{0}$ in Q_2
- (ii) *As $w \rightarrow 0$, the solution $\mathbf{u}_\infty^{out}(\mathbf{y}; \mathbf{e}_i, w)$ converges uniformly in $\mathring{H}(Q_1)$ to the solution $\mathbf{u}_D^i(\mathbf{y})$ of the classical cell problem (1.4).*
- (iii) *For $w \in B_0(1/2E_1^2)$, the difference between the self-permeability $\mathbf{K}(\mathbf{y}; \mathbf{e}_i, w)$ and the classical permeability tensor $\mathbf{K}^{(D)}(\mathbf{y}; \mathbf{e}_i)$ satisfies $|K_{ij} - (K^{(D)})_{ij}| = O(|w|)$, hence $\mathbf{K} \rightarrow \mathbf{K}^{(D)}$ uniformly as $|w| \rightarrow 0$.*

Proof. The uniform convergence allows passing the limit $w \rightarrow 0$ inside the summation of (2.21) to obtain $\mathbf{u}_\infty^{in}(\mathbf{y}; \mathbf{e}_i, 0) = \mathbf{0}$. Similarly, the uniform convergence allows

passing the limit $w \rightarrow 0$ inside the summation of (2.21) to obtain

$$\mathbf{u}_\infty^{out}(\mathbf{y}; \mathbf{e}_i, 0) = \mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}_i)$$

Furthermore, $\mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}_i) \in H(Q_1)$ satisfies (2.46) and in fact $\mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}_i) \in \mathring{H}(Q_1)$ since $\mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}_i)|_\Gamma = \mathbf{0}$, which is identical to the equation for \mathbf{u}_D (1.4). The uniqueness of the solution then ensures that $\mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}_i) = \mathbf{u}_D^i(\mathbf{y})$. Therefore the series $\mathbf{u}_\infty^{out}(\mathbf{y}; \mathbf{e}_i, w) \rightarrow \mathbf{u}_D^i(\mathbf{y})$ uniformly as $|w| \rightarrow 0$ in $\mathring{H}(Q_1)$. For (iii), we note that

$$\begin{aligned} |K_{ij}(w) - K_{ij}^{(D)}| &= \left| \int_Q (\mathbf{u}^i - \chi_1 \mathbf{u}_D^i) \cdot \mathbf{e}_j \, d\mathbf{y} \right| \leq \| \mathbf{u}^i - \chi_1 \mathbf{u}_D^i \|_{L^2(Q)} \\ &\leq \frac{1}{B_1(Q)} \left\| \sum_{k=1}^\infty (\mathbf{u}_k^{in}(\mathbf{y}; \mathbf{e}_i) \chi_2 + \mathbf{u}_k^{out}(\mathbf{y}; \mathbf{e}_i) \chi_1) w^k \right\|_Q \end{aligned}$$

From (2.50), (2.53) and (2.48), we have for $|w| < 1/2E_1^2$, or equivalently $|z| > 2E_1^2$,

$$|K_{ij}(w) - K_{ij}^{(D)}| \leq C_1 \left(\frac{E_1 + 1}{2E_1 B_1(Q)} \right) \frac{2E_1^2 |w|}{1 - 2E_1^2 |w|} \tag{2.55}$$

□

In the following section, we study the behaviour of $\mathbf{K}(z)$ near $z = 0$, i.e. the inclusion is an air bubble.

2.4. Analyticity of the solution for small $|z|$

Let \mathbf{e} be a constant unit vector in \mathbb{R}^n . We seek solutions of the following form

$$\mathbf{u}_{null}^{in}(\mathbf{y}; \mathbf{e}, z) = \sum_{k=0}^\infty \mathbf{u}_k^{in}(\mathbf{y}; \mathbf{e}) z^k, \quad p^{in}(\mathbf{y}; \mathbf{e}, z) = \sum_{k=0}^\infty p_k^{in}(\mathbf{y}; \mathbf{e}) z^k \text{ in } Q_2, \tag{2.56}$$

$$\mathbf{u}_{null}^{out}(\mathbf{y}; \mathbf{e}, z) = \sum_{k=0}^\infty \mathbf{u}_k^{out}(\mathbf{y}; \mathbf{e}) z^k, \quad p^{out}(\mathbf{y}; \mathbf{e}, z) = \sum_{k=0}^\infty p_k^{out}(\mathbf{y}; \mathbf{e}) z^k \text{ in } Q_1 \tag{2.57}$$

By a procedure similar to that in § 2.3, the following equations are obtained via collecting terms with respect to the order of z . The PDEs for Q_1 are as follows:

$$O(1) : \quad \text{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_0^{out}) - p_0^{out} \mathbf{I}) = -\mathbf{e} \tag{2.58}$$

$$O(z^k), k \geq 1 : \quad \text{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) = \mathbf{0} \tag{2.59}$$

Similarly, the PDEs for Q_2 are

$$O(1) : \quad -\nabla p_0^{in} = -\mathbf{e} \tag{2.60}$$

$$O(z^k), k \geq 1 : \quad \text{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}_{k-1}^{in}) - p_k^{in} \mathbf{I}) = \mathbf{0} \tag{2.61}$$

The interface condition (2.29) remains the same for the small $|z|$ case while (2.27) and (2.28) now read

$$\mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}_0^{out}) - p_0^{out} \mathbf{I}) \cdot \mathbf{n} - (-p_0^{in} \mathbf{I}) \cdot \mathbf{n})] = \mathbf{0} \tag{2.62}$$

$$\mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}_{k-1}^{in}) - p_k^{in} \mathbf{I})) \cdot \mathbf{n}] = \mathbf{0}, \quad k \geq 1 \tag{2.63}$$

The first equation to be solved is (2.60), whose solution is simply

$$p_0^{in}(\mathbf{y}) = \mathbf{e} \cdot \mathbf{y} + c - \int_{Q_2} (\mathbf{e} \cdot \mathbf{y} + c) \, d\mathbf{y} \text{ in } Q_2 \tag{2.64}$$

where c is a constant. The next problem is the system of (2.58) and (2.62). Similar to the calculation in lemma 2.2, the weak formulation of this system is: find $u_0^{out} \in H(Q_1)$ such that for all $\Phi \in H(Q_1)$ and $\pi_0^{out} := 2\mu_1 e(\mathbf{u}_0^{out}) - p_0^{out} \mathbf{I}$

$$- \int_{\Gamma} (\bar{\Phi} \cdot (\pi_0^{out} + p_0^{in} \mathbf{I}) - p_0^{in} \mathbf{I}) \cdot \mathbf{n} \, dS - \int_{Q_1} 2\mu_1 e(\mathbf{u}_0^{out}) : e(\bar{\Phi}) \, d\mathbf{y} = \int_{Q_1} -\mathbf{e} \cdot \bar{\Phi} \, d\mathbf{y}$$

Since $\Phi \cdot \mathbf{n} = 0$ and $p_0^{in} \mathbf{I} \cdot \mathbf{n}$ is parallel to \mathbf{n} , (2.62) implies the integral on Γ vanishes. Hence by the Lax–Milgram lemma, we have

$$\|\mathbf{u}_0^{out}\|_{Q_1} \leq \frac{\sqrt{|Q_1|}}{B_1(Q_1)} \tag{2.65}$$

The system for \mathbf{u}_{k-1}^{in} , $k \geq 1$ (inner problem) is to find $\mathbf{u}_{k-1}^{in} \in H(Q_2)$ with given $\mathbf{u}_0^{out} \in H(Q_1)$ such that

$$\begin{cases} \operatorname{div} (2\mu_1 e(\mathbf{u}_{k-1}^{in}) - p_k^{in} \mathbf{I}) = \mathbf{0} & \text{in } Q_2 \\ \mathbf{u}_{k-1}^{in}|_{\Gamma} = \mathbf{u}_{k-1}^{out}|_{\Gamma} \end{cases} \tag{2.66}$$

With an argument similar to the derivation of lemma 2.1, the following estimate can be derived for system (2.66)

LEMMA 2.5. *Let Q_2 satisfy the same assumption in lemma 2.1. For any given vector field $\mathbf{u}^{out} \in H(Q_1)$, there exists a unique weak solution $\mathbf{u}^{in}(\mathbf{y}) \in H(Q_2)$ s.t.*

$$\begin{cases} \operatorname{div}_{\mathbf{y}} (2\mu_1 e(\mathbf{u}^{in}) - p^{in} \mathbf{I}) = \mathbf{f}^{in} & \text{in } Q_2 \\ \mathbf{u}^{in} = \mathbf{u}^{out} & \text{on } \Gamma \end{cases} \tag{2.67}$$

$$\|\mathbf{u}^{in}\|_{Q_2} \leq \frac{1}{B_1(Q_2)} \|\mathbf{f}^{in}\|_{L^2(Q_2)} + 2E_2 \|\mathbf{u}^{out}\|_{Q_1}. \tag{2.68}$$

where $E_2 > 1$ is the constant associated with the extension operator T , $\|T(\Phi)\|_Q \leq E_2 \|\Phi\|_{Q_1}$ for all $\Phi \in H(Q_1)$ and $T(\Phi)$ decays rapidly to 0 inside Q_2 . Note that the periodic condition of space $H(Q_1)$ implies $\int_{\Gamma} \mathbf{u}^{out} \cdot \mathbf{n} \, dS = 0$.

The system for \mathbf{u}_k^{out} and p_k^{out} with given $\mathbf{u}_{k-1}^{in} \in H(Q_2)$ and p_k^{in} , $k \geq 1$ is

$$\begin{cases} \operatorname{div} (2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) = \mathbf{0} \\ \mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}_k^{out}) - p_k^{out} \mathbf{I}) - (2\mu_1 e(\mathbf{u}_{k-1}^{in}) - p_k^{in} \mathbf{I})) \cdot \mathbf{n}] = \mathbf{0} \end{cases} \tag{2.69}$$

By an argument similar to the one for lemma 2.2, the system above can be shown to satisfy the following estimate.

LEMMA 2.6. Let Q_2 satisfy the same assumption in lemma 2.1. For any given pair of $(\mathbf{u}^{in}, p^{in}) \in H(Q_2) \times L^2(Q_1)/\mathbb{C}$ that satisfies (2.67), there exists a unique vector $\mathbf{u}^{out}(\mathbf{y}; \mathbf{f}^{out}) \in H(Q_1)$ solving the following system

$$\begin{cases} \operatorname{div}(2\mu_1 e(\mathbf{u}^{out}) - p^{out} \mathbf{I}) = \mathbf{f}^{out} \text{ in } Q_1 \\ \mathbf{n} \times \mathbf{n} \times [((2\mu_1 e(\mathbf{u}^{in}) - p^{in} \mathbf{I}) - (2\mu_1 e(\mathbf{u}^{out}) - p^{out} \mathbf{I})) \cdot \mathbf{n}] = \mathbf{0} \text{ on } \Gamma \end{cases} \tag{2.70}$$

$$\|\mathbf{u}^{out}\|_{Q_1} \leq \frac{E_2}{B_1(Q_1)} \|\mathbf{f}^{out}\|_{L^2(Q_1)} + \frac{E_2}{B_1(Q_2)} \|\mathbf{f}^{in}\|_{L^2(Q_2)} + E_2 \|\mathbf{u}^{in}\|_{Q_2} \tag{2.71}$$

where E_2, B_1 depend only on Q and Γ .

Equation (2.69), lemma 2.5, equation (2.66) and lemma 2.6 imply that for all $k \geq 0$, we have $\|\mathbf{u}_k^{in}\|_{Q_2} \leq 2E_2 \|\mathbf{u}_k^{out}\|_{Q_1}$ and $\|\mathbf{u}_{k+1}^{out}\|_{Q_1} \leq E_2 \|\mathbf{u}_k^{in}\|_{Q_2}$. Therefore,

$$\|\mathbf{u}_k^{in}\|_{Q_2} \leq \frac{(2E_2^2)^{k+1}}{E_2} \|\mathbf{u}_0^{out}\|_{Q_1} \leq (2E_2^2)^{k+1} \left(\frac{|Q_1|}{E_2 B_1(Q_1)} \right) \tag{2.72}$$

$$\|\mathbf{u}_k^{out}\|_{Q_1} \leq (2E_2^2)^k \|\mathbf{u}_0^{out}\|_{Q_1} \leq (2E_2^2)^k \left(\frac{|Q_1|}{B_1(Q_1)} \right) \tag{2.73}$$

Therefore, the series in (2.56) and (2.57) converge uniformly in the disk $|z| < 1/2(E_2)^2$ to an analytic function in Q_2 and Q_1 , respectively. The limit functions $\mathbf{u}_{null}^{in}(\mathbf{y}, \mathbf{e}, z)$, $\mathbf{u}_{null}^{out}(\mathbf{y}, \mathbf{e}, z)$ and the corresponding permeability $K_{ij}(z)$ in (2.12) are analytic at $z = 0$. Define the permeability (‘B’ for ‘bubbles’)

$$K_{ij}^{(B)} := \int_Q [\chi_1 \mathbf{u}_0^{out}(\mathbf{y}; \mathbf{e}_i) + \chi_2 \mathbf{u}_0^{in}(\mathbf{y}; \mathbf{e}_i)] \cdot \mathbf{e}_j \, dy \tag{2.74}$$

then the following estimate, valid for $|z| < 1/2E_2^2$, holds

$$|K_{ij}(z) - K_{ij}^B| \leq \frac{|Q_1|(1 + 2E_2)}{B_1(Q)B_1(Q_1)} \left(\frac{2E_2^2|z|}{1 - 2E_2^2|z|} \right) = O(|z|). \tag{2.75}$$

In conclusion, $\mathbf{K}(z)$ in (2.5) is analytic for $z \in \mathbb{C} \setminus [-2E_1^2, -(1/2E_2^2)]$, $E_1, E_2 \geq 1$. In the next section, and IRF for $\mathbf{K}(z)$ will be derived in two different ways.

3. Integral representation of the permeability $\mathbf{K}(z)$

We first observe two properties of \mathbf{K} implied by (2.12).

PROPOSITION 3.1.

$$\frac{\mathbf{K}(z) - \mathbf{K}^*(z)}{z - \bar{z}} \leq 0 \text{ if } \operatorname{Im}(z) \neq 0 \tag{3.1}$$

$$\mathbf{K}(x) \geq 0 \text{ for } x > 0 \tag{3.2}$$

Proof. Note that $K_{ij}(z) - (K^*)_{ij}(z) = 2\mu_1(\bar{z} - z) \int_{Q_2} e(\mathbf{u}^j(z)) : \overline{e(\mathbf{u}^i(z))} \, d\mathbf{y}$. Hence

$$\frac{K_{ij}(z) - K_{ij}^*(z)}{z - \bar{z}} = -2\mu_1 \int_{Q_2} e(\mathbf{u}^j(z)) : \overline{e(\mathbf{u}^i(z))} \, d\mathbf{y} = -(\mathbf{u}^j, \mathbf{u}^i)_{Q_2} =: -A_{ij}$$

The matrix \mathbf{A} is obviously Hermitian and for any $\boldsymbol{\xi} \in \mathbb{C}^n$, we have $\overline{\xi_i} A_{ij} \xi_j = (\xi_j \mathbf{u}^j, \xi_i \mathbf{u}^i)_{Q_2} \geq 0$. This proves (3.1). Recall that $K_{ij}(x) = ((\mathbf{u}^j, \mathbf{u}^i)_{Q_1} + x(\mathbf{u}^j, \mathbf{u}^i)_{Q_2})$. With a similar argument, (3.2) follows. \square

With these two properties and the fact that \mathbf{K} is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, the characterization theorem for matrix-valued functions belonging to the Stieltjes class [20] implies that there exists a monotonically increasing matrix-valued function $\boldsymbol{\sigma}(t)$ such that the following IRF holds for $z \in \mathbb{C} \setminus (-\infty, 0]$

$$\mathbf{K}(z) = \mathbf{A} + \frac{\mathbf{C}}{z} + \int_{+0}^{\infty} \frac{1}{z+t} \, d\boldsymbol{\sigma}(t)$$

where $\mathbf{A} \geq 0$, $\mathbf{C} \geq 0$, $\int_{+0}^{\infty} \frac{1}{1+t} \, d\boldsymbol{\sigma}(t) := \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{\infty} \frac{1}{1+t} \, d\boldsymbol{\sigma}(t) < \infty$ and $\mathbf{A} + \mathbf{C} + \int_{+0}^{\infty} \frac{1}{1+t} \, d\boldsymbol{\sigma}(t) > 0$. Since $\mathbf{K}(0) = \mathbf{K}^{(B)}$, we must have $\mathbf{C} = \mathbf{0}$. Also, $\mathbf{K}(\infty) = \mathbf{K}^{(D)}$ implies $\mathbf{A} = \mathbf{K}^{(D)}$

$$\mathbf{K}(z) = \mathbf{K}^{(D)} + \int_{1/2E_2^2}^{2E_1^2} \frac{1}{z+t} \, d\boldsymbol{\sigma}(t),$$

where the limits of the integral reflect the fact that $\mathbf{K}(z)$ in (2.5) is analytic for $z \in \mathbb{C} \setminus [-2E_1^2, -(1/2E_2^2)]$, $E_1, E_2 \geq 1$.

Therefore, for real valued z , $\mathbf{K}(z)$ is decreasing as z increases, i.e. $\mathbf{K}(x_1) - \mathbf{K}(x_2)$ is negative semidefinite if $x_1 > x_2$. To study how the measure $d\boldsymbol{\sigma}$ is related to the microstructure, we derive the spectral representation of $\mathbf{K}(z)$ by using the underlying system (2.6).

3.1. Spectral representation of $\mathbf{K}(z)$

Adding $\int_{Q_2} 2\mu_1 e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y}$ to both sides of (2.14), we have

$$\int_Q 2\mu_1 e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y} = -\frac{1}{s} \int_Q 2\mu_1 \chi_2 e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y} + \int_Q \mathbf{e}_k \cdot \bar{\mathbf{v}} \, d\mathbf{y} \tag{3.3}$$

where the new variable s is defined as

$$s := \frac{1}{z - 1}$$

Let $\Delta_{\#}^{-1}$ be the operator that solves for $\mathbf{w}(\mathbf{y}; \mathbf{f}) \in H(Q)$ in the following variational formulation

$$\int_Q 2\mu_1 e(\mathbf{w}) : \overline{e(\mathbf{v})} \, d\mathbf{y} = \int_Q \mathbf{f} \cdot \bar{\mathbf{v}} \, d\mathbf{y} \tag{3.4}$$

where $\mathbf{f} \in L^2(Q)$ and Q -periodic. In other words, solution $\mathbf{w}(\mathbf{y}) = \Delta_{\#}^{-1}\mathbf{f} \in H(Q)$ is a weak solution to the cell problem

$$\begin{cases} -\mu_1 \Delta \mathbf{w} = \mathbf{f} & \text{in } Q_1 \cup Q_2 \\ \llbracket \boldsymbol{\pi} \rrbracket \mathbf{n} = (\llbracket \boldsymbol{\pi} \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} & \text{on } \Gamma \end{cases} \tag{3.5}$$

In order to get the spectral representation, we apply $\Delta_{\#}^{-1}$ on both sides of (3.3) and symbolically represent the resulted equations as

$$\mathbf{w}_1 = -\frac{1}{s} \mathbf{w}_2 + \mathbf{w}_3$$

Then clearly, we have $\mathbf{w}_1 = \mathbf{u}^k$ and $\mathbf{w}_3 = \Delta_{\#}^{-1} \mathbf{e}_k$. Observe that \mathbf{w}_2 solves

$$\int_Q 2\mu_1 e(\mathbf{w}_2) : \overline{e(\mathbf{v})} \, d\mathbf{y} = \int_Q 2\mu_1 \chi_2 e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y} \text{ for all } \mathbf{v} \in H(Q) \tag{3.6}$$

Define the operator Γ_{χ} such that $\mathbf{w}_2 = \Gamma_{\chi} \mathbf{u}^k$ and (3.6) can be expressed as

$$(\Gamma_{\chi} \mathbf{u}^k, \mathbf{v})_Q = \int_Q 2\mu_1 \chi_2 e(\mathbf{u}^k) : \overline{e(\mathbf{v})} \, d\mathbf{y} \text{ for all } \mathbf{v} \in H(Q). \tag{3.7}$$

The subscript χ is used to signify the dependence of Γ_{χ} on χ_2 , the characteristic function of Q_2 . Clearly, Γ_{χ} is self-adjoint with respect to the inner product $(\cdot, \cdot)_Q$ because

$$(\Gamma_{\chi} \mathbf{u}, \mathbf{v})_Q = \overline{\int_Q 2\mu_1 \chi_2 e(\mathbf{v}) : \overline{e(\mathbf{u})} \, d\mathbf{y}} = \overline{(\Gamma_{\chi} \mathbf{v}, \mathbf{u})_Q} = (\mathbf{u}, \Gamma_{\chi} \mathbf{v})_Q.$$

Formally, we have $\Gamma_{\chi} \mathbf{u} = \Delta_{\#}^{-1}(\nabla \cdot \chi_2 e(\mathbf{u}))$. Now (3.3) becomes

$$\mathbf{u}^k = -\frac{1}{s} \Gamma_{\chi} \mathbf{u}^k + \Delta_{\#}^{-1} \mathbf{e}_k \Leftrightarrow \left(I + \frac{\Gamma_{\chi}}{s} \right) \mathbf{u}^k = \Delta_{\#}^{-1} \mathbf{e}_k \tag{3.8}$$

PROPOSITION 3.2. *The self-adjoint operator Γ_{χ} defined in (3.7) is positive and bounded with $\|\Gamma_{\chi}\| \leq 1$.*

Proof. It can be proved by choosing $\mathbf{v} = \mathbf{u}$ in (3.7) and observe that $0 \leq \int_{Q_2} 2\mu_1 e(\mathbf{u}) : \overline{e(\mathbf{u})} \, d\mathbf{y} \leq \int_Q 2\mu_1 e(\mathbf{u}) : \overline{e(\mathbf{u})} \, d\mathbf{y} = (\mathbf{u}, \mathbf{u})_Q$. □

THEOREM 3.3. For $|s| > 1$, the solution $\mathbf{u}^k \in H(Q)$ admits a series representation

$$\mathbf{u}^k(\mathbf{y}; s) = \sum_{m=0}^{\infty} \left(-\frac{1}{s}\right)^m (\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_k \tag{3.9}$$

and the components of \mathbf{K} can be represented by the following IRF

$$K_{kl}(s) = s \int_0^1 \int_Q \frac{(\tilde{M}(d\lambda) \Delta_{\#}^{-1} \mathbf{e}_k)_l}{s + \lambda} d\mathbf{y}, \quad k, l = 1, \dots, n, \tag{3.10}$$

for some projection-valued measures $\tilde{M}(d\lambda)$ and a series representation

$$K_{kl}(s) = \int_Q (\Delta_{\#}^{-1} \mathbf{e}_k)_l d\mathbf{y} + \sum_{m=1}^{\infty} \frac{\tilde{\lambda}_{kl}^m}{(-s)^m} \quad \text{with } \tilde{\lambda}_{kl}^m := \int_Q ((\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_k)_l d\mathbf{y}.$$

Proof. From (3.8), since Γ_χ is self-adjoint with norm bounded by 1, for $|s| > 1$, the spectral theory for self-adjoint operator implies the existence of a projection-valued measure \tilde{M} such that

$$\mathbf{u}^k(\mathbf{y}; s) = \left(I + \frac{\Gamma_\chi}{s}\right)^{-1} \Delta_{\#}^{-1} \mathbf{e}_k = s \int_0^1 \frac{\tilde{M}(d\lambda) (\Delta_{\#}^{-1} \mathbf{e}_k)}{s + \lambda} \tag{3.11}$$

Hence the kl -the element of permeability \mathbf{K} has the following IRF

$$K_{kl}(s) = \int_Q (\mathbf{u}^k)_l d\mathbf{y} = s \int_0^1 \int_Q \frac{(\tilde{M}(d\lambda) \Delta_{\#}^{-1} \mathbf{e}_k)_l}{s + \lambda} d\mathbf{y} \tag{3.12}$$

On the other hand, for $|s| > 1$, the geometric expansion of the middle term near $s = \infty$ in (3.11) results in the following expression

$$K_{kl}(s) = \int_Q \left[\sum_{m=0}^{\infty} \left(-\frac{1}{s}\right)^m (\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_k \right] \cdot \mathbf{e}_l d\mathbf{y} = \sum_{m=0}^{\infty} \frac{\tilde{\lambda}_{kl}^m}{(-s)^m} \tag{3.13}$$

where $\tilde{\lambda}_{kl}^m$ is defined as $\tilde{\lambda}_{kl}^m := \int_Q ((\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_k)_l d\mathbf{y}$. □

For the three-dimensional space $n = 3$, the expansion (3.12) can be cast in the matrix form

$$\mathbf{K}(s) = \sum_{m=0}^{\infty} \frac{\tilde{\Lambda}_m}{(-s)^m} \tag{3.14}$$

with the matrix-valued moments defined as

$$\tilde{\Lambda}_m := \left(\int_Q (\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_1 d\mathbf{y} \quad \int_Q (\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_2 d\mathbf{y} \quad \int_Q (\Gamma_\chi)^m \Delta_{\#}^{-1} \mathbf{e}_3 d\mathbf{y} \right) \tag{3.15}$$

3.2. Relationships between two representations and characterization of the microstructural information on permeability

The calculations in the previous section reveal that the variable $s := 1/(z - 1)$ is the natural one to use. Because of this, we will consider \mathbf{K} as a function of s . Note that s maps $(-\infty, 0]$ on the z -plane to $[-1, 0]$ on the s -plane. The following properties of $\mathbf{K}(s)$ can be easily deduced from the results in proposition 3.1.

- (i) $\mathbf{K}(s)$ is holomorphic in $\mathbb{C} \setminus \left[-\frac{2E_2^2}{1+2E_2^2}, -\frac{1}{1+2E_1^2}\right]$.
- (ii) $\frac{\mathbf{K}(s) - (\mathbf{K}(s))^*}{s - \bar{s}} \geq 0$ for all $Im(s) \neq 0$
- (iii) $\mathbf{K}(s) \geq 0$ for $\mathbb{R} \ni s > 0$ because $s > 0$ iff $\mathbb{R} \ni z > 1$.

Then by the representation theorem in [20, theorem 3.1], there exists a monotonically increasing matrix-valued function $\boldsymbol{\sigma}(t)$, matrices $\mathbf{A} \geq 0$ and $\mathbf{C} \geq 0$ such that $\int_{+0}^{\infty} \frac{d\boldsymbol{\sigma}}{1+t} < \infty$, $\mathbf{A} + \mathbf{C} + \int_{+0}^{\infty} \frac{d\boldsymbol{\sigma}}{1+t} > 0$ and

$$\mathbf{K}(s) = \mathbf{A} + \mathbf{C}s + \int_{+0}^{\infty} \frac{s}{s+t} d\boldsymbol{\sigma}(t), \tag{3.16}$$

As $s \rightarrow \infty$, $z \rightarrow 1$ and hence $\mathbf{K} \rightarrow \mathbf{K}(z = 1)$. Therefore, we must have $\mathbf{C} = \mathbf{0}$. Moreover, $\mathbf{A} = \mathbf{K}(s = 0) = \mathbf{K}^{(D)}$. Also, since $\mathbf{K}(s)$ is holomorphic in $\mathbb{C} \setminus \left[-\frac{2E_2^2}{1+2E_2^2}, -\frac{1}{1+2E_1^2}\right]$, we have

$$\mathbf{K}(s) = \mathbf{K}^{(D)} + \int_{1/(1+2E_1^2)}^{2E_2^2/(1+2E_2^2)} \frac{s}{s+t} d\boldsymbol{\sigma}(t), \tag{3.17}$$

which is valid for all $s \in \mathbb{C} \setminus \left[-\frac{2E_2^2}{1+2E_2^2}, -\frac{1}{1+2E_1^2}\right] \subset (-1, 0)$. To compare with (3.14), which is valid only for $|s| > 1$, we expand (3.17) near $s = \infty$ to obtain the following series expansion

$$\mathbf{K}(s) = \mathbf{K}^{(D)} + \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{s}\right)^{m+1} \boldsymbol{\mu}_m^\sigma \tag{3.18}$$

where $\boldsymbol{\mu}_m^\sigma$ is the m -th moment of the measure $d\boldsymbol{\sigma}$. Equating the coefficients term by term with (3.14) leads to the following relation between $\boldsymbol{\mu}_m^\sigma$ and the ‘geometrical information’ coefficients in (3.15)

$$\mathbf{K}^{(D)} + \boldsymbol{\mu}_0^\sigma = \tilde{\boldsymbol{\Lambda}}_0 = \mathbf{K}(s = \infty), \text{ i.e. } \boldsymbol{\mu}_0^\sigma = \mathbf{K}(z = 1) - \mathbf{K}^{(D)} \tag{3.19}$$

$$\boldsymbol{\mu}_m^\sigma = \tilde{\boldsymbol{\Lambda}}_m, \quad m \geq 1 \tag{3.20}$$

Recall that \mathbf{K} can be regarded as a function of s as well as a function of z , $s := 1/(z - 1)$. In particular, the first moment $\boldsymbol{\mu}_1^\sigma$ can be calculated explicitly as follows

$$\tilde{\lambda}_{kl}^1 = (\Gamma_\chi \mathbf{u}^k(\mathbf{y}; 1), \mathbf{u}^l(\mathbf{y}; 1))_Q = 2\mu_1 \int_Q \chi_2 e(\mathbf{u}^k(\mathbf{y}; 1)) : \overline{e(\mathbf{u}^l(\mathbf{y}; 1))} d\mathbf{y} \tag{3.21}$$

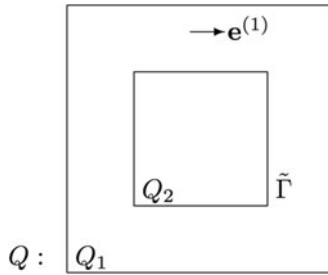


Figure 2. Computational domain.

4. Numerical verification

The computational domain with $Q = (0, 1)^2$, $Q_2 = [1/4, 3/4]^2$ and $Q_1 = Q \setminus Q_2$ is illustrated in figure 2 and is chosen in the first two numerical examples, (4.1) and (4.2).

We consider three cases: (1) Q_2 is a solid obstacle, (2) Q_2 is a bubble and (3) Q_2 is another fluid.

For case (1), we find $(\mathbf{u}_1, p_1) \in \mathbf{V}_1 \times P_1$, such that

$$\begin{cases} (e(\mathbf{u}_1), e(\mathbf{v})) - (p_1, \text{div } \mathbf{v}) = (\mathbf{e}_1, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_1, \\ (q, \text{div } \mathbf{u}_1) = 0 & \forall q \in P_1, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \mathbf{V}_1 &= \{ \mathbf{v} \in H^1(Q_1)^2 \mid \mathbf{v}|_{\partial Q_2} = 0, \text{ vis } Q\text{-periodic} \}, \\ P_1 &= \{ q \in L_0^2(Q_1) \mid q = \text{div } \mathbf{v} \text{ for some } \mathbf{v} \in \mathbf{V}_1 \}. \end{aligned}$$

For case (2), we find $(\mathbf{u}_2, p_2) \in \mathbf{V}_2 \times P_2$, such that

$$\begin{cases} (e(\mathbf{u}_2), e(\mathbf{v})) - (p_2, \text{div } \mathbf{v}) = (\mathbf{e}_1, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_2, \\ (q, \text{div } \mathbf{u}_2) = 0 & \forall q \in P_2, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} \mathbf{V}_2 &= \{ \mathbf{v} \in H^1(Q_1)^2 \mid \mathbf{v} \cdot \mathbf{n}|_{\partial Q_2} = 0, \mathbf{v} \text{ is } Q\text{-periodic} \}, \\ P_2 &= \{ q \in L_0^2(Q_1) \mid q = \text{div } \mathbf{v} \text{ for some } \mathbf{v} \in \mathbf{V}_2 \}. \end{aligned}$$

For case (3), we set $\mu_1 = 1$ and $\mu_2 = \mu$. We find $(\mathbf{u}_3, p_3) \in \mathbf{V}_3 \times P_3$, such that

$$\begin{cases} (\mu e(\mathbf{u}_3), e(\mathbf{v})) - (p_3, \text{div } \mathbf{v}) = (\mathbf{e}_1, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_3, \\ (q, \text{div } \mathbf{u}_3) = 0 & \forall q \in P_3, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} \mathbf{V}_3 &= \{ \mathbf{v} \in H^1(Q)^2 \mid \mathbf{v} \cdot \mathbf{n}|_{\partial Q_2} = 0, \text{ vis } Q\text{-periodic} \}, \\ P_3 &= \{ q \in L_0^2(Q) \mid q = \text{div } \mathbf{v} \text{ for some } \mathbf{v} \in \mathbf{V}_3 \}. \end{aligned}$$

The computation is done on square grids. The first level grid consists of 12 squares, for the first two cases. Each square is subdivided into 4 sub-squares to

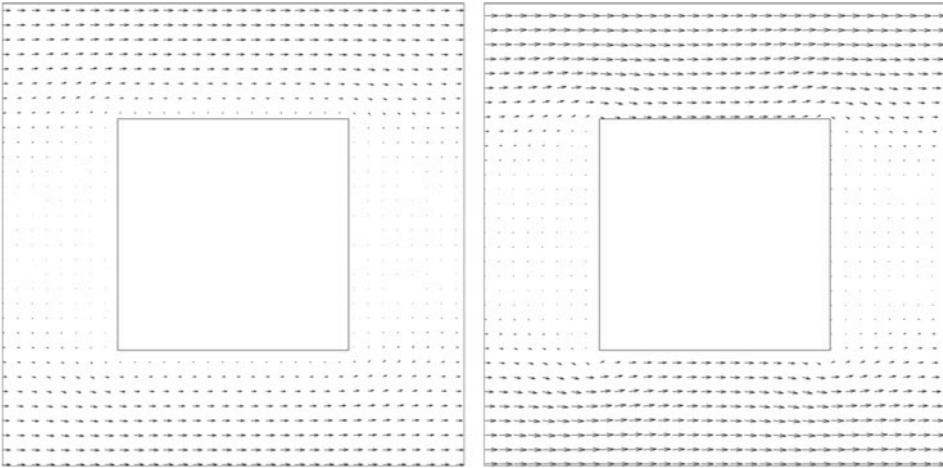


Figure 3. Velocity field \mathbf{u}^1 for a solid obstacle Q_2 (4.1), and for a slippery bubble Q_2 (4.2).

get the next level grid, $\mathcal{T}_h = \{T\}$. We use the $Q_{5,4}^{1,0} \times Q_{4,5}^{0,1}$ velocity finite-element space with the $Q_{4,4}^{0,0}$ pressure finite-element space. Here $Q_{5,4}^{1,0}$ means the space of polynomials of degree at most 5 in y_1 and of degree at most 4 in y_2 which is C^1 in y_1 -direction and C^0 in y_2 -direction. That is,

$$Q_{5,4}^{1,0} = \left\{ u_1|_T = \sum_{i=0}^5 \sum_{j=0}^4 c_{ij} y_1^i y_2^j \mid u_1 \text{ and } \partial_{y_1} u_1 \in C^0(Q_1), \text{ and } Q\text{-periodic} \right\},$$

$$Q_{4,4}^{0,0} = \left\{ p|_T = \sum_{i=0}^4 \sum_{j=0}^4 c_{ij} y_1^i y_2^j \mid p \in C_0^0(Q), \text{ and } Q\text{-periodic} \right\}.$$

We note that $\text{div}(Q_{5,4}^{1,0} \times Q_{4,5}^{0,1}) = Q_{4,4}^{0,0}$. Therefore, the finite element velocity is also pointwise divergence-free. We plot the velocity field of these two problems in figure 3. We can see the magnitude of the latter is much bigger, as the resistance from a slippery bubble is much less.

In figures 4 and 5, we plot the two velocity fields of two-fluid flow (4.3) for two viscosity coefficients μ_2 . When μ_2 is big, the sticky inner fluid flows less and drags the outer fluid near the interface. When μ_2 approaches infinity, the inner fluid stops and it posts a zero Dirichlet boundary condition for on tangential velocity of the outer fluid at the inner boundary $\tilde{\Gamma} = \partial Q_2$. The model of a solid obstacle (4.1) is a limit case of the model of two-fluid (4.3) when $\mu_2 \rightarrow \infty$. We can compare the left chart of figure 3 and the left chart of figure 4.

When μ_2 approaches zero, the inner fluid flow freely which produces little drag on the outer fluid. In theory, the force inside fluid Q_2 may even push outer fluid somewhat. But due to the zero outflow boundary condition on the velocity at ∂Q_2 , such a force would be balanced by its left portion and right portion of an edge of

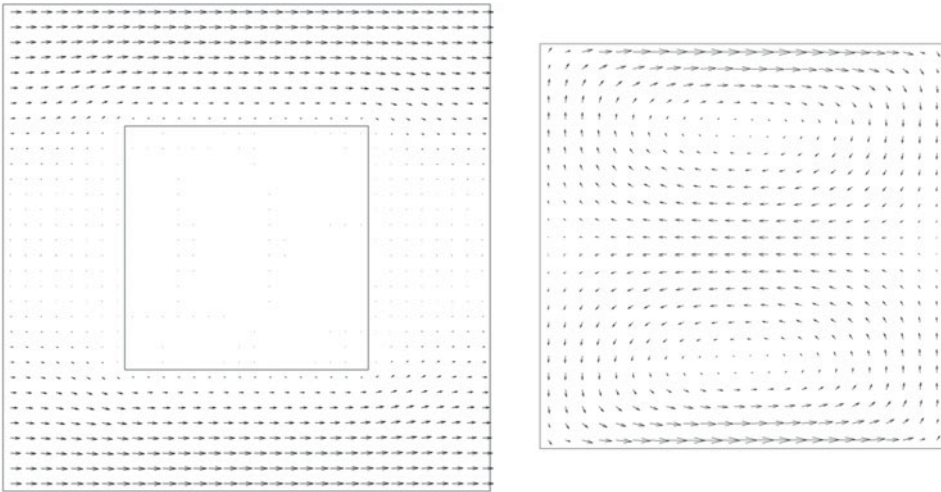


Figure 4. Velocity field \mathbf{u}_3 for two-fluid flow (4.3) with $\mu_2 = 10^2$ on Q (left), on Q_2 (right, scaled by 200).

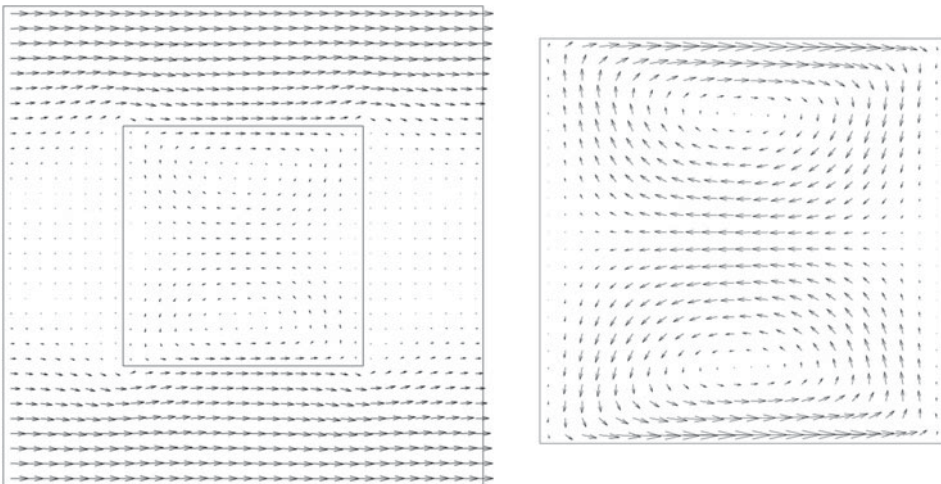


Figure 5. Velocity field \mathbf{u}_3 for two-fluid flow (4.3) with $\mu_2 = 10^{-2}$ on Q (left), on Q_2 (right, scaled by 2).

∂Q_2 . It is equivalent to zero tangential stress boundary on the outer flow. That is, model of a slippery bubble (4.2) is a limit model of two-fluid (4.3) with $\mu_2 \rightarrow 0$. We may compare the right chart of figure 3 and the left chart of figure 5.

Table 1. Computed permeability k_{11} by (4.4)–(4.6)

Level	(4.1)	$\mu_2 = 10^4$	(4.3) $\mu_2 = 1$	$\mu_2 = 10^{-4}$	(4.2)
1	0.0105	0.0105	0.0122	0.0140	0.0140
2	0.0119	0.0119	0.0144	0.0181	0.0181
3	0.0125	0.0125	0.0154	0.0209	0.0209
4	0.0128	0.0128	0.0159	0.0228	0.0228
5	0.0129	0.0129	0.0161	0.0240	0.0240

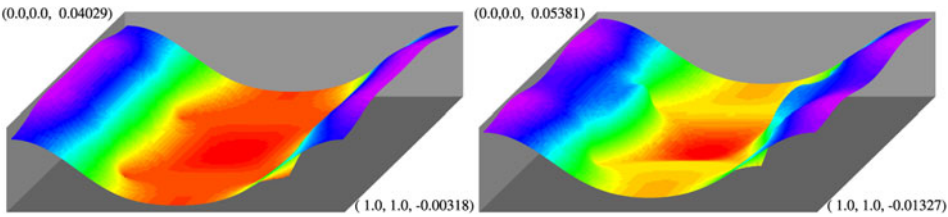


Figure 6. First component of velocity \mathbf{u}_3 , from (4.3), for $\mu = 10^2$ and $\mu_2 = 10^{-2}$.

The homogenized permeability tensor $\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ is computed by

$$k_{11} = \frac{1}{|Q|} \int_{Q \setminus Q_2} \mathbf{u}_1 \cdot \mathbf{e}_1 \, dy \quad \text{for (4.1),} \tag{4.4}$$

$$k_{11} = \frac{1}{|Q|} \int_{Q \setminus Q_2} \mathbf{u}_2 \cdot \mathbf{e}_1 \, dy \quad \text{for (4.2),} \tag{4.5}$$

$$k_{11} = \frac{1}{|Q|} \int_Q \mathbf{u}_3 \cdot \mathbf{e}_1 \, dy \quad \text{for (4.3).} \tag{4.6}$$

Due to the symmetry, in all our examples we have $k_{11} = k_{22}$ and $k_{12} = k_{21} = 0$.

To verify the convergence results stated in (2.55) and (2.75), we solve the two-fluid problem (4.3) with $\mu_1 = 1$ and $\mu_2 = 10^{-4}, 1, 10^4$. In table I, this model is between the two ‘limiting’ models (4.1) and (4.2).

To see how viscosity μ_2 influences the flow, we plot $(\mathbf{u}_3)_1$ in figure 6 for two different μ_2 with $\mu_1 = 1$.

Though the magnitude of $(\mathbf{u}_3)_1$ is way larger than that of $(\mathbf{u}_3)_2$, their corresponding stress are about the same size. In figure 7, we plot them for a comparison. We plot the stress intensity $|\nabla \mathbf{u}^1|$ in figure 8.

Finally we compute the energy of the two-fluid flow,

$$E(Q_2) = \int_{Q_2} \mu(\mathbf{y}) e(\mathbf{u}_3) : e(\mathbf{u}_3) \, dy, \tag{4.7}$$

$$E(Q) = \int_Q \mu(\mathbf{y}) e(\mathbf{u}_3) : e(\mathbf{u}_3) \, dy. \tag{4.8}$$

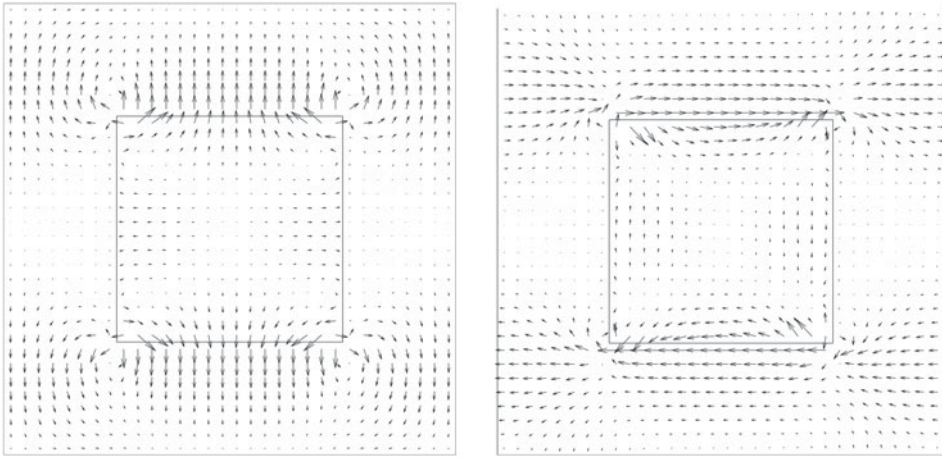


Figure 7. Stress $\nabla(u_3)_1$, and $\nabla(u_3)_2$ for (4.3) with $\mu_2 = 10^2$.

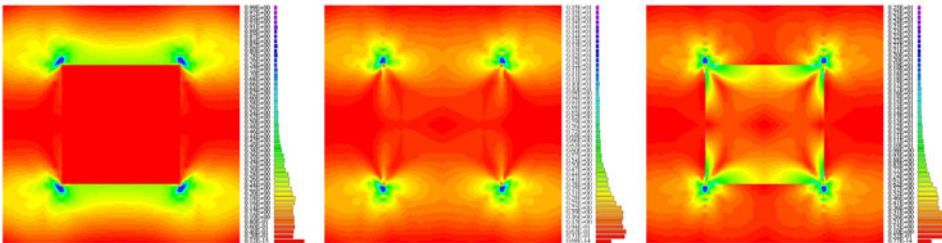


Figure 8. Stress intensity $|e((\mathbf{u}_3)_1)|$ in (4.3) with $\mu_2 = 10^2$, $\mu_2 = 1$, $\mu_2 = 10^{-2}$.

The homogenized permeability can also be computed by the energy,

$$k_{ij} = \frac{1}{|Q|} \int_Q \mu(\mathbf{y}) e(\mathbf{u}_3) : e(\mathbf{u}_3) \, d\mathbf{y}. \tag{4.9}$$

In table II, we demonstrate the equivalence of these two definitions for k_{11} .

5. Conclusion and future work

In this paper, we show that the permeability of a porous material [40] and that of a bubbly fluid [29] are limiting cases of the complexified version of the two-fluid models posed in [29]. We assume the viscosity of the inclusion fluid is $z\mu_1$ and the viscosity of the hosting fluid is μ_1 , $z \in \mathbb{C}$. The proof is carried out by construction of solutions for large $|z|$ and small $|z|$ by an iteration process similar with the one used in [16, 21] and analytic continuation. Moreover, we also show that for a fixed microstructure, the permeabilities of these three cases share the same IRF (3.17) with different values of s , as long as the ‘contrast parameter’ $s := 1/(z - 1)$ is not in the interval $\left[-\frac{2E_2^2}{1+2E_2^2}, -\frac{1}{1+2E_1^2}\right]$, where the constants E_1 and E_2 are the extension constants that depend on the geometry of Q_1 , Q_2 and Q . For the mixture with

Table 2. Computed permeability k_{11} both ways and energy

Level	k_{11} (4.6)	k_{11} (4.9)	$E(Q_2)$ (4.7)	$E(Q_2)/E(Q)$ (4.8)
For \mathbf{u}_3 in (4.3) with $\mu_2 = 10^2$				
1	0.107×10^{-1}	0.107×10^{-1}	0.952×10^{-4}	0.888×10^{-2}
2	0.121×10^{-1}	0.121×10^{-1}	0.811×10^{-4}	0.670×10^{-2}
3	0.127×10^{-1}	0.126×10^{-1}	0.676×10^{-4}	0.534×10^{-2}
4	0.129×10^{-1}	0.129×10^{-1}	0.604×10^{-4}	0.468×10^{-2}
5	0.130×10^{-1}	0.130×10^{-1}	0.570×10^{-4}	0.438×10^{-2}
For \mathbf{u}_3 in (4.3) with $\mu_2 = 1$				
1	0.122×10^{-1}	0.122×10^{-1}	0.752×10^{-3}	0.615×10^{-1}
2	0.144×10^{-1}	0.144×10^{-1}	0.135×10^{-2}	0.936×10^{-1}
3	0.154×10^{-1}	0.154×10^{-1}	0.175×10^{-2}	0.113×10^0
4	0.159×10^{-1}	0.159×10^{-1}	0.200×10^{-2}	0.125×10^0
5	0.162×10^{-1}	0.162×10^{-1}	0.215×10^{-2}	0.132×10^0
For \mathbf{u}_3 in (4.3) with $\mu_2 = 10^{-4}$				
1	0.140×10^{-1}	0.140×10^{-1}	0.432×10^{-6}	0.308×10^{-4}
2	0.181×10^{-1}	0.181×10^{-1}	0.109×10^{-5}	0.602×10^{-4}
3	0.209×10^{-1}	0.209×10^{-1}	0.183×10^{-5}	0.875×10^{-4}
4	0.228×10^{-1}	0.228×10^{-1}	0.254×10^{-5}	0.111×10^{-3}
5	0.240×10^{-1}	0.240×10^{-1}	0.316×10^{-5}	0.131×10^{-3}

bubbles, $s = -1$ and thus

$$K^{(B)} = \mathbf{K}^{(D)} + \int_{1/(1+2E_1^2)}^{2E_2^2/(1+2E_2^2)} \frac{1}{1-t} d\boldsymbol{\sigma}(t) \tag{5.1}$$

Also, we note that the matrix-valued measure in (3.10) has a Dirac measure sitting at $\lambda = 0$ with strength equal to $\mathbf{K}^{(D)}$. The permeability $\mathbf{K}^{(D)}$ is related to the measure in the sense that the zero-th moment of the measure is equal to $\mathbf{K}(z = 1) - \mathbf{K}^{(D)}$.

Clearly, the positive matrix-valued measure $d\boldsymbol{\sigma}$ is independent of s and it characterizes how the geometry influences the permeability. We have shown that this measure is related to the projection measure of the self-adjoint operator Γ_χ and its moments can be computed by equation (3.20).

Because the IRF is valid for most of s on the complex plane, the IRF will be useful in the study of two-fluid mixture with complex viscosities such as dehomogenization for these fluid. Also, the integration limits in the IRF should imply bounds on the permeability tensors. We will explore the results of this paper in these direction in the future.

Acknowledgments

The work of CB and MYO was partially sponsored by the US National Science foundation via grants NSF-DMS-1413039 and NSF-DMS-1821857.

References

- 1 G. Allaire. Homogenization of the Stokes flow in a connected porous medium. *Asymptotic Anal.* **2** (1989), 203–222.
- 2 G. Allaire. Continuity of the Darcy's law in the low-volume fraction limit. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **18** (1991), 475–499.
- 3 G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.* **23** (1992), 1482–1518.
- 4 G. Allaire. *Homogenization in porous media* (France: CEA-EDF-INRIA school on homogenization, 2010).
- 5 J. Auriault, L. Borne and R. Chambon. Dynamics of porous saturated media, checking of the generalized law of Darcy. *J. Acoust. Soc. Am.* **77** (1985), 1641–1650.
- 6 M. Avellaneda and A. J. Majda. Stieltjes integral representation and effective diffusivity bounds for turbulent transport. *Phys. Rev. Lett.* **62** (1989), 753.
- 7 M. Avellaneda and A. J. Majda. An integral representation and bounds on the effective diffusivity in passive advection by laminar and turbulent flows. *Commun. Math. Phys.* **138** (1991), 339–391.
- 8 M. Avellaneda and S. Torquato. Rigorous link between fluid permeability, electrical conductivity, and relaxation times for transport in porous media. *Phys. Fluids A* **3** (1991), 2529–2540.
- 9 A. Y. Beliaev and S. Kozlov. Darcy equation for random porous media. *Commun. Pure Appl. Math.* **49** (1996), 1–34.
- 10 A. Bensoussan, J.-L. Lions and G. Papanicolaou. *Asymptotic analysis for periodic structures*. Vol. 374 (Providence, RI: American Mathematical Society, 2011).
- 11 D. J. Bergman. The dielectric constant of a composite material – a problem in classical physics. *Phys. Rep.* **43** (1978), 377–407.
- 12 M. A. Biot. Mechanics of deformation and acoustic propagation in porous media. *J. Appl. Phys.* **33** (1962), 1482–1498.
- 13 S. Brenner and R. Scott. *The mathematical theory of finite element methods*. Vol. 15 (New York: Springer Science & Business Media, 2007).
- 14 H. Brinkman. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Flow Turbul. Combust.* **1** (1949), 27–34.
- 15 O. P. Bruno. The effective conductivity of strongly heterogeneous composites. *Proc. R. Soc. London, Ser. A* **433** (1991), 353–381.
- 16 O. P. Bruno and P. H. Leo. On the stiffness of materials containing a disordered array of microscopic holes or hard inclusions. *Arch. Ration. Mech. Anal.* **121** (1993), 303–338.
- 17 D. Cioranescu, P. Donato and H. I. Ene. Homogenization of the Stokes problem with non-homogeneous slip boundary conditions. *Math. Methods Appl. Sci.* **19** (1996), 857–881.
- 18 C. Conca. On the application of the homogenization theory to a class of problems arising in fluid mechanics. *J. Math. Pures Appl.* **64** (1985), 31–75.
- 19 H. P. G. Darcy. *Les fontaines publiques de la ville de dijon. Exposition et application des principes à suivre et des formules à employer dans les questions de distribution d'eau, etc* (Paris: V. Dalamont (Ed.), 1856).
- 20 Y. Dyukarev and V. Katsnelson. Multiplicative and additive classes of Stieltjes analytic matrix valued functions, and interpolation problems associated with them. *Am. Math. Soc. Transl.* **131** (1986), 55–70.
- 21 K. Golden and G. Papanicolaou. Bounds for effective parameters of heterogeneous media by analytic continuation. *Commun. Math. Phys.* **90** (1983), 473–491.
- 22 D. L. Johnson, J. Koplik and R. Dashen. Theory of dynamic permeability and tortuosity in fluid-saturated porous media. *J. Fluid Mech.* **176** (1987), 379–402.
- 23 Y. Kantor and D. J. Bergman. Elastostatic resonances – a new approach to the calculation of the effective elastic constants of composites. *J. Mech. Phys. Solids* **30** (1982), 355–376.
- 24 T. Kato, M. Mitrea, G. Ponce and M. Taylor. Extension and representation of divergence-free vector fields on bounded domains. *Math. Res. Lett.* **7** (2000), 643–650.
- 25 J. B. Keller. Darcy's law for flow in porous media and the two-space method. Tech. Rep. (CA: Stanford University, 1980).

- 26 T. Lévy. Fluid flow through an array of fixed particles. *Int. J. Eng. Sci.* **21** (1983), 11–23.
- 27 B. V. Limaye. Banach space-valued analytic functions. In *Spectral Perturbation and Approximation with Numerical Experiments*. Centre for Mathematical Analysis, pp. 44–60 (Canberra, Australia: The Australian National University, 1987).
- 28 J.-L. Lions. *Some methods in the mathematical analysis of systems and their control* (Beijing: Science Press, 1981).
- 29 R. Lipton and M. Avellaneda. Darcy’s law for slow viscous flow past a stationary array of bubbles. *Proc. R. Soc. Edinburgh Sect. A: Math.* **114** (1990), 71–79.
- 30 T. S. Lundgren. Slow flow through stationary random beds and suspensions of spheres. *J. Fluid Mech.* **51** (1972), 273–299.
- 31 G. W. Milton. *The Theory of Composites* (ed. G. W. Milton), p. 748, ISBN 0521781256 (Cambridge, UK: Cambridge University Press, 2002).
- 32 S. P. Neuman. Theoretical derivation of Darcy’s law. *Acta Mech.* **25** (1977), 153–170.
- 33 G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.* **20** (1989), 608–623.
- 34 O. Oleinik, A. S. Shamaev and G. A. Yosifian. *Mathematical problems in elasticity and homogenization*, 1 edn. Studies in mathematics and its applications, vol. 26 (Amsterdam: North Holland, 1992).
- 35 M.-J. Y. Ou. On reconstruction of dynamic permeability and tortuosity from data at distinct frequencies. *Inverse Probl.* **30** (2014), 095002.
- 36 M. Poreh and C. Elata. An analytical derivation of Darcy law. *Israel J. Tech.* **4** (1966), 214–217.
- 37 P. G. Saffman. On the boundary condition at the surface of a porous medium. *Stud. Appl. Math.* **50** (1971), 93–101.
- 38 E. Sanchez-Palencia. *Non-homogeneous media and vibration theory*. Lecture Notes in Physics (Berlin, Heidelberg: Springer, 1980).
- 39 C. K. Tam. The drag on a cloud of spherical particles in low Reynolds number flow. *J. Fluid Mech.* **38** (1969), 537–546.
- 40 L. Tartar. Incompressible fluid flow in a porous medium-convergence of the homogenization process. *Non-Homogeneous Media and Vibration Theory* (1980).
- 41 I. Tice. *From stokes flow to Darcy’s law* (Pittsburgh: CNA Working Group on Homogenization, Carnegie Mellon University, 2014).