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CHARACTERIZATION OF PROJECTIVE QUANTALES

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Abstract

It is proved that a quantale is projective if and only if it is isomorphic to a derived tensor quantale over a completely distributive sup-lattice. Furthermore, an intrinsic characterization of projectivity is given in terms of inertial sup-lattices and derivations of quantales.

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1. Introduction

Quantales (alias 'quantum locales') were introduced by Mulvey [16] as models for the logic of quantum mechanics, and as a substitute for the spectrum in the case of a noncommutative C^*-algebra [6, 11, 13, 17, 18]. Their relationship to Girard's linear logic [9], and its noncommutative version sketched by Girard in his seminar lectures (Montreal 1987), was clarified by Yetter [30] and pursued, for example, by Rosenthal [24, 25]. Furthermore, quantales have been shown to provide a proper language for the study of Penrose tilings [7, 19], noncommutative topology [5, 8], étale groupoids and inverse semigroups [23], and process semantics [1, 22].

Joyal and Tierney [10] introduced the category **Sup** of sup-lattices, a rigid tensor category with strong analogies to the category **Ab** of abelian groups. Quantales can be regarded as semigroup objects in **Sup**, just as associative rings are the semigroup objects in **Ab**. In other words, replacing abelian groups by sup-lattices is all that has to be done in order to switch from the additive world of ring theory to that of quantales.

The problem of classifying projective quantales was raised in 2002 by Li *et al.* [14]. Two decades earlier, several characterizations of projective sup-lattices were given by Niefield [20]. Banaschewski [2] extended Scott's characterization of injective

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T_0-spaces [28] to locales by identifying projective frames as stably supercontinuous complete lattices. Banaschewski and Niefield's succinct proof [4] of this result led to the natural question whether a similar characterization might be valid for projective quantales [14]. Li *et al.* [14] derived a necessary criterion (complete distributivity and weak stability), while Kruml and Paseka [12] proved that these conditions cannot suffice. Paseka [21] proposed a general view in the spirit of Banaschewski [3], but the missing condition for projectivity of quantales has not been found (see [12, Section 4]).

In this paper we close this gap by providing a complete characterization of projective quantales. Guided by the analogy between quantales and associative rings, some new concepts for quantales are developed for that purpose. We introduce semidirect products for sup-lattices and analyse their relationship to modules over locales (Propositions 3.7 and 4.4). Proposition 3.7 states that every ideal I of a projective sup-lattice L gives rise to a semidirect product $L = A \ltimes I$ with a unique sup-lattice A. This will be applied to quantales Q with an *inertial* sup-lattice A, which means that $Q = A \ltimes Q^2$, and that an associated graded quantale $\mathrm{gr}\, Q$ can be constructed. Moreover, there is an epimorphism $p : \mathrm{gr}\, Q \twoheadrightarrow Q$ of quantales. Then we focus our attention upon a special case where p is determined by a derivation of Q which lifts to what we call a *radical map* of $\mathrm{gr}\, Q$. This leads to an explicit description of *derived quantales*, that is, special retracts of graded quantales with some analogy to associative algebras twisted by a derivation (Theorem 4.6). If $\mathrm{gr}\, Q$ is a tensor quantale, we speak of a *derived tensor quantale*.

Our main result (Theorem 5.4) states that, up to isomorphism, projective quantales Q coincide with derived tensor quantales with Q/Q^2 projective as a sup-lattice. Even if Q/Q^2 is the two-element lattice, this characterization yields a multitude of nonfree projective quantales, one for each subsemigroup N of the additive semigroup $\mathbb{N}^+$ of positive integers. In the simplest case $N = \mathbb{N}^+$, the projective quantale Q is an infinite chain $\{1 > 1^2 > 1^3 > \cdots > 0\}$. The more general case $|Q/Q^2| = 2$ with $n\mathbb{N}^+ \subset N$ for some $n > 0$ is related to ideal lattices of orders in a skew-field (see Example 5.7; and [26, Satz 15.1]). Our characterization exhibits an analogy between projective quantales and completed tensor algebras over a projective bimodule which are projective in a category of algebras over a local ring (see [27, Theorem 3]). An intrinsic characterization of projective quantales Q will be given in terms of a derivation $d : Q \to Q^2$ (Corollary 5.5).

2. The category of sup-lattices

Let **Sup** denote the category of *sup-lattices* [10]. Objects of **Sup** are complete lattices; morphisms are maps $f : L \to M$ which satisfy $f(\bigvee A) = \bigvee f(A)$ for subsets $A \subset L$. For a sup-lattice L, we set $1 = 1_L := \bigvee L$ and $0 = 0_L := \bigvee \varnothing$ for the greatest and smallest element, respectively. The morphisms $L \to M$ between two sup-lattices form a sup-lattice $Q(L, M)$ with pointwise supremum $(\bigvee F)(x) := \bigvee\{f(x) \mid f \in F\}$ for $F \subset Q(L, M)$. The composition of morphisms respects joins:

$$\Big(\bigvee f_i\Big)g = \bigvee(f_i g), \quad f\Big(\bigvee g_i\Big) = \bigvee(f g_i). \tag{2.1}$$

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There is a natural duality

$$Q(L,M) \cong Q(M^\circ, L^\circ), \tag{2.2}$$

where M° denotes the sup-lattice with the dual ordering. In fact, every morphism $f : L \to M$ in **Sup** has a right adjoint morphism $f^\circ : M^\circ \to L^\circ$ which is uniquely determined by the equivalence

$$f(x) \leqslant y \;\Leftrightarrow\; x \leqslant f^\circ(y)$$

for all $x \in L$ and $y \in M$. Thus $(f^\circ)^\circ = f$. The self-dual sup-lattice $\mathbb{1} := \{0,1\}$ with $0 < 1$ satisfies $Q(\mathbb{1}, L) \cong L$ and $Q(L, \mathbb{1}) \cong L^\circ$.

As observed in [10], the category **Sup** behaves somewhat similarly to the category **Ab** of abelian groups. The intuition for the proof of our main result (Theorem 5.4) stems from this analogy. Therefore, it might be helpful to sketch the main points of the relationship.

Equations (2.1) correspond to the bilinearity of composition, where the join $\bigvee$ interprets a kind of infinite sum. Accordingly, the biproduct in **Ab** (see [15, VIII.2]) admits an infinite analogue in **Sup**, namely, the cartesian product $\prod_{i\in I} L_i$ with the *projections* $p_j \in Q(\prod_{i\in I} L_i, L_j)$ and *injections* $e_j \in Q(L_j, \prod_{i\in I} L_i)$ determined by $e_j^\circ = p_j$. In fact, these maps satisfy the equations

$$p_i e_j = 0 \quad \text{for } i \neq j, \qquad p_i e_i = 1, \qquad \bigvee_{i\in I} e_i p_i = 1 \tag{2.3}$$

for all $i, j \in I$, in complete analogy with the biproduct in **Ab**. The term *biproduct* is justified by the following proposition.

Proposition 2.1. *The cartesian product $\prod_{i\in I} L_i$ of sup-lattices L_i is a product with respect to the projections p_i, and a coproduct with respect to the injections e_i.*

Proof. With $L := \prod_{i\in I} L_i$, let $f_i : M \to L_i$ be a morphism for each $i \in I$. Then $f := \bigvee_{i\in I} e_i f_i$ satisfies $p_i f = f_i$ for all $i \in I$. Together with (2.3), the latter equations yield $f = (\bigvee_{i\in I} e_i p_i) f = \bigvee_{i\in I} e_i f_i$. So the $p_i : L \to L_i$ define a product. The assertion for the coproduct follows by symmetry. □

Because of this self-duality, we write $\bigoplus_{i\in I} L_i$ instead of $\prod_{i\in I} L_i$, or $L_1 \oplus L_2$ if $I = \{1,2\}$. If $L_i = L$ for all $i \in I$, we keep the notation L^I. The neutral element with respect to the biproduct is the zero sup-lattice $0 := \{0\}$. Morphisms between biproducts are given by infinite matrices:

$$Q\Big(\bigoplus_{i\in I} L_i, \bigoplus_{j\in J} M_j\Big) \cong \bigoplus_{i\in I} \bigoplus_{j\in J} Q(L_i, M_j).$$

As **Sup** has difference kernels, the category **Sup** is complete, hence cocomplete by duality. Recall that a monomorphism is said to be *regular* if it arises as a difference kernel (similarly for epimorphisms). We write $\rightarrowtail$ (respectively, $\twoheadrightarrow$) for monomorphisms (epimorphisms) in **Sup**.

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Proposition 2.2. *Monomorphisms and epimorphisms in* **Sup** *are regular.*

Proof. Let $f : L \twoheadrightarrow M$ be epic in **Sup**. For any $z \in M$, we have $ff^\circ(z) \leqslant z$. Hence $f(x) \leqslant z$ implies $f(x) = ff^\circ f(x) \leqslant ff^\circ(z)$ for all $x \in M$. Choose $g, h \in Q(M, \mathbb{1})$ with $g^\circ(0) = z$ and $h^\circ(0) = ff^\circ(z)$. Then $gf = hf$, and thus $g = h$. So we get $z = ff^\circ(z)$ for all $z \in M$, which proves that f is surjective.

Next we take a kernel pair $p, q : K \to L$ of f, that is, a pullback of f along f. For all $x, y \in L$ with $f(x) = f(y)$, this implies that $x = p(z)$ and $y = q(z)$ for some $z \in K$. Hence f is the difference cokernel of p, q. By duality, this completes the proof. □

In particular, Proposition 2.2 shows that monomorphisms (epimorphisms) in **Sup** are injective (surjective). Furthermore, every $f \in Q(M, N)$ admits a factorization $f : M \overset{p}{\twoheadrightarrow} \operatorname{Im} f \overset{i}{\rightarrowtail} N$ into a regular epimorphism p and a regular monomorphism i.

Any pair L, M of sup-lattices has a *tensor product* $L \otimes M$ in **Sup**, generated universally by elements $x \otimes y$ with $x \in L$ and $y \in M$ such that

$$x \otimes \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \otimes y_i), \quad \Big(\bigvee_{i \in I} x_i\Big) \otimes y = \bigvee_{i \in I} (x_i \otimes y)$$

for all $x, x_i \in L$ and $y, y_i \in M$. In other words, if we define a *bimorphism* $\beta : L \times M \to N$ with $L, M, N \in$ **Sup** to be a map whose partial maps $\beta(x, -)$ and $\beta(-, y)$ are morphisms in **Sup**, the tensor product is characterized by a natural bijection

$$\operatorname{Bi}(L \times M, N) \cong Q(L \otimes M, N), \tag{2.4}$$

where $\operatorname{Bi}(L \times M, N)$ stands for the set of bimorphisms $L \times M \to N$. Hence

$$L \otimes M \cong M \otimes L.$$

On the other hand, the tensor product can be represented as

$$L \otimes M \cong Q(L, M^\circ)^\circ. \tag{2.5}$$

That is, the simple tensors $x \otimes y \in L \otimes M$ can be regarded as morphisms $L \to M^\circ$ via

$$(x \otimes y)(z) := \begin{cases} 1 & \text{for } z = 0, \\ y & \text{for } 0 < z \leqslant x, \\ 0 & \text{for } z \nleqslant x, \end{cases} \tag{2.6}$$

and then every $f \in Q(L, M^\circ)^\circ$ can be written as

$$f = \bigvee_{x \in L} x \otimes f(x). \tag{2.7}$$

Thus (2.6) and (2.7) yield

$$x \otimes y \leqslant f \iff y \leqslant f(x)$$

for all $x \in L$, $y \in M$, and $f \in L \otimes M$. In particular, this implies that the equivalence

$$x \otimes y = 0 \iff (x = 0 \text{ or } y = 0)$$

holds for all $x \in L$ and $y \in M$.

The following result shows that tensor products of sup-lattices are easier to handle than tensor products of modules.

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Proposition 2.3. *Let L and M be sup-lattices. Then:*

(a) $0 \neq x \otimes y \leqslant x' \otimes y'$ *implies* $x \leqslant x'$ *and* $y \leqslant y'$;

(b) $x \otimes \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \otimes y_i)$;

(c) $x \neq 0,\ x \otimes y \leqslant x \otimes y'$ *implies* $y \leqslant y'$;

for all $x, x' \in L$ *and* $y, y', y_i \in M$, *and* $I \neq \varnothing$.

Proof. Assume that $0 \neq x \otimes y \leqslant x' \otimes y'$. Then $x \neq 0$ implies that $0 < y = (x \otimes y)(x) \leqslant (x' \otimes y')(x)$. Hence $x \leqslant x'$, and thus $y \leqslant (x' \otimes y')(x) = y'$. This proves (a). The remaining assertions follow immediately by (a). □

Together with (2.5) and (2.2), the natural isomorphism

$$Q(L, Q(M, N)) \cong Q(M, Q(L, N))$$

implies

$$Q(L \otimes M, N) \cong Q(L, Q(M, N)),$$

which also gives

$$(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$$

for all $L, M, N \in \mathbf{Sup}$. Furthermore, (2.5) shows that the tensor product satisfies $\mathbb{1} \otimes L \cong L \otimes \mathbb{1} \cong L$ and distributes over the biproduct:

$$\Big(\bigoplus_{i \in I} L_i\Big) \otimes L \cong \bigoplus_{i \in I} (L_i \otimes L). \tag{2.8}$$

A sup-lattice L is said to be *completely distributive* if arbitrary joins in L distribute over arbitrary meets, that is,

$$\bigvee \Big\{ \bigwedge A_i \,\Big|\, i \in I \Big\} = \bigwedge \Big\{ \bigvee_{i \in I} a_i \,\Big|\, \forall i \in I : a_i \in A_i \Big\}$$

for any family of subsets $A_i \subset L$. Recall that a sup-lattice M is said to be *projective* if every epimorphism $L \twoheadrightarrow M$ splits.

The following proposition is essentially contained in [20]. The equivalence of (a) and (b) has been generalized by several authors (see [29]). Note that a left adjoint $f_\circ : M \to L$ of a morphism $f : L \to M$ in $\mathbf{Sup}$ must be of the form

$$f_\circ(y) := \bigwedge \{ x \in L \mid f(x) \geqslant y \}.$$

Though $f_\circ$ can always be defined, the equivalence of $f_\circ(y) \leqslant x$ and $y \leqslant f(x)$ for all $x \in L$ and $y \in M$ need not be satisfied. Unless f respects meets, there is no way back from $f_\circ$ to f.

Proposition 2.4. *For a sup-lattice L, the following are equivalent:*

(a) *L is projective;*

(b) *L is completely distributive;*

(c) *for any morphism $f \in Q(L, M)$ in* $\mathbf{Sup}$, *the map $f_\circ$ belongs to $Q(M, L)$;*

(d) *L is injective.*

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Proof. (a) implies (b): Every $a \in L$ determines a unique morphism $\mathbb{1} \to L$ which maps 1 to a. So we get an epimorphism $q : \mathbb{1}^L \twoheadrightarrow L$ which splits since L is projective. Hence there is a morphism $j : L \to \mathbb{1}^L$ with $qj = 1$. Now let $(A_i)_{i\in I}$ be a family of subsets $A_i \subset L$. If T denotes the set of transversals $\{a_i \mid i \in I\}$ with $a_i \in A_i$, we have to verify that $\bigvee\{\bigwedge A_i \mid i \in I\} \geqslant \bigwedge\{\bigvee A \mid A \in T\}$, the other inequality being trivial. Since $\mathbb{1}^L$ is completely distributive, we have

$$j\Big(\bigwedge\Big\{\bigvee A\,\Big|\,A \in T\Big\}\Big) \leqslant \bigwedge j\Big(\Big\{\bigvee A\,\Big|\,A \in T\Big\}\Big) = \bigwedge\Big\{\bigvee j(A)\,\Big|\,A \in T\Big\}$$
$$= \bigvee\Big\{\bigwedge j(A_i)\,\Big|\,i \in I\Big\}.$$

Applying q, this gives

$$\bigwedge\Big\{\bigvee A\,\Big|\,A \in T\Big\} \leqslant q\Big(\bigvee\Big\{\bigwedge j(A_i)\,\Big|\,i \in I\Big\}\Big) = \bigvee q\Big(\Big\{\bigwedge j(A_i)\,\Big|\,i \in I\Big\}\Big)$$
$$\leqslant \bigvee\Big\{\bigwedge A_i\,\Big|\,i \in I\Big\}.$$

(b) implies (c): Let $f \in Q(L, M)$ be given, and let X be a subset of M. Then

$$f_\circ\Big(\bigvee X\Big) = \bigwedge\Big\{a \in L\,\Big|\,f(a) \geqslant \bigvee X\Big\} \leqslant \bigwedge\Big\{\bigvee_{x\in X} a_x\,\Big|\,\forall x \in X : a_x \in L, f(a_x) \geqslant x\Big\}$$
$$= \bigvee_{x\in X} \bigwedge\Big\{a \in L\,\Big|\,f(a) \geqslant x\Big\} = \bigvee f_\circ(X).$$

(c) implies (d): Let $f \in Q(L, M)$ be monic. Then $f_\circ \in Q(M, L)$, and $f_\circ f = 1$. Hence L is injective.

(d) implies (a): By duality, L° is projective. Hence L° is completely distributive, and thus L° is injective. Again by duality, L is projective. □

For a set X, consider the map $\delta_X : X \to \mathbb{1}^X$ with $\delta_X(x)(y) = 1$ if and only if $x = y$. Then every map $f : X \to L$ into a sup-lattice L extends uniquely to a morphism $f' : \mathbb{1}^X \to L$ of sup-lattices such that $f' \circ \delta_X = f$. Thus $X \mapsto \mathbb{1}^X$ is a functor **Set** $\to$ **Sup** which is left adjoint to the forgetful functor **Sup** $\to$ **Set**.

3. Quantales

Recall that an associative ring can be viewed as a semigroup object in the category **Ab** of abelian groups. Similarly, a semigroup object in **Sup** is said to be a *quantale*. Explicitly, this means that a quantale Q is a sup-lattice with a semigroup structure such that the left and right multiplications are morphisms in **Sup**. A *morphism* $f : Q \to Q'$ of quantales is a morphism in **Sup** which is a homomorphism of semigroups. The category of quantales will be denoted by **Quant**. A quantale Q is *unital* if it admits an element u which satisfies $ua = au = a$ for all $a \in Q$.

There is a perfect analogy between modules over a ring and modules over a quantale. By (2.1), every sup-lattice L gives rise to a unital quantale $Q(L) := Q(L, L)$

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with the composition of maps as multiplication, the *endomorphism quantale* of M. A *(left) module* over a quantale Q is a sup-lattice M together with a morphism $\rho : Q \to Q(M)$ of quantales (see [1, 12]). For $a \in Q$ and $x \in M$, we write $ax := \rho(a)(x)$. Equivalently, a Q-module M is given by a morphism $Q \otimes M \to M$, induced by $a \otimes x \mapsto ax$, which satisfies $(ab)x = a(bx)$ for all $a, b \in Q$ and $x \in M$. A module M over a unital quantale Q is said to be *unital* if $ux = x$ for all $x \in M$.

Example 3.1. Let L be a sup-lattice. By $\mathrm{Sub}(L)$ we denote the set of subsets $A \subset L$ such that $A \hookrightarrow L$ belongs to $Q(A, L)$ (that is, A is a sub-sup-lattice of L.) For a family $(A_i)_{i\in I}$ of $A_i \in \mathrm{Sub}(L)$, we define

$$\bigvee_{i\in I} A_i := \Big\{\bigvee a_i \,\Big|\, a_i \in A_i\Big\}.$$

This makes $\mathrm{Sub}(L)$ into a sup-lattice. We call $I \in \mathrm{Sub}(L)$ an *order ideal* of L if $a \leqslant b \in I$ implies $a \in I$. For $A \in \mathrm{Sub}(L)$, the order ideal of L generated by A coincides with the downset $\downarrow A := \{a \in L \mid \exists b \in A : a \leqslant b\}$. Note that every order ideal I of L is principal: $I = \downarrow\{\bigvee I\}$. For an order ideal I of L, we denote the interval $\{x \in L \mid \bigvee I \leqslant x \leqslant 1\}$ by L/I. There is a natural epimorphism of sup-lattices

$$p : L \twoheadrightarrow L/I \tag{3.1}$$

with $p(a) := a \vee \bigvee I$.

For a quantale Q and $A, B \in \mathrm{Sub}(Q)$, we set

$$AB := \Big\{\bigvee S \,\Big|\, S \subset \{ab \mid a \in A, b \in B\}\Big\}. \tag{3.2}$$

With (3.1) and (3.2), $\mathrm{Sub}(Q)$ becomes a quantale. We write A^n for the n-fold product $A \cdots A$ of $A \in \mathrm{Sub}(Q)$. Note that the Q^n form a descending sequence of subquantales of Q. If Q is unital, then $Q^2 = Q$.

To define a semidirect product of sup-lattices, we need a suitable kind of action.

Definition 3.2. Let L, M be sup-lattices. We define a *dual action* of L on M to be a map $L \times M \to M$ which satisfies:

(O1) $0x = x$;
(O2) $a \leqslant b$ implies $ax \leqslant bx$;
(O3) $x \leqslant y$ implies $ax \leqslant ay$;
(O4) $a(ax) = ax$,

for all $a, b \in L$ and $x, y \in M$.

Example 3.3. Recall that a quantale L is said to be a *frame* if its multiplication coincides with the meet, or equivalently, if $a \cdot a = a \cdot 1 = 1 \cdot a = a$ holds for all $a \in L$. Such a quantale is unital with unit 1. If L and M are sup-lattices such that L° is a frame

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and M° is a unital L°-module, the induced map $L \times M \to M$ satisfies (O1)–(O4). We call such a dual action *localic*. Explicitly, this means that $(a \vee b)x = a(bx)$, $0x = x$, and

$$\Big(\bigwedge a_i\Big) \vee a = \bigwedge (a_i \vee a), \quad \Big(\bigwedge a_i\Big)x = \bigwedge (a_i x), \quad a\Big(\bigwedge x_i\Big) = \bigwedge (a x_i)$$

for all $a, b, a_i \in L$ and $x, x_i \in M$. Note that (O1) and (O2) imply

$$x \leqslant ax \tag{3.3}$$

for all $a \in L$ and $x \in M$. In what follows, we write $a \oplus x$ for the elements $a \vee x \in L \oplus M$ with $a \in L$ and $x \in M$.

Lemma 3.4. *Let $L \times M \to M$ be a dual action of sup-lattices. Then the subset*

$$L \ltimes M := \{a \oplus ax \mid a \in L,\ x \in M\}$$

of $L \oplus M$ is $\bigwedge$-closed.

Proof. For a family of elements $a_i \oplus a_i x_i \in L \oplus M$, inequality (3.3) gives

$$\bigwedge a_i x_i \leqslant \Big(\bigwedge a_i\Big)\Big(\bigwedge a_i x_i\Big) \leqslant a_i(a_i x_i) = a_i x_i$$

for all i, hence $\bigwedge a_i x_i = (\bigwedge a_i)(\bigwedge a_i x_i)$. Therefore, $\bigwedge(a_i \oplus a_i x_i) = (\bigwedge a_i) \oplus (\bigwedge a_i x_i) = (\bigwedge a_i) \oplus (\bigwedge a_i)(\bigwedge a_i x_i) \in L \ltimes M$. □

By Lemma 3.4, any subset S of $L \ltimes M$ admits a supremum $p(\bigvee S)$ in $L \ltimes M$, where $x := \bigvee S$ is the supremum in $L \oplus M$, and $p(x)$ denotes the smallest element greater than or equal to x in $L \ltimes M$. Thus every dual action $L \times M \to M$ gives rise to an epimorphism

$$p : L \oplus M \twoheadrightarrow L \ltimes M$$

of sup-lattices. We call $L \ltimes M$ the *semidirect product* of L by M. There are natural embeddings $L \hookrightarrow L \ltimes M$ and $M \hookrightarrow L \ltimes M$ given by $a \mapsto a \oplus a0$ and $x \mapsto 0 \oplus x$, respectively. Note that M is an order ideal of $L \ltimes M$ such that

$$(L \ltimes M)/M \cong L. \tag{3.4}$$

The pairs (L, M) with a dual action of L on M form a category $\mathbf{Sup}^{(2)}$. Morphisms $(L, M) \to (L', M')$ are pairs (f, g) of maps $f \in Q(L, L')$ and $g \in Q(M, M')$ such that $f(a)g(ax) = f(a)g(x)$ for $a \in L$ and $x \in M$.

Example 3.5. Every sup-lattice L admits a natural dual action on itself, given by $a \cdot b := a \vee b$. This gives a functor

$$\Delta : \mathbf{Sup} \longrightarrow \mathbf{Sup}^{(2)}$$

with $\Delta(L) := (L, L)$. If L° is a frame, the dual action of $\Delta(L)$ is localic. On the other hand, the semidirect product gives a functor

$$S : \mathbf{Sup}^{(2)} \longrightarrow \mathbf{Sup}$$

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with $S(L, M) := L \ltimes M$. It is easy to verify that S is a left adjoint of Δ. Note that a dual action $L \times M \to M$ can be recovered from the diagram $L \hookrightarrow L \ltimes M \hookleftarrow M$ in **Sup**. That is, for $a \in L$ and $x \in M$,

$$ax = (a \vee x) \wedge 1_M \in M. \tag{3.5}$$

Proposition 3.6. *Let I be an order ideal of a sup-lattice L, and $A \in \mathrm{Sub}(L)$. With $A \vee I := \{a \vee x \in L \mid a \in A, x \in I\}$, the following are equivalent:*

(a) $L = A \ltimes I$;

(b) *$L = A \vee I$, and the natural morphism $\alpha : A \hookrightarrow L \twoheadrightarrow L/I$ is injective;*

(c) *$L = A \vee I$, and the natural morphism $\alpha : A \hookrightarrow L \twoheadrightarrow L/I$ is bijective.*

Proof. That (a) implies (b) follows by the identification $A \vee I = A \ltimes I$. Furthermore, $L = A \vee I$ implies that α is surjective. Hence (b) implies (c). Conversely, let (c) be satisfied. Then (3.5) defines a dual operation of A on I, with $ax = (a \vee x) \wedge 1_I$ for all $a \in A$ and $x \in I$. Clearly, properties (O1)–(O3) are satisfied. By definition, $x \leqslant ax \leqslant a \vee x$. Hence $ax \leqslant a(ax) \leqslant a \vee ax \leqslant a \vee x$, and thus $a(ax) = ax$. So the elements of L admit a unique representation $a \vee x$ with $a \in A$ and $x = ax \in I$, which proves (a). □

Proposition 3.7. *Let I be an order ideal of a projective sup-lattice L. There exists a unique $A \in \mathrm{Sub}(L)$ with $L = A \ltimes I$. Furthermore, A and I are retracts of L, and the dual action of A on I is localic.*

Proof. Define $q \in Q(L, I)$ by $q(x) := x \wedge 1_I$. Then q is a retraction onto I. The implication

$$x \leqslant y \;\Rightarrow\; q(x) = x \wedge q(y) \tag{3.6}$$

holds for all $x, y \in L$. Furthermore, we define a map $r : L \to L$ by

$$r(x) := \bigwedge \{a \in L \mid a \vee q(x) = x\}.$$

Then $q(x) \vee r(x) = x$. By (3.6), the inequality $x \leqslant y$ yields $(x \wedge r(y)) \vee q(x) = (x \wedge r(y)) \vee (x \wedge q(y)) = x \wedge (r(y) \vee q(y)) = x \wedge y = x$. Hence $r(x) \leqslant x \wedge r(y) \leqslant r(y)$, which shows that r is monotonic. To show that $r \in Q(L)$, let X be a subset of L. Then $\bigvee X = \bigvee_{x \in X}(r(x) \vee q(x)) \leqslant \bigvee r(X) \vee \bigvee q(X) = \bigvee r(X) \vee q(\bigvee X) \leqslant \bigvee X$. So we get $r(\bigvee X) \leqslant \bigvee r(X)$, which implies that r is a morphism of sup-lattices. For any $x \in L$, we have $r(r(x)) \leqslant r(x)$, and $r(r(x)) \vee q(x) = r(r(x)) \vee q(r(x)) \vee q(x) = r(x) \vee q(x) = x$. Hence $r(r(x)) \geqslant r(x)$, and thus r is a retraction onto $A := r(L)$. In particular, A is a projective sup-lattice, and A° is a frame.

Next we show that $L = A \ltimes I$. Every $x \in L$ has a representation $x = r(x) \vee q(x) \in A \vee I$. Furthermore, r provides a retraction of the natural morphism $\alpha : A \hookrightarrow L \twoheadrightarrow L/I$. In fact, if $a \in A$, then $r(a \vee 1_I) = r(a) \vee r(1_I) = a$. Hence $L = A \ltimes I$.

To show the uniqueness of A, let $L = B \ltimes I$ be another representation with $B \in \mathrm{Sub}(L)$. Then any $a \in A$ can be written as $a = b \vee x$ with $b \in B$ and $x \in I$. Hence

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$a = b \vee q(a)$, and thus $a = r(a) \leqslant b \leqslant a$, which gives $A \subset B$. Conversely, every $b \in B$ satisfies $b = q(b) \vee r(b)$ with $r(b) \in A \subset B$. Whence $b = r(b)$.

By (3.5), the dual action of A on I must be $ax := q(a \vee x)$. Then (3.6) gives

$$\begin{aligned} a \vee bx &= a \vee q(b \vee x) = a \vee ((b \vee x) \wedge q(a \vee b \vee x)) \\ &= (a \vee b \vee x) \wedge (a \vee q(a \vee b \vee x)) = a \vee q(a \vee b \vee x) \end{aligned}$$

for all $a, b \in A$ and $x \in I$. Applying q yields $a(bx) = (a \vee b)x$. Furthermore, $0x = q(x) = x$. Since q respects arbitrary meets, the map $A \times I \to I$ makes I° into a unital A°-module. Therefore, the dual action is localic. □

Remark 3.8. The referee has pointed out that the existence of a decomposition $L = A \ltimes I$ holds under the more general hypothesis that L° is a frame. (The same applies to the uniqueness.) The distributivity of the sup-lattice L does not suffice. For example, consider the sublattice of $\mathbb{R} \times \mathbb{R}$ given by $L := \{(x, 0) \mid 0 \leqslant x \leqslant 1\} \cup \{(x, 1) \mid 0 < x \leqslant 1\}$. This is a distributive sup-lattice with an order ideal $I := \{(x, 0) \mid 0 \leqslant x \leqslant 1\}$. However, L cannot be written as a semidirect product $A \ltimes I$.

4. Derived quantales

In this section, we introduce a class of quantales similar to graded algebras A twisted by a derivation. The Hausdorff property $\bigcap_{n=0}^{\infty} \mathrm{Rad}^n A = 0$ has to be reformulated in an appropriate manner.

Definition 4.1. We call a quantale Q *separated* if the Q^n are order ideals and the Q/Q^n cogenerate Q, that is, the natural morphism $Q \to \bigoplus_{n=1}^{\infty} Q/Q^n$ is injective. We say that Q *splits* if Q is separated with $Q^n = A^n \ltimes Q^{n+1}$ for all $n > 0$ and some $A \in \mathrm{Sub}(Q)$ which will be called an *inertial sup-lattice* of Q.

For a separated quantale Q, we can form the *associated graded quantale*

$$\mathrm{gr}\, Q := \bigoplus_{n=1}^{\infty} Q^n/Q^{n+1}$$

with multiplication induced by the maps $Q^m/Q^{m+1} \otimes Q^n/Q^{n+1} \twoheadrightarrow Q^{m+n}/Q^{m+n+1}$. Note that $\mathrm{gr}\, Q$ is a splitting quantale with inertial sup-lattice Q/Q^2.

Lemma 4.2. *Let Q be a separated quantale with $Q = A \vee Q^2$ for some $A \in \mathrm{Sub}(Q)$. Then $Q = \bigvee_{n=1}^{\infty} A^n$.*

Proof. By induction, we get $Q^n = A^n \vee Q^{n+1}$, and thus $Q = \bigvee_{i<n} A^i \vee Q^n$ for all $n > 0$. Therefore, every $x \in Q$ can be written as $x = a_n \vee x_n$ with $a_n \in \bigvee_{i<n} A^i$ and $x_n \in Q^n$ for all $n > 0$. Hence $x = (\bigvee_{i=1}^{\infty} a_i) \vee x_n$ for all n. Since Q is separated, this yields $x = \bigvee_{i=1}^{\infty} a_i \in \bigvee_{i=1}^{\infty} A^i$. □

Let Q be a splitting quantale with inertial sup-lattice A. By (3.4), there are natural isomorphisms $Q^n/Q^{n+1} \cong A^n$ for all $n > 0$, and by Lemma 4.2, they induce an epimorphism of quantales

$$p : \mathrm{gr}\, Q \twoheadrightarrow Q. \tag{4.1}$$

If p is an isomorphism, we call Q a *graded* quantale.

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Definition 4.3. Let L be a quantale. We define a *derivation* of L to be a sup-lattice morphism $d \in Q(L)$ which satisfies

$$d(xy) = d(x)y \vee xd(y)$$

for all $x, y \in L$. The derivations of L form a sup-lattice $\mathrm{Der}(L) \in \mathrm{Sub}(Q(L))$. For an order ideal I of L, the derivations with image in I will be denoted by $\mathrm{Der}(L, I)$. We call a derivation d of L *convex* if $d^2 \leqslant d$.

Thus, derivations of quantales are quite analogous to derivations of rings, while convexity has no counterpart in ring theory. Note that the set of convex derivations d of a quantale Q is $\bigwedge$-closed in $\mathrm{Der}(Q)$. In fact, if $d_i \in \mathrm{Der}(Q)$ are convex for all $i \in I$, then $(\bigwedge d_i)(\bigwedge d_j x) \leqslant d_i(d_i x) \leqslant d_i x$ holds for all $x \in Q$ and $i \in I$, whence $(\bigwedge d_i)^2 \leqslant \bigwedge d_i$. So the convex derivations of Q form a sup-lattice $\mathrm{Der}^{\mathrm{c}}(Q)$, and there is a natural epimorphism of sup-lattices

$$\mathrm{Der}(Q) \twoheadrightarrow \mathrm{Der}^{\mathrm{c}}(Q). \tag{4.2}$$

Proposition 4.4. *Let d be a convex derivation of a quantale Q. Then $1 \vee d$ is an idempotent endomorphism of Q.*

Proof. For $x, y \in Q$, we have $dx \cdot dy \leqslant d^2x \cdot y \vee dx \cdot dy = d(dx \cdot y) \leqslant d(dx \cdot y \vee x \cdot dy) = d^2(xy) \leqslant d(xy)$. Hence $xy \vee d(xy) = xy \vee dx \cdot y \vee x \cdot dy \vee dx \cdot dy = (x \vee dx)(y \vee dy)$, and thus $1 \vee d$ is an endomorphism of Q. Furthermore, $(1 \vee d)^2 x = (1 \vee d)(x \vee dx) = (x \vee dx) \vee (dx \vee d^2x) = x \vee dx$, which shows that $1 \vee d$ is idempotent. □

The converse of Proposition 4.4 does not hold (see Example 5.8).

Definition 4.5. Let Q be a quantale. We define a *radical map* of Q to be a morphism $\delta : Q \to Q^2$ of sup-lattices such that $1 \vee \delta$ is an idempotent endomorphism of Q. For a radical map δ, we define the *derived quantale* Q^δ to be the retract $(1 \vee \delta)(Q)$ of Q. If Q is graded, we call Q^δ a *derived graded quantale*.

Our next result gives an intrinsic description of derived graded quantales. Recall that a splitting quantale Q with inertial sup-lattice A satisfies $Q^n = A^n \ltimes Q^{n+1}$ for all $n > 0$. Hence there are dual actions $A^n \times Q^{n+1} \to Q^{n+1}$ in the sense of Definition 3.2. For $a \in A^n$ and $y \in Q^{n+1}$, we will write $(a, y) \mapsto {}^a y$ for this action, to distinguish it from the multiplication in Q.

Theorem 4.6. *For a quantale Q, the following are equivalent:*

(a) *Q is a derived graded quantale;*
(b) *Q splits, and there exist an inertial sup-lattice A, and a convex derivation d of Q, such that ${}^a y = da \vee y$ for all $a \in A^n$, $y \in Q^{n+1}$, and $n > 0$;*
(c) *Q splits, and the dual p° of the morphism (4.1) is a morphism of quantales.*

Proof. (a) implies (b): Let $Q = \bigoplus_{n=1}^\infty M^n$ be a graded quantale, and let $\delta : Q \to Q^2$ be a radical map. We show that $Q^\delta := (1 \vee \delta)(Q)$ satisfies (b). Define $A := (1 \vee \delta)(M)$. Then $Q^\delta = \bigvee_{n=1}^\infty A^n$. It follows from $(Q^\delta)^n = Q^n \cap Q^\delta$ that the $(Q^\delta)^n$ are order ideals of Q^δ.

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Since Q is separated, Q^δ is separated, too. The elements of A^n are of the form $x \vee \delta(x)$ with $x \in M^n$, and $\delta(x) \in Q^{n+1}$. Therefore, the morphisms $A^n \hookrightarrow (Q^\delta)^n \twoheadrightarrow (Q^\delta)^n/(Q^\delta)^{n+1}$ are injective. This proves that Q^δ splits.

Since $1 \vee \delta$ is idempotent, we have $x \vee \delta(x) = (1 \vee \delta)(x \vee \delta(x)) = x \vee \delta(x) \vee \delta^2(x)$ for all $x \in M^n$ and $n > 0$, which gives $\delta^2(x) \leqslant \delta(x)$ for $x \in M^n$, and thus $\delta^2 \leqslant \delta$. Hence $\delta(x \vee \delta(x)) = \delta(x)$ for all $x \in Q$. For $x \in M^m$ and $y \in M^n$, the multiplicativity of $1 \vee \delta$ implies that $xy \vee \delta(xy) = (x \vee \delta(x))(y \vee \delta(y))$, which yields $\delta(xy) = \delta(x)y \vee x\delta(y) \vee \delta(x)\delta(y) = \delta(x)(y \vee \delta(y)) \vee (x \vee \delta(x))\delta(y)$. Therefore, with $a := x \vee \delta(x)$ and $b := y \vee \delta(y)$, we get $\delta(ab) = \delta(a)b \vee a\delta(b)$. So the morphism δ restricts to a convex derivation d of Q^δ. Assume that $a \in A^n$ and $y \in (Q^\delta)^{n+1}$ for some $n > 0$. Then $a = x \vee \delta(x)$ for some $x \in M^n$, and $da = \delta(x) \in Q^\delta \cap Q^{n+1} = (Q^\delta)^{n+1}$. If $z \leqslant a \vee y$ with $z \in (Q^\delta)^{n+1}$, then $z \leqslant \delta(x) \vee y = da \vee y$. Hence (3.5) yields ${}^a y = da \vee y$. This proves (b).

(b) implies (c): By Lemma 4.2, we have a surjection (4.1). We identify Q^n/Q^{n+1} with A^n. To show that $p^\circ : Q \to \operatorname{gr} Q$ is a morphism of quantales, let $(x_i)_{i \in I}$ be a family of $x_i \in Q$. Assume that $p^\circ(x_i) = \bigoplus_{n=1}^\infty x_{in}$ with $x_{in} \in A^n$. Then $x_i = \bigvee_{n=1}^\infty x_{in}$ for all $i \in I$, and the x_{in} are maximal with this property. This gives $\bigvee_{i \in I} x_i = \bigvee_{n=1}^\infty \bigvee_{i \in I} x_{in}$. To verify that p° is a morphism of sup-lattices, we have to show that $p^\circ(\bigvee_{n=1}^\infty \bigvee_{i \in I} x_{in}) = \bigoplus_{n=1}^\infty \bigvee_{i \in I} x_{in}$. This means that every $z \in A^n$ with $z \leqslant \bigvee_{n=1}^\infty \bigvee_{i \in I} x_{in}$ satisfies $z \leqslant \bigvee_{i \in I} x_{in}$. Since $Q^n = A^n \ltimes Q^{n+1}$, this is equivalent to

$$\Big(\bigvee_{j=1}^{\infty} \bigvee_{i \in I} x_{ij}\Big) \wedge \bigvee Q^n = \bigvee_{j \geqslant n} \bigvee_{i \in I} x_{ij}.$$

Using induction over n, it suffices to verify

$$\Big(\bigvee_{j \geqslant n} \bigvee_{i \in I} x_{ij}\Big) \wedge \bigvee Q^{n+1} = \bigvee_{j > n} \bigvee_{i \in I} x_{ij}$$

for all $n > 0$. Now (3.5) yields

$$\begin{aligned}
\Big(\bigvee_{j \geqslant n} \bigvee_{i \in I} x_{ij}\Big) \wedge \bigvee Q^{n+1} &= \Big(\bigvee_{i \in I} x_{in}\Big)\Big(\bigvee_{j > n} \bigvee_{i \in I} x_{ij}\Big) = d\Big(\bigvee_{i \in I} x_{in}\Big) \vee \bigvee_{j > n} \bigvee_{i \in I} x_{ij} \\
&= \bigvee_{i \in I} dx_{in} \vee \bigvee_{j > n} \bigvee_{i \in I} x_{ij} = \bigvee_{i \in I}\Big(dx_{in} \vee \bigvee_{j > n} x_{ij}\Big) \\
&= \bigvee_{i \in I}\Big(x_{in} \cdot \bigvee_{j > n} x_{ij}\Big) = \bigvee_{i \in I}\Big(\bigvee_{j \geqslant n} x_{ij} \wedge \bigvee Q^{n+1}\Big) = \bigvee_{i \in I} \bigvee_{j > n} x_{ij}.
\end{aligned}$$

It remains to be shown that p° is multiplicative. Every element of Q is of the form $a \vee x$ with $a \in A^m$ and $x \in Q^{m+1}$ for some $m > 0$. Thus if $b \in A^n$ and $y \in Q^{n+1}$, we have to verify that

$$p^\circ((a \vee x)(b \vee y)) = p^\circ(a \vee x) \cdot p^\circ(b \vee y). \tag{4.3}$$

We will show that there exist $x_i \in Q^{m_i}$ and $y_i \in Q^{n_i}$ with $m_i \leqslant m$ and $n_i \leqslant n$ such that $m_i + n_i > m + n$ and

$$p^\circ((a \vee x)(b \vee y)) = ab \oplus \bigvee_{i=1}^{3} p^\circ(x_i y_i), \quad p^\circ(a \vee x) \cdot p^\circ(b \vee y) = ab \oplus \bigvee_{i=1}^{3} p^\circ(x_i)p^\circ(y_i).$$

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By induction, this will imply (4.3). Now $p^\circ(a \vee x) = a \oplus p^\circ({}^a x) = a \oplus p^\circ(da \vee x)$. Hence

$$\begin{aligned} p^\circ((a \vee x)(b \vee y)) = p^\circ(ab \vee ay \vee xb \vee xy) &= ab \oplus p^\circ(d(ab) \vee ay \vee xb \vee xy) \\ &= ab \oplus p^\circ(da \cdot b \vee a \cdot db \vee ay \vee xb \vee xy) \\ &= ab \oplus p^\circ(a(db \vee y)) \vee p^\circ((da \vee x)b) \vee p^\circ(xy), \end{aligned}$$

and $p^\circ(a \vee x) \cdot p^\circ(b \vee y) = (a \oplus p^\circ(da \vee x))(b \oplus p^\circ(db \vee y)) = ab \oplus z$ with

$$\begin{aligned} z &= a \cdot p^\circ(db \vee y) \vee p^\circ(da \vee x) \cdot b \vee p^\circ(da \vee x) \cdot p^\circ(db \vee y) \\ &= a \cdot p^\circ(db \vee y) \vee p^\circ(da \vee x)b \vee p^\circ(da)p^\circ(db \vee y) \vee p^\circ(da \vee x)p^\circ(db) \vee p^\circ(x)p^\circ(y) \\ &= p^\circ(a)p^\circ(db \vee y) \vee p^\circ(da \vee x)p^\circ(b) \vee p^\circ(x)p^\circ(y). \end{aligned}$$

(c) implies (a): Let A be an inertial sup-lattice of Q. For each $x \in Q^n/Q^{n+1} \subset \operatorname{gr} Q$, we have $p^\circ p(x) = x \oplus \delta_n(x)$ with $\delta_n(x) \in (\operatorname{gr} Q)^{n+1}$. Since p° is a morphism of quantales, $\delta_n : Q^n/Q^{n+1} \to (\operatorname{gr} Q)^{n+1}$ is a morphism of sup-lattices. So the δ_n extend to a morphism $\delta : \operatorname{gr} Q \to (\operatorname{gr} Q)^2$ of sup-lattices which satisfies

$$p^\circ p(x) = x \oplus \delta x$$

for all $x \in \operatorname{gr} Q$. Hence δ is a radical map. □

5. Projective quantales

In this section we determine the projective quantales. A quantale Q is said to be *projective* if every quantale morphism $Q \to N$ lifts along regular epimorphisms $M \twoheadrightarrow N$ of quantales. Note that the regular epimorphisms in **Quant** are just the surjective ones.

Let M be a sup-lattice. We define the *tensor quantale* $T(M)$ as follows. First, we consider the tensor powers $M^{\otimes n} := M \otimes \cdots \otimes M$ (n times). Then we define

$$T(M) := \bigoplus_{n=1}^{\infty} M^{\otimes n} \tag{5.1}$$

with the multiplication induced by

$$M^{\otimes m} \times M^{\otimes n} \to M^{\otimes(m+n)}.$$

So we have

$$T(M)^n = \bigoplus_{i=n}^{\infty} M^{\otimes i}, \tag{5.2}$$

and in particular, $T(M)^2 \neq T(M)$ if $M \neq 0$. Thus every tensor quantale $T(M)$ is graded, hence splitting with inertial sup-lattice M.

Every morphism $f : M \to N$ in **Sup** induces morphisms

$$f^{\otimes n} : M^{\otimes n} \to N^{\otimes n}$$

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for all $n > 0$ via (2.4) which make up a quantale morphism

$$T(f) : T(M) \to T(N).$$

This gives a functor

$$T : \mathbf{Sup} \to \mathbf{Quant}$$

which is left adjoint to the forgetful functor **Quant** → **Sup**. In particular, every sup-lattice morphism $\mathbb{1} \to Q$ into a quantale Q extends to a quantale morphism $T(\mathbb{1}) \to Q$, which implies that regular epimorphisms in **Quant** are surjective.

We call an element $x \in T(M)$ *homogenous* if it belongs to some $M^{\otimes n}$. Thus any $x \in T(M)$ admits a unique decomposition

$$x = \bigvee_{n=1}^{\infty} x_n$$

into its homogenous components $x_n \in M^{\otimes n}$.

The free quantale over a set X can be obtained in two steps as the tensor quantale $T(\mathbb{1}^X)$ of the free sup-lattice $\mathbb{1}^X$. The two steps can be interchanged. Note that the free semigroup $S(X)$ over X looks rather similar to the tensor quantale (5.1), being the disjoint union

$$S(X) = \bigsqcup_{n=1}^{\infty} X^n.$$

Thus (2.8) yields $T(\mathbb{1}^X) \cong \bigoplus_{n=1}^{\infty} (\mathbb{1}^X)^{\otimes n} \cong \bigoplus_{n=1}^{\infty} \mathbb{1}^{X^n} \cong \mathbb{1}^{S(X)}$ as a sup-lattice. The multiplication is induced by the multiplication in $S(X)$.

Definition 5.1. Let H be a semigroup and Q a quantale. We define the *semigroup quantale* $Q[H]$ to be the sup-lattice Q^H with the convolution product

$$(f \cdot g)(z) := \bigvee \{f(x)g(y) \mid x, y \in H,\ xy = z\}.$$

If Q is unital, we have a natural embedding as a subsemigroup $H \hookrightarrow Q[H]$, where $x \in H$ has to be regarded as a function $x : H \to Q$ with $x(y) = 0$ for $y \neq x$ and $x(x) = u$, the unit in Q. With this identification, the elements $f \in Q[H]$ admit a unique expression

$$f = \bigvee_{x \in H} a_x x$$

with $a_x \in Q$.

Now the unital quantale $Q(\mathbb{1})$ has $\mathbb{1}$ as underlying sup-lattice. So we get another representation of the free quantale over a set X:

$$T(\mathbb{1}^X) \cong Q(\mathbb{1})[S(X)]. \tag{5.3}$$

For a tensor quantale $T(M)$, every morphism $d \in Q(M, T(M))$ admits a unique extension to a derivation of $T(M)$. In particular, there is a natural embedding $Q(M) \hookrightarrow \mathrm{Der}(T(M))$. Therefore, we have a natural isomorphism of sup-lattices

$$\mathrm{Der}(T(M)) \cong Q(M, T(M)) \cong Q(M) \oplus \mathrm{Der}(T(M), T(M)^2). \tag{5.4}$$

As an immediate consequence of (5.3), we get the following proposition (see [14]).

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Proposition 5.2. *Every projective quantale is projective as a sup-lattice.*

Proof. For a projective quantale Q, the natural epimorphism $Q(\mathbb{1})[S(Q)] \twoheadrightarrow Q$ splits. Since $Q(\mathbb{1})[S(Q)]$ is free as a sup-lattice, the statement follows. $\square$

Definition 5.3. We define a *derived tensor quantale* to be a derived quantale Q for which $\operatorname{gr} Q$ is a tensor quantale.

For a sup-lattice M, every morphism $d \in \mathcal{Q}(M, T(M)^2)$ gives rise to a derived tensor quantale with inertial sup-lattice M. In fact, (5.4) shows that d extends to a derivation of $T(M)$, and by (4.2), there is a smallest convex derivation $\delta \in \operatorname{Der}(T(M), T(M)^2)$ with $\delta \geqslant d$, hence a derived tensor quantale $T^\delta(M) := T(M)^\delta$.

Now we are ready to prove our main result.

Theorem 5.4. *A quantale Q is projective if and only if Q is isomorphic to a derived tensor quantale $T^\delta(M)$ over a projective sup-lattice M.*

Proof. Let Q be projective. The identity map $Q \to Q$ induces an epimorphism $p : T(\mathbb{1}^Q) \twoheadrightarrow Q$ of quantales. So there is a section $s : Q \to T(\mathbb{1}^Q)$. By (5.2), the $T(\mathbb{1}^Q)^n$ are order ideals. Hence $Q^n = p(T(\mathbb{1}^Q)^n)$ is an order ideal of Q for each $n > 0$. Since $T(\mathbb{1}^Q)$ is separated, it follows that Q is separated.

By Proposition 5.2, Q is projective as a sup-lattice. Therefore, Proposition 3.7 implies that there is a projective $A \in \operatorname{Sub}(Q)$ with $Q = A \ltimes Q^2$. So the inclusion map $A \hookrightarrow Q$ extends to a quantale morphism $q : T(A) \to Q$ which is surjective by Lemma 4.2. Since Q is projective, there is a morphism $t : Q \to T(A)$ of quantales with $qt = 1$. Hence tq is an idempotent endomorphism of $T(A)$. For any $a \in A$, we have $qtq(a) = q(a) = a$, which implies that $tq(a) = a \oplus \delta_1(a)$ with $\delta_1(a) \in T(A)^2$. As tq is a morphism of sup-lattices, it follows that $\delta_1 \in \mathcal{Q}(A, T(A)^2)$. Furthermore, if $a \in A^{\otimes n}$, we obtain $tq(a) = a \oplus \delta_n(a)$ with a unique $\delta_n \in \mathcal{Q}(A^{\otimes n}, T(A)^{n+1})$. So the δ_n extend to a radical map $\delta \in \mathcal{Q}(T(A), T(A)^2)$ with $1 \vee \delta = tq$. Hence $Q \cong T^\delta(A)$.

Conversely, assume that Q is isomorphic to a derived tensor quantale $T^\delta(M)$ over a projective sup-lattice M. By Definition 4.5, Q is a retract of $T(M)$. Since M is projective, it is a retract of the free sup-lattice $\mathbb{1}^M$. Applying the functor T, we infer that $T(M)$ is a retract of the free quantale $T(\mathbb{1}^M)$. Hence $T(M)$ is projective, and thus Q is a projective quantale. $\square$

Corollary 5.5. *A quantale Q is projective if and only if the following conditions are satisfied:*

(a) *Q/Q^2 is a projective sup-lattice;*

(b) *Q splits, and there exist an inertial sup-lattice A, and a convex derivation d of Q, such that ${}^a y = da \vee y$ for all $a \in A^n$, $y \in Q^{n+1}$, and $n > 0$;*

(c) $\operatorname{gr} Q \cong T(Q/Q^2)$.

Proof. By Theorem 4.6, Q is a derived tensor quantale if and only if (b) and (c) hold. Therefore, the corollary follows immediately by Theorem 5.4. $\square$

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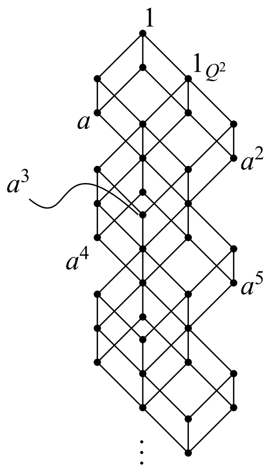

FIGURE 1. A nonfree projective quantale.

Using property (c) of Theorem 4.6, we get the following characterization of projective quantales.

COROLLARY 5.6. *A quantale Q is projective if and only if Q splits, the dual p° of the morphism* (4.1) *is a quantale morphism, the sup-lattice Q/Q^2 is projective, and* gr Q *is a tensor quantale.*

For a tensor quantale $T(M)$, there are plenty of morphisms $d : M \to T(M)^2$ of sup-lattices. So there are many (convex) derivations δ of $T(M)$, even for $M = \mathbb{1}$. The following example describes the class of nonfree projective quantales $T^\delta(\mathbb{1})$ obtained in this case.

EXAMPLE 5.7. Let N be any subsemigroup of the additive semigroup of positive integers. To avoid confusion, we denote the greatest element of $\mathbb{1}$ by e. Consider the derivation δ of $T(\mathbb{1})$ given by $\delta(e) := \bigvee_{n\in N} e^{n+1}$. Then the semigroup property of N implies that δ is a convex derivation with $\delta(e^m) := \bigvee_{n\in N} e^{m+n}$ for all $m > 0$. By Theorem 5.4, the derived tensor quantale $T^\delta(\mathbb{1})$ is projective.

Let us consider the (very) special case $N = \{k \in \mathbb{N} \mid k \geqslant n\}$ for a fixed positive integer n. Then $T^\delta(\mathbb{1})$ is generated by $a := e \vee \delta(e) = e \vee \bigvee_{k>n} e^k$, and for any $m > 0$, the number n is the smallest integer $k > 0$ with $a^{m+k} \leqslant a^m$. For $n = 1$, we get a chain $T^\delta(\mathbb{1}) = \{1 > 1^2 > 1^3 > \cdots > 0\}$. For $n = 3$, the quantale $Q := T^\delta(\mathbb{1})$ satisfies $a \vee a^2 \vee a^3 = 1$ and is depicted in Figure 1.

If N contains a subsemigroup $\{k \in \mathbb{N} \mid k \geqslant m\}$ for some $m > 0$, the quantale $T^\delta(\mathbb{1})$ can be interpreted as an ideal lattice for some order in a skew-field (see [26, Satz 15.1]). We illustrate this for the above example with $N = \{3, 4, 5, \ldots\}$. Let D be a skew-field with a discrete valuation $v : D^\times \twoheadrightarrow \mathbb{Z}$. Let $\Delta := \{a \in D \mid v(a) \geqslant 0\}$ be the valuation ring and $\Pi := \operatorname{Rad}\Delta$ its radical. Assume that Λ is an order in D such that Δ is totally

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ramified over Λ with $\operatorname{Rad}\Lambda = \Pi^3$. Thus $\Lambda + \Pi = \Delta$ and $\Lambda \cap \Pi = \Pi^3$. To any ideal I of Λ we can associate a *characteristic* [26], that is, a sequence $\chi(I) = (\chi_i(I))_{i\in\mathbb{Z}}$ with

$$\chi_i(I) := \operatorname{length}_\Lambda((I \cap \Pi^i) + \Pi^{i+1}/\Pi^{i+1}).$$

In our special case, $\chi_i(I) \in \{0, 1\}$ for all $i \in \mathbb{Z}$. Since Λ is representation-finite, two ideals I, J of Λ have the same characteristic if and only if $I = J\alpha$ for some $\alpha \in \Delta^\times$. Hence, if $\mathscr{I}$ denotes the lattice of ideals $I \subset \Pi$ of Λ, the map $I \mapsto \chi(I)$ makes $\mathscr{I}$ into a quantale $\mathscr{I}/\Delta^\times$. In fact, $\mathscr{I}/\Delta^\times$ coincides with the quantale of Figure 1.

Example 5.8. Let H be the free commutative semigroup with two generators a and b. Consider the semigroup quantale $Q := Q(\mathbb{1})[H]$. Define a morphism $\delta : Q \to Q^2$ of sup-lattices by

$$\delta(a^m) = \bigvee_{i>m}^{\infty} a^i, \quad \delta(b^n) = \bigvee_{j>n}^{\infty} b^j, \quad \delta(a^m b^n) = \bigvee \{a^i b^j \mid i \geqslant m, j \geqslant n, (i, j) \neq (m, n)\}$$

for $m, n > 0$. It is easy to verify that $1 \vee \delta$ is an idempotent quantale endomorphism of Q. Hence δ is a radical map, and we can form the derived graded quantale Q^δ. However, δ is not a derivation since $\delta(a)b \vee a\delta(b) = \bigvee_{n=2}^{\infty}(a^n b \vee ab^n)$, which is strictly smaller than $\delta(ab)$.

References

[1] S. Abramsky and S. Vickers, ‘Quantales, observational logic and process semantics’, *Math. Structures Comput. Sci.* **3**(2) (1993), 161–227.

[2] B. Banaschewski, ‘On the topologies of injective spaces’, in: *Continuous Lattices and their Applications (Bremen, 1982)*, Lecture Notes in Pure and Applied Mathematics, 101 (Dekker, New York, 1985), 1–8.

[3] B. Banaschewski, ‘Projective frames: a general view’, *Cah. Topol. Géom. Différ. Catég.* **46**(4) (2005), 301–312.

[4] B. Banaschewski and S. B. Niefield, ‘Projective and supercoherent frames’, *J. Pure Appl. Algebra* **70**(1–2) (1991), 45–51.

[5] F. Borceux and G. van den Bossche, ‘An essay on noncommutative topology’, *Topology Appl.* **31**(3) (1989), 203–223.

[6] F. Borceux, J. Rosický and G. Van den Bossche, ‘Quantales and C^*-algebras’, *J. Lond. Math. Soc. (2)* **40**(3) (1989), 398–404.

[7] A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, CA, 1994).

[8] M. E. Coniglio and F. Miraglia, ‘Non-commutative topology and quantales’, *Stud. Log.* **65**(2) (2000), 223–236.

[9] J.-Y. Girard, ‘Linear logic’, *Theoret. Comput. Sci.* **50** (1987), 1–102.

[10] A. Joyal and M. Tierney, ‘An extension of the Galois theory of Grothendieck’, *Mem. Amer. Math. Soc.* **51**(309) (1984), vii+71 pages.

[11] D. Kruml, J. W. Pelletier, P. Resende and J. Rosický, ‘On quantales and spectra of C^*-algebras’, *Appl. Categ. Structures* **11**(6) (2003), 543–560.

[12] D. Kruml and J. Paseka, ‘Algebraic and categorical aspects of quantales’, in: *Handbook of Algebra*, Vol. 5 (Elsevier/North-Holland, Amsterdam, 2008), 323–362.

[13] D. Kruml and P. Resende, ‘On quantales that classify C^*-algebras’, *Cah. Topol. Géom. Différ. Catég.* **45**(4) (2004), 287–296.

https://doi.org/10.1017/S1446788715000506 Published online by Cambridge University Press

[14] Y.-M. Li, M. Zhou and Z.-H. Li, 'Projective and injective objects in the category of quantales', *J. Pure Appl. Algebra* **176**(2–3) (2002), 249–258.

[15] S. Mac Lane, *Categories for the Working Mathematician* (Springer, New York, 1971).

[16] C. J. Mulvey, 'Second topology conference (Taormina, 1984)', *Rend. Circ. Mat. Palermo (2) Suppl. (12)* (1986), 99–104.

[17] C. J. Mulvey and J. W. Pelletier, 'On the quantisation of points', *J. Pure Appl. Algebra* **159**(2–3) (2001), 231–295.

[18] C. J. Mulvey and J. W. Pelletier, 'On the quantisation of spaces, Special volume celebrating the 70th birthday of Professor Max Kelly', *J. Pure Appl. Algebra* **175**(1–3) (2002), 289–325.

[19] C. J. Mulvey and P. Resende, 'A noncommutative theory of Penrose tilings', *Internat. J. Theoret. Phys.* **44**(6) (2005), 655–689.

[20] S. B. Niefield, 'Exactness and projectivity', in: *Category Theory (Gummersbach, 1981)*, Lecture Notes in Mathematics, 962 (Springer, Berlin, 1982), 221–227.

[21] J. Paseka, 'Projective quantales: a general view', *Internat. J. Theoret. Phys.* **47**(1) (2008), 291–296.

[22] P. Resende, 'Quantales, finite observations and strong bisimulation', *Theoret. Comput. Sci.* **254**(1–2) (2001), 95–149.

[23] P. Resende, 'Étale groupoids and their quantales', *Adv. Math.* **208**(1) (2007), 147–209.

[24] K. I. Rosenthal, 'A note on Girard quantales', *Cah. Topol. Géom. Différ. Catég.* **31**(1) (1990), 3–11.

[25] K. I. Rosenthal, *Quantales and their Applications*, Pitman Research Notes in Mathematics Series, 234 (Longman Scientific & Technical, Harlow, 1990).

[26] W. Rump, 'Irreduzible und unzerlegbare Darstellungen klassischer Ordnungen', *Bayreuth. Math. Schriften* **32** (1990), 1–405.

[27] W. Rump, 'Inertial algebras, inertial bimodules, and projective covers of algebras', *Comm. Algebra* **27**(11) (1999), 5303–5331.

[28] D. Scott, 'Continuous lattices', in: *Toposes, Algebraic Geometry and Logic (Conf., Dalhousie Univ., Halifax, N. S., 1971)*, Lecture Notes in Mathematics, 274 (Springer, Berlin, 1972), 97–136.

[29] I. Stubbe, 'Towards "dynamic domains": totally continuous cocomplete $\mathcal{Q}$-categories', *Theoret. Comput. Sci.* **373**(1–2) (2007), 142–160.

[30] D. N. Yetter, 'Quantales and (noncommutative) linear logic', *J. Symbolic Logic* **55**(1) (1990), 41–64.

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