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On Residues of Intertwining Operators in Cases with Prehomogeneous Nilradical

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Abstract. Let $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ be a Levi decomposition of a maximal parabolic subgroup of a connected reductive group G over a p-adic field F. Assume that there exists $w_0 \in \mathrm{G}(F)$ that normalizes M and conjugates P to an opposite parabolic subgroup. When N has a Zariski dense $\mathrm{Int}\,\mathrm{M}$-orbit, F. Shahidi and X. Yu described a certain distribution D on $\mathrm{M}(F)$, such that, for irreducible unitary supercuspidal representations π of $\mathrm{M}(F)$ with $\pi \cong \pi \circ \mathrm{Int}\, w_0$, $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ is irreducible if and only if $D(f) \neq 0$ for some pseudocoefficient f of π. Since this irreducibility is conjecturally related to π arising via transfer from certain twisted endoscopic groups of M, it is of interest to realize D as endoscopic transfer from a simpler distribution on a twisted endoscopic group H of M. This has been done in many situations where N is abelian. Here we handle the standard examples in cases where N is nonabelian but admit a Zariski dense $\mathrm{Int}\,\mathrm{M}$-orbit.

1 Introduction

Let G be a quasi-split connected reductive group over a p-adic field F, and $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ a Levi decomposition of a maximal parabolic subgroup of G. Given an irreducible unitary supercuspidal representation π of $\mathrm{M}(F)$ (inflated in the obvious way to $\mathrm{P}(F)$), one often considers the question of when the parabolically induced representation $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ (induction normalized to preserve unitarity) is irreducible. This question is especially interesting when the following condition is satisfied.

- There exists $w_0 \in \mathrm{G}(F)$ that normalizes M and takes P to a parabolic subgroup opposed to it, such that, as representations of $\mathrm{M}(F)$, $\pi \cong \pi \circ \mathrm{Int}\, w_0$.

When this condition is not satisfied, one knows by purely harmonic analytic means that $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ is irreducible. What makes the above question interesting when the condition is satisfied is that the answer is conjecturally related to L-functions thanks to Langlands' conjecture on normalization of intertwining operators [Sha90].

Let us explain this a bit more: when the above condition is satisfied, we know, thanks to Harish-Chandra, that the irreducibility of $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ is equivalent to a certain meromorphic family $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ of intertwining operators having a pole at $s = 0$; and by Langlands' conjecture on Plancherel measures, $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ is expected to have a pole at $s = 0$ if and only if some member of a certain finite set of (as yet conjectural) L-functions $L(s, \pi, r_i)$ has a pole at $s = 0$.

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In keeping with this is a program, pioneered by F. Shahidi [Sha92] and developed further by him as well as D. Goldberg and S. Spallone, with contributions by L. Cai, W.-W. Li, B. Xu, and X. Yu [GS98, Spa08, Li13, CX15] to:

(i) obtain a "formula" for the residue of the family $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ at 0,
(ii) endoscopically interpret the result therein.

Here by a formula for the residue, one roughly means a distribution D on $\mathrm{M}(F)$ such that for all π satisfying the conditions above, $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ has a pole at $s = 0$ if and only if $D(f) \neq 0$ for some $f \in C_c^\infty(\mathrm{M}(F))$ such that

$$m \mapsto \int_{\mathrm{Z_M}(F)} f(zm)\omega_\pi(z^{-1})\,dz,$$

$\mathrm{Z_M}$ denoting the center of M and ω_π the central character of π, is a matrix coefficient for π. By *endoscopically interpreting* this result, one refers to finding a twisted endoscopic group H for M and interpreting the above distribution D in terms of endoscopic transfer from H (we are abusing notation here and conflating H with a twisted endoscopic datum).

Let us discuss (i) first, restricting to situations that concern us. In the special case where N is abelian, Shahidi [Sha00] came up with a simple and elegant expression for a distribution D as above. There are two features that make this special case considerably simpler than the general case. First, in these cases N has a Zariski-dense Int M-orbit. Second, the connected stabilizer of a point in this Zariski-dense orbit is also the connected twisted centralizer of a suitable point in M [Sha00, Lemma 2.1]. Together, these features make it easier to handle the intertwining operator involved (roughly speaking, this is because its prescription contains an integral over $\mathrm{N}(F)$, whose transfer to a twisted orbital integral for $\mathrm{M}(F)$ is facilitated by these features).

X. Yu [Yu09] observed that the first of the two features above, namely, the fact that N has a dense Int M-orbit, already forces the second, and that this allows for extending the formula for the distribution D in [Sha00] to cases where N is not necessarily abelian but still has a dense Int M-orbit.

Work on (ii), in contrast, has been done on a case-by-case basis. More specifically, it is easy to see that one can write $\mathrm{G} = (\mathrm{G}' \times \mathrm{Res}_{E/F}\,\mathrm{G}_1)/\mathrm{Z}$ and $\mathrm{P} = (\mathrm{G}' \times \mathrm{Res}_{E/F}\,\mathrm{P}_1)/\mathrm{Z}$ where G_1 is absolutely simple, simply connected, and quasi-split, E/F is a finite extension, G' is quasi-split, and Z is a central subgroup of $\mathrm{G}' \times \mathrm{Res}_{E/F}\,\mathrm{G}_1$. Corresponding to the Levi decomposition $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ in G is a Levi decomposition $\mathrm{P}_1 = \mathrm{M}_1\,\mathrm{N}_1$ in G_1; N is abelian if and only if N_1 is, and N has an open Int M-orbit if and only if N_1 has an open Int M_1-orbit. For each possible $(\mathrm{G}_1, \mathrm{M}_1)$ that is of interest, *i.e.*, where M_1 is "self-associate", Shahidi studied a specific, convenient, choice of (G, M) that relates to it in the manner just stated. In particular, he takes $E = F$, and G' to be trivial in most cases. With these choices, Shahidi observed [Sha00] that for most of the cases of concern, D is simply the endoscopic transfer from a twisted endoscopic group H of the distribution $f^{\mathrm{H}} \mapsto f^{\mathrm{H}}(1)$ on $\mathrm{H}(F)$.

However, merely requiring N to have an open Int M-orbit, as in [Yu09], throws additional possibilities into play as follows.

Case 1: $(\mathrm{G}_1, \mathrm{M}_1) \cong (\mathrm{Sp}_{2n}, \mathrm{GL}_1 \times \mathrm{Sp}_{2n-2})$ $(n \geq 2)$.

Case 2: $\mathrm{G}_1 \cong \mathrm{Spin}_{4N+1}$, M_1 is a double cover of GL_{2N}.

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Case 3: $\mathrm{G}_1 \cong \mathrm{Spin}_{4N+3}$, M_1 is a double cover of GL_{2N+1}.

In fact, a computation of the distribution D in Cases 2 and 3, where one assumes $\mathrm{G} = \mathrm{SO}_{4N+1}$ or SO_{4N+3}, was already done [Sha92]. However, an endoscopic interpretation was discussed only in Case 2 with $N = 1$, *i.e.*, for $\mathrm{G} = \mathrm{SO}_5$. (See [Sha92, §10]; further, immediately following Remark 2 of that section, Case 2 has been mentioned to be much more complicated than the other cases considered there as it features non-semisimple orbital integrals.) Similarly, while Yu derived an expression for D in all three cases above, he avoided the use of endoscopy [Yu09].

The main purpose of the present paper is to provide an endoscopic interpretation for the residue in Cases 1–3, under the simplifying assumptions that $E = F$ and that G' is trivial (so that G equals either G_1 or its adjoint form).

More precisely, in Cases 1 and 2 (resp., Case 3), we find a twisted endoscopic group (strictly speaking, a twisted endoscopic datum) H such that the distribution D is the twisted endoscopic transfer of the distribution, say D^{H}, on $\mathrm{H}(F)$, given by $f^{\mathrm{H}} \mapsto f^{\mathrm{H}}(1)$ (resp., $f^{\mathrm{H}} \mapsto f^{\mathrm{H}}(-1)$; this also explains why we are separating Case 3 from Case 2).

Even in Cases 1 and 2, where the statement of the final result bears resemblance to analogous results in [Sha00], the proofs differ, since, as mentioned above, the twisted orbital integrals that contribute to the formula of [Sha92] and [Yu09] are no longer semisimple, and the twisted endoscopic data are no longer basic in the sense of Shelstad's appendix to [Sha00]. In particular, the transfer factors are not as simple as the ones mentioned at the beginning of Section 3 of Shelstad's appendix to [Sha00]. Instead one studies the endoscopic transfer of D^{H} using Waldspurger's descent for transfer factors [Wal08], which tells us how transfer factors behave near matching conjugacy classes. With this done, the behavior of nilpotent orbital integrals under scaling and what at least *prima facie* has the appearance of a fortuitous coincidence involving dimensions of minimal nilpotent orbits in symplectic groups (see Remark 3.5) constrain the twisted endoscopic transfer of D^{H} to be supported in a certain finite union of twisted $\mathrm{M}(F)$-conjugacy classes, in fact in a single orbit for action under a closely related group. Certain equivariance properties of transfer factors then pin this endoscopic transfer down up to a scalar, forcing it to equal (a multiple of) D.

We should also add that, at least in Case 1 when $\mathrm{G} = \mathrm{G}_1$ (as opposed to its adjoint form), and in Cases 2 and 3 when G is the adjoint form of G_1, our conclusion concerning irreducibility is not new as it follows from Arthur's work [Art13, §6.6]. However, we have chosen to include these cases because Shahidi's method of computing residues of intertwining operators, which we follow, is of interest. For instance, we hope that understanding this method in the simpler situations where N has a Zariski dense $\mathrm{Int}\,\mathrm{M}$-orbit (to which we are restricting ourselves) will inform approaches to more difficult situations where this is not the case; and some of these more difficult situations are expected to feature interesting L-functions such as the symmetric cube L-function for GL_2. Including the known cases mentioned above, also has the advantage of serving to illustrate how the method functions across groups that are isogenous.

While our proofs are not case-independent, we have attempted to separate the parts where we rely on case-by-case computation from the rest. Our prescription for the endoscopic data, for instance, is uniform. We have identified [Sha90, Lemma 2.1 (b)] applied to the dual group of G in place of G, as the reason why L-function considerations in our cases of interest are naturally related to certain endoscopic data (see Remark 2.4).

Let us now discuss the contents of this paper. In Section 2, we set some notation and define the twisted endoscopic data for M (and a certain automorphism of it coming from conjugation by the element w_0 discussed earlier) that will concern us. These data, in general, depend not only on G, P, and M, but also on the restriction of the central character of the representation π of interest to $\mathrm{A}(F)$, A being the connected center of M. In Section 3, we assume certain lemmas about the behavior of the twisted endoscopic transfer that concerns us and use these to interpret the residue formula of Shahidi (due to [Yu09] in some of our cases) in terms of endoscopic transfer. Finally, in Section 4, we prove the lemmas that are assumed in Section 3.

2 The Groups and Endoscopic Data

2.1 Some Notations

We begin by digressing to fix notational conventions that will be in force throughout this paper. For an algebraic group G', $\mathrm{Z}_{\mathrm{G}'}$ will denote its center. If G' acts on a variety X and $x \in \mathrm{X}$, G'^{x} will stand for the stabilizer of x in G' and G'_x for the identity component of G'^{x}.

Recall [Wal08, §1.2] the notion of a twisted space $\widetilde{\mathrm{G}}'$ for a connected reductive group G'. It is a bitorsor for G'. When we talk of $\mathrm{G}'^{\widetilde{\eta}}$ or $\mathrm{G}'_{\widetilde{\eta}}$, where $\widetilde{\eta}$ belongs to G' or the twisted space $\widetilde{\mathrm{G}'}$, the reference will be understood to be to the action of G' by conjugation (on itself or on $\widetilde{\mathrm{G}'}$ as the case may be).

Given $\widetilde{\eta} \in \widetilde{\mathrm{G}'}$, η will stand for the automorphism $\operatorname{Int}\widetilde{\eta}$ of G', *i.e.*, $\widetilde{\eta}g = \eta(g)\cdot\widetilde{\eta}$ for all $g \in \mathrm{G}'$. Given an automorphism η of G', G'^{η} will stand for its group of fixed points, and G'_η for the identity component of G'^{η}. Thus, we have

$$\mathrm{G}'^{\widetilde{\eta}} = \mathrm{G}'^{\eta} = \mathrm{G}'^{\operatorname{Int}\widetilde{\eta}}, \quad \mathrm{G}'_{\widetilde{\eta}} = \mathrm{G}'_{\eta} = \mathrm{G}'_{\operatorname{Int}\widetilde{\eta}}.$$

For a diagonalizable group T, $X^*(\mathrm{T})$ and $X_*(\mathrm{T})$ will denote, respectively, the character group and cocharacter group of the base change $\mathrm{T}\times_F\overline{F}$ of T to $\overline{F}$. Fraktur letters will be used for Lie algebras according to standard conventions, *e.g.*, $\mathfrak{g}' = \operatorname{Lie}\mathrm{G}'$. A superscript of 0 will stand for *identity component*. For a reductive group G', $\mathrm{G}'_{\mathrm{sc}}$ and $\mathrm{G}'_{\mathrm{ad}}$ will denote the simply connected cover of the derived group of G' and its adjoint form, respectively. However, we will override this convention in the context of a maximal torus T of such a G', by using T_{sc} and T_{ad} to denote the maximal tori corresponding to T in $\mathrm{G}'_{\mathrm{sc}}$ and $\mathrm{G}'_{\mathrm{ad}}$, respectively. If G' is an algebraic group over $\mathbb{C}$, we will use G' and $\mathrm{G}'(\mathbb{C})$ interchangeably. In Section 2.2 below, F will denote a field of characteristic zero, while from Section 2.3 onwards, F will denote a p-adic field.

2.2 Some Useful Computations

Before we describe the groups in which we are interested, we recall a few well-known computations of interest.

For this subsection alone, we allow F to be any field of characteristic zero; we will use the computations below for groups over a p-adic field (that we will later use F to denote), and for groups over $\mathbb{C}$.

Let G be a connected reductive group over F and $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ a Levi decomposition of a maximal parabolic subgroup of G. Let $w_0 \in \mathrm{G}(F)$ be such that $\mathrm{P}\,w_0\,\mathrm{P}$ is of the largest possible dimension (and hence is open in G). Further, we assume that w_0 can be chosen to normalize M (namely, the parabolic P is self-associate), which will be satisfied in all the cases that we consider. Although w_0 is not unique, the set $\mathrm{M}\,w_0$ is. This makes $\widetilde{\mathrm{M}} := w_0\,\mathrm{M} = \mathrm{M}\,w_0 = \mathrm{M}\,w_0^{-1} \subset \mathrm{G}$ a twisted space for M; it has actions of M coming from left and right multiplication that make it a bitorsor for M.

Shahidi's approach to computing the residues of intertwining operators involves transferring integrals over $\mathrm{N}(F)$ to those over suitable subsets of $\widetilde{\mathrm{M}}(F)$, by means of a map $\mathrm{N}' \to \widetilde{\mathrm{M}}$, N' being an open subset of N. Let us review this map: $\mathrm{N}^- := w_0\,\mathrm{N}\,w_0^{-1}$ is the unipotent radical of the parabolic subgroup of G opposite to P and containing M. Note that $\mathrm{N}' := \mathrm{N} \cap \widetilde{\mathrm{M}}\,\mathrm{N}\,\mathrm{N}^- = \mathrm{N} \cap \mathrm{M}\,\mathrm{N}^-\,w_0\,\mathrm{N}^-$ is open in N and also nonempty, as the intersection $\mathrm{M}\,\mathrm{N}^-\,\mathrm{N} \cap \mathrm{M}\,\mathrm{N}^-\,w_0\,\mathrm{N}^-$ of nonempty open subsets of G is. Further $\mathrm{N}'(F)$ is nonempty as $\mathrm{N}(F)$ is Zariski dense in $\mathrm{N}(\overline{F})$. We have a map $\mathrm{N}'(F) \to \widetilde{\mathrm{M}}(F)$, defined by

$$n \mapsto \widetilde{m}, \text{ where } n \in \widetilde{m}\,\mathrm{N}\,\mathrm{N}^- . \tag{2.1}$$

We would like to discuss the above considerations more explicitly when G is the connected isometry group of a nondegenerate quadratic or symplectic form $\langle \cdot, \cdot \rangle$ on a vector space V. Excluding the case where $\dim V$ is twice an odd number and $\langle \cdot, \cdot \rangle$ is quadratic (see [Sha92, Lemma 3.4]), a case that will not concern us in this paper, it is well known that maximal parabolic subgroups of G are all self-associate and in one-to-one correspondence with isotropic subspaces W' of $(V, \langle \cdot, \cdot \rangle)$. This correspondence maps P to W', where P is the stabilizer of W'. Levi subgroups M of such a P then correspond to orthogonal decompositions

$$V = W' \oplus X \oplus W, \tag{2.2}$$

where X is a nondegenerate subspace of V, and W is an isotropic subspace in duality with W' under $\langle \cdot, \cdot \rangle$.

Fix $V, \langle \cdot, \cdot \rangle, W, X, W'$, G, P, and M as in the above paragraph. Let N be the unipotent radical of P and N^- that of the parabolic subgroup opposite to P and containing M. As in [MS16], any element $n \in \mathrm{N}(F)$ can be specified by a pair (ξ, η) with $\xi \in \mathrm{Hom}(X, W')$ and $\eta \in \mathrm{Hom}(W, W')$ such that $\xi\xi^* = -(\eta + \eta^*)$ as elements of $\mathrm{Hom}(W, W')$ (where the $*$ is taken using $\langle \cdot, \cdot \rangle$). This element, which we denote by $n(\xi, \eta)$, is pictorially represented on the left-hand side of (2.3) below: it acts as the identity on W', as $x \mapsto x + \xi(x)$ on X, and as $w \mapsto w - \xi^*(w) + \eta(w)$ on W.

The map $n \mapsto \widetilde{m}$ is then easily read off from the following assertion, which is straightforward to verify (or see, *e.g.*, [MS16]). An element $n = n(\xi, \eta) \in \mathrm{N}$ belongs to N' if and only if $\eta \in \mathrm{Isom}(W, W')$, in which case the unique decomposition

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$n = \widetilde{m} n' n^-$, with $\widetilde{m} \in \widetilde{\mathrm{M}}$, $n' \in \mathrm{N}$, and $n^- \in \mathrm{N}^-$, takes the form

$$(2.3)\qquad \begin{pmatrix} 1 & \xi & \eta \\ & 1 & -\xi^* \\ & & 1 \end{pmatrix} = \begin{pmatrix} & & \eta \\ & 1 + \xi^* \eta^{-1} \xi & \\ \eta^{-*} & & \end{pmatrix} \times \begin{pmatrix} 1 & -\eta^* \eta^{-1} \xi & \eta^* \\ & 1 & \xi^* \eta^{-*} \eta \\ & & 1 \end{pmatrix} \times \begin{pmatrix} 1 & & \\ -\xi^* \eta^{-*} & 1 & \\ \eta^{-1} & \eta^{-1} \xi & 1 \end{pmatrix}$$

(with $\eta^{-*} = (\eta^*)^{-1} = (\eta^{-1})^*$).

Note that $\mathrm{M} \cong \mathrm{GL}(W) \times \mathrm{Aut}^0(X, \langle \cdot, \cdot \rangle)$ (the $\mathrm{GL}(W)$ factor identifying with the bottom right, and not the top left, entry). Then $\widetilde{\mathrm{M}}$ identifies naturally with the set of self-maps of V that takes W and W' into each other and preserve X (and lie in $\mathrm{Aut}^0(V, \langle \cdot, \cdot \rangle)$), *i.e.*, with $\mathrm{Isom}(W, W') \times \widetilde{\mathrm{Aut}}(X, \langle \cdot, \cdot \rangle)$, as a twisted space for $\mathrm{GL}(W) \times \mathrm{Aut}^0(X, \langle \cdot, \cdot \rangle)$. Here $\widetilde{\mathrm{Aut}}(X, \langle \cdot, \cdot \rangle)$ is a connected component of

$$\mathrm{Aut}(X, \langle \cdot, \cdot \rangle):$$

specifically, the identity component if either $\langle \cdot, \cdot \rangle$ is symplectic or $\dim W$ is even, and the non-identity-component otherwise. Here, if $g_1, g_2 \in \mathrm{GL}(W) \subset \mathrm{M}$ and $T \in \mathrm{Isom}(W, W')$, then $g_1 T g_2 = (g_1^{-1})^* \circ T \circ g_2$, where $g_1^* \in \mathrm{GL}(W')$ is such that $\langle g_1 w, w' \rangle = \langle w, g_1^* w' \rangle$ for all $w \in W, w' \in W'$.

We choose the identification of $\mathrm{Isom}(W, W')$ with the twisted space $\widetilde{\mathrm{GL}}(W)$ of nondegenerate bilinear forms on W that takes $T \in \mathrm{Isom}(W, W')$ to

$$(w_1, w_2) \mapsto \langle w_1, T w_2 \rangle.$$

This lets us view $\widetilde{\mathrm{M}}$ as the product twisted space $\widetilde{\mathrm{GL}}(W) \times \widetilde{\mathrm{Aut}}(X, \langle \cdot, \cdot \rangle)$ for the group $\mathrm{M} = \mathrm{GL}(W) \times \mathrm{Aut}^0(X, \langle \cdot, \cdot \rangle)$, where $\widetilde{\mathrm{GL}}(W)$ is viewed as a twisted space for $\mathrm{GL}(W)$ via the action $(g_1 B g_2)(w_1, w_2) = B(g_1^{-1} w_1, g_2 w_2)$. To realize the twisted space in this form, we followed [Li13] in choosing P to be the stabilizer of W', rather than of W.

2.3 The Description of G and P

We will deal with a connected reductive group G over F, with a self-associate maximal parabolic subgroup having Levi decomposition $\mathrm{P} = \mathrm{M}\,\mathrm{N}$. We assume that

$$(\mathrm{G}, \mathrm{P}, \mathrm{M}, \mathrm{N})$$

equals either $(\mathrm{G}_1, \mathrm{P}_1, \mathrm{M}_1, \mathrm{N}_1)$ or $(\mathrm{G}_2, \mathrm{P}_2, \mathrm{M}_2, \mathrm{N}_2)$ where the $(\mathrm{G}_i, \mathrm{P}_i, \mathrm{M}_i, \mathrm{N}_i)$ are as in one of three cases that we denote by Cases 1–3 below.

- Case 1. Let $n \in \mathbb{N}, n \geq 2$. Let $(V, \langle \cdot, \cdot \rangle)$ be a $2n$-dimensional symplectic space over F. Set $\mathrm{G}_1 = \mathrm{Sp}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{Sp}_{2n}/F, \mathrm{G}_2 = \mathrm{PSp}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{PSp}_{2n}/F$. We have an obvious isogeny $\mathrm{G}_1 \to \mathrm{G}_2$. Fix a decomposition $V = W' \oplus X \oplus W$ as in (2.2), such that $\dim W = 1$. For $i = 1, 2$, this defines a Levi decomposition $\mathrm{P}_i = \mathrm{M}_i \mathrm{N}_i$ of a maximal parabolic subgroup of G_i such that $\mathrm{M}_1 \cong \mathrm{GL}_1 \times \mathrm{Sp}_{2n-2}/F$ and $\mathrm{M}_2 = \mathrm{GSp}_{2n-2}/F$ as per a usual identification.
- Cases 2 and 3. Let $n \in \mathbb{N}$. Let $(V, \langle \cdot, \cdot \rangle)$ be a quadratic space over F of dimension $2n+1$ such that the associated groups $\mathrm{G}_1 = \mathrm{Spin}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{Spin}_{2n+1}/F$ and $\mathrm{G}_2 = \mathrm{SO}(V, \langle \cdot, \cdot \rangle) \cong \mathrm{SO}_{2n+1}/F$ are split. If n is even, we set $n = 2N$ and say

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that we are in Case 2. Else we set $n = 2N + 1$ and say that we are in Case 3. We have an obvious isogeny $\mathrm{G}_1 \to \mathrm{G}_2$. Fix a decomposition $V = W' \oplus X \oplus W$ as in (2.2), such that $\dim W = n$. This decomposition defines, for $i = 1, 2$, parabolic subgroups $\mathrm{P}_i = \mathrm{M}_i \mathrm{N}_i$ of G_i, together with an identification $\mathrm{M}_2 = \mathrm{GL}(W)/F$. It is easy to check, *e.g.*, by first computing the Langlands dual $\widehat{\mathrm{M}}_1$, that we have an obvious identification of algebraic groups over F:

$$\mathrm{M}_1 = \{(x, g) \mid x \in \mathbb{G}_m, g \in \mathrm{GL}(W), x^2 = \det g\}. \tag{2.4}$$

Note that the unipotent radical N of P can be identified naturally with either of N_1 or N_2.

Let A_i $(i = 1, 2)$ denote the identity component of the center of M_i, and A that of M.

Remark 2.1 In both Case 1 and Case 2, the obvious isogeny $\mathrm{M}_1 \to \mathrm{M}_2$ induces an isomorphism $\mathrm{A}_1 \to \mathrm{A}_2$, so that we have obvious identifications $\mathrm{A} = \mathrm{A}_1 = \mathrm{A}_2 = \mathbb{G}_m$. In Case 3, however, $\mathrm{A}_1 \to \mathrm{A}_2$ is an isogeny with kernel of order two. In this case, A_1 is realized via (2.4) as the set of all (a^n, a^2), which identifies with $\mathbb{G}_m$ since $(n, 2) = 1$.

2.4 An Endoscopic Datum for M

We can identify $\widehat{\mathrm{G}}_i$, where i equals 1 (in Case 1) or 2 (in Cases 2 and 3) with the connected isometry group of a nondegenerate symplectic or quadratic space $(\widehat{V}, \widehat{q})$ over $\mathbb{C}$. Thus, as in Section 2.2, we may choose a decomposition $\widehat{V} = \widehat{W}' \oplus \widehat{X} \oplus \widehat{W}$ corresponding to a Levi decomposition of a parabolic subgroup of $\widehat{\mathrm{G}}_i$ featuring $\widehat{\mathrm{M}}_i = \mathrm{GL}(\widehat{W}) \times \mathrm{Aut}^0(\widehat{X}, \widehat{q}|_{\widehat{X}})$ as the underlying Levi subgroup. Dual to the chain of isogenies $\mathrm{M}_1 \to \mathrm{M} \to \mathrm{M}_2$, we get a chain of isogenies:

$$\widehat{\mathrm{M}}_2 \to \widehat{\mathrm{M}} \to \widehat{\mathrm{M}}_1. \tag{2.5}$$

Note that we have:

$$\widehat{\mathrm{M}}_2 = \begin{cases} \mathrm{GSpin}(\widehat{X}, \widehat{q})(\mathbb{C}) \cong \mathrm{GSpin}_{2n-1}(\mathbb{C}) & \text{in Case 1,} \\ \mathrm{GL}(\widehat{W})(\mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C}) & \text{in Cases 2 and 3,} \end{cases}$$

while

$$\widehat{\mathrm{M}}_1 = \begin{cases} \mathrm{GL}(\widehat{W})(\mathbb{C}) \times \mathrm{SO}(\widehat{X})(\mathbb{C}) \cong \mathrm{GL}_1(\mathbb{C}) \times \mathrm{SO}_{2n-1}(\mathbb{C}) & \text{in Case 1,} \\ \mathrm{GL}(\widehat{W})(\mathbb{C})/\{\pm 1\} \cong \mathrm{GL}_n(\mathbb{C})/\{\pm 1\} & \text{in Cases 2 and 3.} \end{cases}$$

Further, for $i = 1, 2$, W_F acts trivially on $\widehat{\mathrm{M}}_i$ so that ${}^L\mathrm{M}_i = \widehat{\mathrm{M}}_i \times W_F$.

2.5 Twisted Space and Endoscopic Data of Interest

Recall the central subgroup $\mathbb{G}_m \cong \mathrm{A} \subset \mathrm{M}$ of M. Fix a quadratic character $\omega\colon \mathrm{A}(F) \cong F^\times \to \mathbb{C}^\times$. If we are in Case 3 and additionally $\mathrm{M} = \mathrm{M}_1$, we require ω to be trivial. Now we wish to define an endoscopic datum $(\mathrm{H} = \mathrm{H}_\omega, \mathcal{H}, s, \widehat{\xi})$, depending on ω, for $(\widetilde{\mathrm{M}}, \mathbb{1})$, as in [KS99, Wal08] (our notations will be closer to the latter), where we write $\mathbb{1}$ to denote the trivial character, in this case of $\mathrm{M}(F)$.

2.5.1 Defining $\widehat{\mathrm{H}}, s$, and $\widehat{\theta}$

Before explicating $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$, we briefly describe the general construction of the dual group $\widehat{\mathrm{H}}$ of H. Exactly the same prescription as with w_0 allows us to consider an element $\widehat{w}_0 \in \widehat{\mathrm{G}}$, and a twisted space $\widetilde{\widehat{\mathrm{M}}}$ for $\widehat{\mathrm{M}}$. Set $\widetilde{\widehat{\theta}} := \widehat{w}_0^{-1} \in \widetilde{\widehat{\mathrm{M}}}(F)$. We choose $\widehat{w}_0$ such that $\widehat{\theta} = \operatorname{Int} \widehat{w}_0^{-1}$ preserves a splitting of $\widehat{\mathrm{M}}$. Then the outer automorphism defined by $\widehat{\theta}$ is dual to that defined by $\operatorname{Int} \widetilde{m}$ for any $\widetilde{m} \in \widetilde{\mathrm{M}}(F)$. Recall that for a suitable open subset $\widehat{\mathrm{N}}' \subset \widehat{\mathrm{N}}$, we have a map $\widehat{\mathrm{N}}' \to \widetilde{\widehat{\mathrm{M}}}$ as in (2.1). In all our cases, $\widehat{\mathrm{M}}$ has a unique open orbit in $\widehat{\mathrm{N}}$, which is contained in $\widehat{\mathrm{N}}'$. Fix $\widehat{n}$ belonging to this open orbit. Write the image of $\widehat{n}$ under the above map as $s\widetilde{\widehat{\theta}}$ with $s \in \widehat{\mathrm{M}}$. Then $\widehat{\mathrm{H}}$ is simply the connected centralizer of $\operatorname{Int} s \circ \widehat{\theta}$ in $\widehat{\mathrm{M}}$, *i.e.*, $\widehat{\mathrm{H}} = \widehat{\mathrm{M}}_{s\widetilde{\widehat{\theta}}}$. This does not yet define $\mathrm{H} = \mathrm{H}_\omega$, as we have not specified the action of W_F on $\widehat{\mathrm{H}}$ (which is not in general the obvious one).

We wish to describe $\widehat{\mathrm{H}}$ explicitly and specify Borel pairs $(\widehat{\mathrm{B}}, \widehat{\mathrm{T}})$ in $\widehat{\mathrm{M}}$ and $(\widehat{\mathrm{B}}_{\mathrm{H}}, \widehat{\mathrm{T}}_{\mathrm{H}})$ in $\widehat{\mathrm{H}}$, such that $s \in \widehat{\mathrm{T}}$ and $\widehat{\mathrm{T}}_{\mathrm{H}} = \widehat{\mathrm{T}}$. This will also involve fixing choices and notation to work with later in computations. Using (2.3), it is easy to check that we may choose things as follows.

Case 1: We have (since $\dim \widehat{W} = 1$ is odd and $\widehat{q}$ is orthogonal)

$$\widehat{\mathrm{M}}_1 \widetilde{\widehat{\theta}} = \widetilde{\mathrm{GL}}(\widehat{W}) \times (\mathrm{O}(\widehat{X}) \smallsetminus \mathrm{SO}(\widehat{X})).$$

If $\mathrm{G} = \mathrm{G}_1$, the image of $s\widetilde{\widehat{\theta}}$ in $\widehat{\mathrm{M}}_1 \widetilde{\widehat{\theta}}$ is of the form (a, s_0), where s_0 acts as the identity on a nondegenerate $2n-2$-dimensional subspace $\widehat{X}_1$ of $\widehat{X}$, and as -1 on its orthogonal complement. Therefore we conclude that $\widehat{\mathrm{H}}$ identifies with $\mathrm{SO}(\widehat{X}_1)(\mathbb{C}) \cong \mathrm{SO}_{2n-2}(\mathbb{C})$ if $\mathrm{G} = \mathrm{G}_1$, and with $\mathrm{Spin}(\widehat{X}_1)(\mathbb{C}) \cong \mathrm{Spin}_{2n-2}(\mathbb{C})$ if $\mathrm{G} = \mathrm{G}_2$. Choose an ordered basis $\widehat{e}_1, \dots, \widehat{e}_{2n-1}$ of $\widehat{X}$ such that $\widehat{q}(\widehat{e}_i, \widehat{e}_j) = (-1)^i \delta_{i,(2n-j)}$ for $1 \le i \le 2n-1$, and such that $\widehat{e}_n$ is the orthogonal complement of $\widehat{X}_1$ (here, $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise). We choose $(\widehat{\mathrm{B}}, \widehat{\mathrm{T}})$ to be the Borel pair determined by the ordered basis $\widehat{e}_1, \dots, \widehat{e}_{2n-1}$ of $\widehat{X}$, and $(\widehat{\mathrm{B}}_{\mathrm{H}}, \widehat{\mathrm{T}}_{\mathrm{H}})$ to be that determined by the ordered basis $\widehat{e}_1, \dots, \widehat{e}_{n-1}, \widehat{e}_{n+1}, \dots, \widehat{e}_{2n-1}$ of $\widehat{X}_1$. We take $\widetilde{\widehat{\theta}}$ to have image $(a, -1) \in \widehat{\mathrm{M}}_1$, forcing $s \in \widehat{\mathrm{T}}$.

Case 2: Assume first that $\mathrm{G} = \mathrm{G}_2$. Write $\widehat{\mathrm{M}}_2$ as $\mathrm{GL}(\widehat{W})$, so one has a natural identification $\widehat{\mathrm{M}}_2 \widetilde{\widehat{\theta}} = \widetilde{\mathrm{GL}}(\widehat{W})$. Thus, the elements $\widetilde{\widehat{\theta}} = \widehat{w}_0^{-1}, s\widetilde{\widehat{\theta}} \in \widetilde{\mathrm{GL}}(\widehat{W})$ are now bilinear forms on $\widehat{W}$ (see §2.2). We choose $\widetilde{\widehat{\theta}}$ so that $\widetilde{\widehat{\theta}}(\widehat{e}_i, \widehat{e}_j) = (-1)^i \delta_{i,(2N+1-j)}$ for a basis $\widehat{e}_1, \dots, \widehat{e}_{2N}$ of $\widehat{W}$ (thus, $\widetilde{\widehat{\theta}}$ is a symplectic form): this choice is consistent with the requirement that $\widehat{\theta}$ preserve a splitting of $\widehat{\mathrm{M}}$, see, *e.g.*, [Art13, (1.2.1)]. Since $\widehat{\mathrm{M}}$ is a Siegel–Levi subgroup of $\widehat{\mathrm{G}} = \mathrm{Sp}(\widehat{V})$, (2.3) gives us that $s\widetilde{\widehat{\theta}}$ is a nondegenerate quadratic form on W. Hence, we can assume that $s\widetilde{\widehat{\theta}}(\widehat{e}_i, \widehat{e}_j) = \delta_{i,(2N+1-j)}$ (after changing it within the orbit for $\mathrm{GL}(\widehat{W})(\mathbb{C})$-conjugation on $\widetilde{\mathrm{GL}}(\widehat{W})(\mathbb{C})$, which identifies with the obvious action of $\mathrm{GL}(\widehat{W})(\mathbb{C})$ on the space of nondegenerate bilinear forms on $\widehat{W}$). When $\mathrm{G} = \mathrm{G}_1$, we choose s and $\widetilde{\widehat{\theta}}$ to be the images of the corresponding choices for G_2. Note that $\widehat{\mathrm{H}} = \mathrm{SO}(\widehat{W}, s\widetilde{\widehat{\theta}})(\mathbb{C}) \cong \mathrm{SO}_{2N}(\mathbb{C})$ if $\mathrm{G} = \mathrm{G}_2$, and

$\widehat{\mathrm{H}} = \mathrm{PSO}(\widehat{W}, s\widetilde{\widetilde{\widehat{\theta}}})(\mathbb{C}) \cong \mathrm{PSO}_{2N}(\mathbb{C})$ if $\mathrm{G} = \mathrm{G}_1$. We choose the Borel pairs $(\widehat{\mathrm{B}}, \widehat{\mathrm{T}})$ for $\widehat{\mathrm{M}}$ and $(\widehat{\mathrm{B}}_{\mathrm{H}}, \widehat{\mathrm{T}}_{\mathrm{H}})$ for $\widehat{\mathrm{H}}$ determined by the ordered basis $\widehat{e}_1, \dots, \widehat{e}_{2N}$ of $\widehat{W}$. Clearly $s \in \widehat{\mathrm{T}}$.

Case 3: This is similar to Case 2, except that here we assume that $s\widetilde{\widetilde{\widehat{\theta}}} = \widetilde{\widetilde{\widehat{\theta}}}$ and that $\widetilde{\widetilde{\widehat{\theta}}}(\widehat{e}_i, \widehat{e}_j) = (-1)^i \delta_{i,(2N+2-j)}$ for all i, j. Then $\widehat{\mathrm{H}} = \mathrm{SO}(\widehat{W}, s\widetilde{\widetilde{\widehat{\theta}}})(\mathbb{C}) \cong \mathrm{SO}_{2N+1}(\mathbb{C})$, both when $\mathrm{G} = \mathrm{G}_1$ and when $\mathrm{G} = \mathrm{G}_2$. The Borel pairs $(\widehat{\mathrm{B}}, \widehat{\mathrm{T}})$ for $\widehat{\mathrm{M}}$ and $(\widehat{\mathrm{B}}_{\mathrm{H}}, \widehat{\mathrm{T}}_{\mathrm{H}})$ for $\widehat{\mathrm{H}}$ are defined using the ordered basis $\widehat{e}_1, \dots, \widehat{e}_{2N+1}$, and we again have $s \in \widehat{\mathrm{T}}$.

2.5.2 The Endoscopic Datum

Now we define a subgroup $\mathcal{H}' \subset {}^L\mathrm{G} = \widehat{\mathrm{G}} \rtimes W_F$ that will be the image of $\mathcal{H}$ under $\widehat{\xi}$. We have a surjection $\widehat{\mathrm{M}} \to \widehat{\mathrm{A}} \cong \mathbb{C}^\times$ dual to $\mathbb{G}_m \cong \mathrm{A} \hookrightarrow \mathrm{M}$. In Case 1, this map is obtained by precomposing the map $\widehat{\mathrm{M}} \to \widehat{\mathrm{M}}_1$ (*cf.* (2.5)) with the map $\widehat{\mathrm{M}}_1 \to \mathbb{C}^\times$ that is a projection onto the $\mathrm{GL}(\widehat{W})(\mathbb{C})(\cong \mathbb{C}^\times)$-factor. In Cases 2 and 3, when $\mathrm{G} = \mathrm{G}_2$, this map is the determinant. In Case 2 when $\mathrm{G} = \mathrm{G}_1$, this map is induced by the determinant. In Case 3, when $\mathrm{G} = \mathrm{G}_1$, this map is induced by the square of the determinant on $\mathrm{GL}_{2N+1}(\mathbb{C})$. Both $\widehat{\theta}$ (which equals $\mathrm{Int}\, \widetilde{\widetilde{\widehat{\theta}}}$) and $\mathrm{Int}(s\widetilde{\widetilde{\widehat{\theta}}})$ induce the automorphism $x \mapsto x^{-1}$ of $\widehat{\mathrm{A}}$. Using $\widehat{\theta}$ to also denote this automorphism of $\widehat{\mathrm{A}}$, we have $\widehat{\mathrm{A}}^{\widehat{\theta}} \cong \{\pm 1\}$.

We fixed an element $\widehat{n}$ in Section 2.5.1. Let $\widehat{\mathrm{M}}^{\widehat{n}}$ denote the stabilizer of $\widehat{n}$ in $\widehat{\mathrm{M}}$. Thus, we have $\widehat{\mathrm{M}}^{\widehat{n}} \subset \widehat{\mathrm{M}}^{s\widetilde{\widetilde{\widehat{\theta}}}}$ [Sha00, Lemma 2.1(a)]. Moreover, using [Sha00, Lemma 2.1(b)], we have that $\widehat{\mathrm{M}}^{\widehat{n}}$ has the same identity component as $\widehat{\mathrm{M}}^{s\widetilde{\widetilde{\widehat{\theta}}}}$, *i.e.*, $\widehat{\mathrm{M}}_{\widehat{n}} = \widehat{\mathrm{M}}_{s\widetilde{\widetilde{\widehat{\theta}}}} = \widehat{\mathrm{H}}$.

The following lemma is crucial for pinning down our endoscopic datum completely.

Lemma 2.2 (i) *The restriction of the map* $\widehat{\mathrm{M}} \to \widehat{\mathrm{A}}$ *induces an injection*

$$\Lambda\colon \widehat{\mathrm{M}}^{\widehat{n}} / \widehat{\mathrm{M}}_{s\widetilde{\widetilde{\widehat{\theta}}}} \hookrightarrow \widehat{\mathrm{A}}^{\widehat{\theta}},$$

which is an isomorphism except in Case 3 *when* $\mathrm{M} = \mathrm{M}_1$.

(ii) *The homomorphism* $\widehat{\mathrm{M}}^{\widehat{n}} \to \widehat{\mathrm{M}}^{\widehat{n}} / \widehat{\mathrm{M}}_{s\widetilde{\widetilde{\widehat{\theta}}}}$ *has a group theoretic section, whose image can be chosen to fix a given splitting of* $\widehat{\mathrm{H}} = \widehat{\mathrm{M}}_{s\widetilde{\widetilde{\widehat{\theta}}}}$.

Proof (i) is immediately verified when either $\mathrm{G} = \mathrm{G}_1$ and we are in Case 1, or when $\mathrm{G} = \mathrm{G}_2$ and we are in Cases 2 or 3. For the rest of Cases 1 and 2 (i) follows from the easily verified fact that in these cases the obvious maps $\mathrm{A}_1 \to \mathrm{A}_2$ and $\widehat{\mathrm{M}}_2^{\widehat{n}} / \widehat{\mathrm{M}}_{2, s\widetilde{\widetilde{\widehat{\theta}}}} \to \widehat{\mathrm{M}}_1^{\widehat{n}} / \widehat{\mathrm{M}}_{1, s\widetilde{\widetilde{\widehat{\theta}}}}$ are isomorphisms (for Case 1 when $\mathrm{G} = \mathrm{G}_2$, use the fact that $\widehat{\mathrm{M}}_2^{\widehat{n}} \to \widehat{\mathrm{M}}_1^{\widehat{n}}$ is necessarily surjective and the fact that the group of fixed points of the semisimple automorphism $\mathrm{Int}(s\widetilde{\widetilde{\widehat{\theta}}})$ on the simply connected group $\mathrm{Spin}(\widehat{X})$ is necessarily connected [Ste68, Theorem 8.1]). When we are in Case 3 and $\mathrm{G} = \mathrm{G}_1$, (i) is trivial as $\widehat{\mathrm{M}}_1^{\widehat{n}}$ being necessarily the image of $\widehat{\mathrm{M}}_2^{\widehat{n}}$ is easily verified to be connected.

Now let us prove (ii). In Case 3, if $\mathrm{G} = \mathrm{G}_1$, the result is trivial, so let us assume that this is not the case. In what follows we will assume notation from Section 2.5.1 without further comment. Note that $\widehat{\mathrm{M}}^{\widehat{n}}/\widehat{\mathrm{M}}_{s\widetilde{\widehat{\theta}}}$ has order two, so it is enough to produce an element $\widehat{m}_0$ of $\widehat{\mathrm{M}}^{\widehat{n}} \smallsetminus \widehat{\mathrm{M}}_{s\widetilde{\widehat{\theta}}}$ preserving a fixed splitting of $\widehat{\mathrm{M}}_{s\widetilde{\widehat{\theta}}}$ and having order two. We first focus on Case 2 and Case 3. We may and do assume that $\mathrm{G} = \mathrm{G}_2$. In Case 2, $\widehat{m}_0$ may be taken to be the element of $\widehat{\mathrm{M}}_2 = \mathrm{GL}(\widehat{W})(\mathbb{C})$ that swaps $\widehat{e}_N$ and $\widehat{e}_{N+1}$ and fixes every other $\widehat{e}_i$. In Case 3, it suffices instead to take $\widehat{m}_0 = -1 \in \mathrm{GL}(\widehat{W})(\mathbb{C})$.

In Case 1, we first consider the case where $\mathrm{G} = \mathrm{G}_1$. In this case we take $\widehat{m}_0$ to be the element that acts on $\widehat{W}$ and on $\widehat{e}_n$ by multiplication by -1, swaps $\widehat{e}_{n-1}$ and $\widehat{e}_{n+1}$, and acts on every other $\widehat{e}_i$ by fixing it. It is easy to see that $\widehat{m}_0$ fixes a splitting of $\widehat{\mathrm{M}}_{s\widetilde{\widehat{\theta}}}$ that may be in an obvious way associated to the ordered basis $\widehat{e}_1, \dots, \widehat{e}_{n-1}, \widehat{e}_{n+1}, \dots, \widehat{e}_{2n-1}$ of $\widehat{X}_1$.

If $\mathrm{G} = \mathrm{G}_2$, take $\widehat{m}_0$ to be a lift of the corresponding element considered when $\mathrm{G} = \mathrm{G}_1$. This is easily seen to work, provided we verify that $\widehat{m}_0^2 = 1$. View $\widehat{\mathrm{M}}_2 = \mathrm{GSpin}(\widehat{X})(\mathbb{C})$ as a quotient $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{Spin}(\widehat{X})(\mathbb{C})/\{(1,1),(-1,c)\}$, where c is the unique nontrivial element in the kernel of the map $\mathrm{Spin}(\widehat{X})(\mathbb{C}) \to \mathrm{SO}(\widehat{X})(\mathbb{C})$ (see [Asg02]). It suffices to show that any preimage (a,h) of $\widehat{m}_0$ in $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{Spin}(\widehat{X})(\mathbb{C})$ satisfies $a^2 = -1$, $h^2 = c$. The image of (a,h) in $\widehat{\mathrm{M}}_1 = \mathrm{GL}(\widehat{W})(\mathbb{C}) \times \mathrm{SO}(\widehat{X})(\mathbb{C})$ equals $(a^2, \overline{h})$, where $\overline{h}$ is the image of h in $\mathrm{SO}(\widehat{X})(\mathbb{C})$. That (a,h) maps to $\widehat{m}_0$, which acts on $\widehat{W}$ by -1, shows that $a^2 = -1$. It is clear that h^2 maps to the identity element of $\mathrm{SO}(\widehat{X})(\mathbb{C})$, so it suffices to show that h^2 is nontrivial in $\mathrm{Spin}(\widehat{X})(\mathbb{C})$. Note that h^2 depends only on the image $\overline{h}$ of h in $\mathrm{SO}(\widehat{X})(\mathbb{C})$, and in fact (thanks to h^2 being central) only on the conjugacy class of the image of h in $\mathrm{SO}(\widehat{X})(\mathbb{C})$. This is the unique conjugacy class of $\mathrm{SO}(\widehat{X})(\mathbb{C})$ defined by the condition of having a -1-eigenspace of dimension two and a 1-eigenspace of the complementary dimension. A member of this class may be written as $\alpha^\vee(\sqrt{-1})$ for some square root $\sqrt{-1}$ of -1, where α is a short root of $\mathrm{SO}(\widehat{X})$ (note that we have written $\alpha^\vee(\sqrt{-1})$ and not $\alpha^\vee(-1)$ because $\alpha^\vee(t)$ has a t^2-eigenspace of dimension 1, a t^{-2}-eigenspace of dimension 1, and a 1-eigenspace of dimension $2n-3$). Thus, the condition $h^2 \neq 1$ is equivalent to the condition $\alpha^\vee(-1) \neq 1$ in $\mathrm{Spin}(\widehat{X})(\mathbb{C})$, which follows as $\mathrm{Spin}(\widehat{X})$ is simply connected. ∎

Since ω is a quadratic character of $\mathrm{A}(F) \cong F^\times$ and since $\widehat{\theta}$ acts on $\widehat{\mathrm{A}}$ by $x \mapsto x^{-1}$, the Langlands parameter φ_ω of ω is an element of $\mathrm{Hom}(W_F, \widehat{\mathrm{A}}^{\widehat{\theta}})$. Thus, using Lemma 2.2 (i), ω defines an element a_ω of $\mathrm{Hom}(W_F, \mathrm{Out}(\widehat{\mathrm{H}}))$ (recall that we have required ω to be trivial if we are in Case 3 and $\mathrm{M} = \mathrm{M}_1$). We take $\mathrm{H} = \mathrm{H}_\omega$ to be the quasi-split reductive group, which exists and is unique up to an isomorphism, such that we can identify $\mathcal{H} := {}^L\mathrm{H} = \widehat{\mathrm{H}} \rtimes W_F$, where W_F acts on $\widehat{\mathrm{H}}$ through the map $W_F \to \mathrm{Out}(\widehat{\mathrm{H}}) \subset \mathrm{Aut}(\widehat{\mathrm{H}})$. Here, this last inclusion is defined using the chosen splitting on $\widehat{\mathrm{H}}$, and the former map is a_ω. Lemma 2.2 (ii) allows us to lift φ_ω to a map $b \in \mathrm{Hom}(W_F, \widehat{\mathrm{M}}^{\widehat{n}})$. Then the prescription $(h,w) \mapsto (hb(w), w)$ defines an embedding $\widehat{\xi}\colon {}^L\mathrm{H} \hookrightarrow {}^L\mathrm{M}$. Then $(\mathrm{H}, {}^L\mathrm{H}, s, \widehat{\xi})$ is an endoscopic datum for $(\widetilde{\mathrm{M}}, \mathbb{1})$ (indeed, Conditions (2.1.1) and (2.1.2) of [KS99] hold by construction, Condition (2.1.3) because we have checked in each case that $s\widetilde{\widehat{\theta}}$ is semisimple as an element of $\widetilde{\widehat{\mathrm{M}}}$, and Condition (2.1.4) because $\widehat{\mathrm{H}} = \widehat{\mathrm{M}}_{s\widetilde{\widehat{\theta}}}$ and because each $hb(w)$, $h \in \widehat{\mathrm{H}}, w \in W_F$, is contained in $\widehat{\mathrm{M}}^{\widehat{n}} \subset \widehat{\mathrm{M}}^{s\widetilde{\widehat{\theta}}}$). The notion of

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equivalence between endoscopic data (*cf.* [KS99, p. 18]) is such that changing the choice of the splitting of $\widehat{\mathrm{H}}$ that we fixed does not change the equivalence class of our endoscopic datum.

Remark 2.3 We conclude that if $d\colon \widehat{\mathrm{M}} \to \widehat{\mathrm{A}}$ is the obvious map and $(g_w, w) \in \widehat{\xi}({}^L\mathrm{H})$ with $w \in W_F$ and $g_w \in \mathrm{M}^{\widehat{n}}$, then $d(g_w) = \varphi_\omega(w)$.

Remark 2.4 Let us briefly discuss the motivation behind the construction of the above endoscopic datum. If π is an irreducible unitary supercuspidal representation of $\mathrm{M}(F)$ with $\pi \circ \mathrm{Int}\, w_0 \cong \pi$, then it is expected that $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ is irreducible precisely when the (conjectural) L-function $L(s, \mathrm{Ad} \circ \varphi)$ has a pole at $s = 0$, where $\varphi\colon W_F \to \widehat{\mathrm{M}}$ represents the (conjectural) Langlands parameter of π and $\mathrm{Ad} \circ \varphi$ stands for the representation $w \mapsto \mathrm{Ad}\, \varphi(w)$ of W_F on $\mathrm{Lie}\, \widehat{\mathrm{N}}$. This condition is equivalent to $\mathrm{Ad}\, \varphi$ having a fixed vector on $\mathrm{Lie}\, \widehat{\mathrm{N}}$. One can check that the stabilizer in $\widehat{\mathrm{M}}$ of any point of $\mathrm{Lie}\, \widehat{\mathrm{N}}$ outside its unique Zariski open $\widehat{\mathrm{M}}$-orbit is contained in a proper parabolic subgroup of $\widehat{\mathrm{M}}$, through which φ, being a discrete parameter, cannot factor. Thus, $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ is irreducible if and only if φ has a representative that factors through $\widehat{\mathrm{M}}^{\widehat{n}}$, explaining our choice of $\widehat{\mathrm{H}}$. This does not in general determine our choice of H or the rest of the endoscopic datum, which, however, is forced by the expected behavior of the local Langlands correspondence with respect to central characters (in our case, restricted to $\mathrm{A}(F)$).

Write $\mathrm{SO}_{2n,\omega}$ for the quasi-split form of SO_{2n} that is split if ω is trivial, and is non-split but split over the quadratic extension of F defined by ω otherwise. We can similarly talk of $\mathrm{PSO}_{2n,\omega}$ etc. Thus, we conclude that H sits in a chain of isogenies (as one of the extremal elements):

$$\begin{cases} \mathrm{SO}_{2n-2,\omega} \to \mathrm{H} \to \mathrm{PSO}_{2n-2,\omega} & \text{in Case 1,} \\ \mathrm{Spin}_{2N,\omega} \to \mathrm{H} \to \mathrm{SO}_{2N,\omega} & \text{in Case 2,} \\ \mathrm{Sp}_{2N} \to \mathrm{H} \to \mathrm{Sp}_{2N} & \text{in Case 3.} \end{cases} \tag{2.6}$$

Here, $\mathrm{Spin}_{2,\omega}$ and $\mathrm{PSO}_{2,\omega}$ are one-dimensional tori isomorphic to $\mathrm{SO}_{2,\omega}$.

3 Endoscopic Interpretation of Shahidi's Residue Formula

3.1 Semisimple Elements of Interest in $\widetilde{\mathrm{M}}(F)$

Let $\epsilon \in \mathrm{H}(F)$ be the identity element in Cases 1 and 2 and let it equal $-1 \in \mathrm{H}(F) = \mathrm{Sp}_{2N}(F)$ in Case 3 (see (2.6)). Let π be an irreducible unitary supercuspidal representation of $\mathrm{M}(F)$ such that we have an isomorphism $\Theta\colon \pi \cong \pi \circ \mathrm{Int}\, w_0^{-1}$. Let ω be the restriction to $\mathrm{A}(F) \cong F^\times$ of the central character of π. Since $\mathrm{Int}\, w_0$ acts on A according to $x \mapsto x^{-1}$, ω is quadratic. We extend π to a representation of the twisted space $\widetilde{\mathrm{M}}(F)$ by setting $\pi(w_0^{-1}) = \Theta$ (for the notion of a representation of a twisted space and other aspects of twisted harmonic analysis that we will use in this paper, see [Li13, §3]). This extension is well defined up to the choice of Θ, which is well defined up to a nonzero scalar. Unless we are in Case 3 with $\mathrm{G} = \mathrm{G}_1$ and $\omega \neq \mathbb{1}$, let $(\mathrm{H} = \mathrm{H}_\omega, \mathcal{H}, s, \widehat{\xi})$ denote

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the endoscopic datum attached to $\widetilde{\mathrm{M}}$ and ω in Section 2.5. Associated to π is a certain family $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ of intertwining operators.

The notation introduced in the above paragraph will hold throughout the rest of this paper.

A distribution D that essentially describes the residue of the family $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ at $s = 0$ was computed in [Sha92, Sha00, Yu09]. The main purpose of this paper is to relate D to the endoscopic transfer of the distribution $f^{\mathrm{H}} \mapsto f^{\mathrm{H}}(\epsilon)$ on $\mathrm{H}(F)$ (for the datum $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$, when it is defined; otherwise, *i.e.*, in Case 3 with $\mathrm{G} = \mathrm{G}_1$ and $\omega \neq \mathbb{1}$, we will see that $D = 0$).

For this we will need to study the behaviour of endoscopic transfer near semisimple conjugacy classes in $\widetilde{\mathrm{M}}(F)$ that are associated with $\epsilon \in \mathrm{H}(F)$. Shahidi's approach of computing residues picks out a few of these that we now discuss.

Henceforth, N' will denote the open orbit of M in N. It is in general a proper subset of what was denoted by N' in Section 2.2 . Recall that, as in (2.1), we have a map $n \mapsto \widetilde{m}$ defined on N', where $n = \widetilde{m} n' n^{-}$, with $\widetilde{m} \in \widetilde{\mathrm{M}} = w_0^{-1}\mathrm{M}, n' \in \mathrm{N}$ and $n^{-} \in \mathrm{N}^{-}$. Write $\mathfrak{t}\colon \mathrm{N}' \to \widetilde{\mathrm{M}}$ for this map. Let $\mathfrak{t}_s\colon \mathrm{N}'(\overline{F}) \to \widetilde{\mathrm{M}}(\overline{F})$ be the map that takes n to the semisimple part of $\mathfrak{t}(n)$. Here, by *semisimple part*, we are referring to the Jordan decomposition in the normalizer of M in G, whose unipotent subvariety is contained in its identity component M (so that this Jordan decomposition restricts to a Jordan decomposition, in an obvious sense, for the twisted space $\widetilde{\mathrm{M}}$). Both $\mathfrak{t}$ and $\mathfrak{t}_s$ are equivariant for the conjugation action of M, and $\mathfrak{t}_s$ takes $\mathrm{N}'(F)$ to $\widetilde{\mathrm{M}}(F)$.

Remark 3.1 Note that the conjugation action of M_1 on $\widetilde{\mathrm{M}}$ factors through the map $\mathrm{M}_1 \to \mathrm{M}_2$, because its kernel is central in G (and not just in M). Henceforth, we will be interested in the action of $\mathrm{A} \times \mathrm{M}_2$ on $\widetilde{\mathrm{M}}$, where the first factor acts by left multiplication and the second factor by conjugation. Since $\operatorname{Int} w_0^{-1}$ acts as $x \mapsto x^{-1}$ on A and since $\mathrm{A} \cong \mathbb{G}_m$ is central in M, this action preserves each stable conjugacy class in $\widetilde{\mathrm{M}}(F)$ (recall that $\widetilde{\eta}, \widetilde{\eta}' \in \widetilde{\mathrm{M}}(F)$ are said to be stably conjugate if $\operatorname{Int} m(\widetilde{\eta}) = \widetilde{\eta}'$ for some $m \in \mathrm{M}(\overline{F})$ such that, for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, the element $\sigma(m)^{-1} m \in \mathrm{M}^{\widetilde{\eta}}(\overline{F})$ actually belongs to $\mathrm{M}_{\widetilde{\eta}}(\overline{F})\, \mathrm{Z}_{\mathrm{M}}^{\widetilde{\eta}}(\overline{F})$).

3.2 Three Crucial Lemmas

In this subsection we state Lemmas 3.3, 3.6, and 3.7, whose proofs are postponed to Section 4. These are the main inputs that, while seeming to admit analogues in greater generality, *i.e.*, whenever M has a Zariski-dense orbit in N, have thus far resisted our attempts at a classification-free approach.

Remark 3.2 Suppose we have a symplectic space V' over F. Using [CM93, Proposition 4.3.3], one sees that a linear transformation T in $\mathrm{Sp}(V')(F)$ lies in a minimal (nonzero) unipotent orbit of $\mathrm{Sp}(V')(F)$ if and only $T-1$ has a one-dimensional image (this makes T automatically unipotent and ensures that $T-1$ has a matrix representation with the upper right entry as the unique nonzero one). Now one can check that the minimal (nonzero) unipotent orbits in $\mathrm{Sp}(V')(F)$ form a single orbit under $\mathrm{PSp}(V')(F)$; denote by $\mathcal{O}_{\mathrm{Sp}(V')}$ their union.

Recall from Section 2.1 that, for $\widetilde{\eta} \in \widetilde{\mathrm{M}}(F)$, we are writing $\eta = \operatorname{Int} \widetilde{\eta}$ and $\mathrm{M}_\eta = \mathrm{M}_{\widetilde{\eta}}$.

Lemma 3.3 (i) *In all of our three cases,* $\mathrm{M}_2(F)$ *acts transitively on* $\mathrm{N}'(F)$*, and* $\mathfrak{t}_s(\mathrm{N}'(F))$ *is contained in a unique stable conjugacy class* $\mathcal{O}_s$ *of* $\widetilde{\mathrm{M}}(F)$*.*

(ii) *Write any given* $\widetilde{m} = \mathfrak{t}(n) \in \mathfrak{t}(\mathrm{N}'(F))$ *as* $\widetilde{\eta} \cdot \widetilde{m}_u$*, with* $\widetilde{\eta} = \mathfrak{t}_s(n)$*.*

(1) *In Case 1, we have a unique identification* $\mathrm{M}_\eta = \mathrm{Sp}(X)$*, induced by either of the chains* $\mathrm{Sp}(X) \hookrightarrow \mathrm{M}_1 \to \mathrm{M}$ *or* $\mathrm{M} \to \mathrm{M}_2 \supset \mathrm{Sp}(X)$*. Moreover,* $\widetilde{m}_u \in \mathcal{O}_{\mathrm{M}_\eta} = \mathcal{O}_{\mathrm{Sp}(X)}$*.*

(2) *In Case 2,* $\widetilde{\eta}$ *defines a symplectic form on* W *and, using* $\widetilde{\eta}$ *to denote this form as well, the composite* $\mathrm{M} \to \mathrm{M}_2 = \mathrm{GL}(W)$ *identifies* $\mathrm{M}_\eta = \mathrm{Sp}(W, \widetilde{\eta})$*. Further,* $\widetilde{m}_u \in \mathcal{O}_{\mathrm{Sp}(W,\widetilde{\eta})} = \mathcal{O}_{\mathrm{M}_\eta}$*.*

(3) *In Case 3,* $\widetilde{m}_u = 1$*, and* $\widetilde{m} = \widetilde{\eta}$ *defines a decomposition of* W *into a* $2N$*-dimensional symplectic space and a one-dimensional quadratic space. In particular,* $\mathrm{M}_\eta \cong \mathrm{Sp}_{2N}$*.*

(iii) *The action of* $\mathrm{A}(F) \times \mathrm{M}_2(F)$ *on* $\mathcal{O}_s$ *(see Remark 3.1) is transitive.*

Remark 3.4 In Cases 1 and 2, set $\mathcal{O}_{\widetilde{\eta}} = \mathcal{O}_{\mathrm{M}_\eta} \subset \mathrm{M}_\eta(F)$; we may do this for all $\widetilde{\eta} \in \mathcal{O}_s$ by Lemma 3.3 (iii). In Case 3, set $\mathcal{O}_{\widetilde{\eta}} = \{1\} \subset \mathrm{M}_\eta(F)$, for all $\widetilde{\eta} \in \mathcal{O}_s$. It is straightforward to see, using Lemma 3.3 (ii) and the fact that $\mathrm{GSp}(X)(F)$ and $\mathrm{GSp}(W, \widetilde{\eta})(F)$ surject to $\mathrm{PSp}(X)(F)$ and $\mathrm{PSp}(W, \widetilde{\eta})(F)$, respectively, that the union $\mathcal{O}$ of the $\mathcal{O}_{\widetilde{\eta}}$, $\widetilde{\eta}$ varying over $\mathcal{O}_s$, is an $\mathrm{A}(F) \times \mathrm{M}_2(F)$-orbit.

Remark 3.5 The notations $\mathcal{O}_s, \mathcal{O}$, and $\mathcal{O}_{\widetilde{\eta}}$ ($\widetilde{\eta} \in \mathcal{O}_s$) will be used throughout the rest of this paper. A coincidence crucial to making our arguments work is that for all $\widetilde{\eta} \in \mathcal{O}_s$, $\mathcal{O}_{\widetilde{\eta}}$ is the unique unipotent $\mathrm{M}_{\eta,\mathrm{ad}}(F)$-orbit in $\mathrm{M}_\eta(F)$ with

$$\dim \mathcal{O}_{\widetilde{\eta}} = \dim \mathrm{M}_\eta - \dim \mathrm{H} = \dim \mathrm{M}_\eta - \dim \mathrm{H}_\epsilon$$

uniformly across our cases. This follows easily from [CM93, Theorem 4.3.5] together with Lemma 3.3.

Henceforth until Section 3.4, assume that $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$ is defined, *i.e.*, the combination Case 3, $\mathrm{G} = \mathrm{G}_1$, $\omega \neq 1$ is excluded.

3.2.1 A Very Brief Review of Some Facts Concerning Endoscopy

Recall that strongly regular elements of $\widetilde{\mathrm{M}}(\overline{F})$ (resp., $\mathrm{H}(\overline{F})$) are those semisimple elements whose centralizers in $\mathrm{M}(\overline{F})$ (resp., $\mathrm{H}(\overline{F})$) are abelian. Kottwitz and Shelstad [KS99, Lemma 3.3.A] defined a map $\mathcal{A}_{\mathrm{H}/\widetilde{\mathrm{M}}}$ from the set of semisimple $\mathrm{H}(\overline{F})$-conjugacy classes in $\mathrm{H}(\overline{F})$ to the set of semisimple $\mathrm{M}(\overline{F})$-conjugacy classes in $\widetilde{\mathrm{M}}(\overline{F})$. A semisimple element of $\mathrm{H}(\overline{F})$ is called strongly $\widetilde{\mathrm{M}}$-regular if the image of its $\mathrm{H}(\overline{F})$-conjugacy class under $\mathcal{A}_{\mathrm{H}/\widetilde{\mathrm{M}}}$ consists of strongly regular elements in $\widetilde{\mathrm{M}}(\overline{F})$. For each $\gamma \in \mathrm{H}(F)$ that is strongly $\widetilde{\mathrm{M}}$-regular and each $\delta \in \widetilde{\mathrm{M}}(F)$ that is strongly regular, [KS99, Chapter 4] defines a complex number $\Delta(\gamma, \delta)$ that is nonzero if and only if the $\mathrm{H}(\overline{F})$-conjugacy class of γ maps to the $\mathrm{M}(\overline{F})$-conjugacy class of δ under $\mathcal{A}_{\mathrm{H}/\widetilde{\mathrm{M}}}$. We say that γ and δ match if this condition is satisfied. The function Δ, defined on a subset

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of $\mathrm{H}(F)\times\widetilde{\mathrm{M}}(F)$, is known as a *transfer factor*. In fact, its definition [KS99, Chapter 4] is only well defined up to multiplication by a complex number of absolute value 1. We now fix such a (noncanonical) choice.

Functions $f\in C_c^\infty(\widetilde{\mathrm{M}}(F))$ and $f^{\mathrm{H}}\in C_c^\infty(\mathrm{H}(F))$ are said to have matching orbital integrals (sometimes we may say instead that f^{H} is an endoscopic transfer of f) if and only if for all strongly $\widetilde{\mathrm{M}}$-regular $\gamma\in\mathrm{H}(F)$, we have an equality

$$\sum_{\gamma'} O(\gamma', f^{\mathrm{H}}, dh/dt_{\gamma'}) = \sum_{\delta}\Delta(\gamma,\delta)O(\delta, f, dm/dt_\delta), \tag{3.1}$$

whose meaning we now proceed to explain. Here γ' runs over a set of representatives for the $\mathrm{H}(F)$-conjugacy classes in $\mathrm{H}(F)\cap\mathrm{Int}\,\mathrm{H}(\overline{F})(\gamma)$, and δ runs over a set of representatives in $\widetilde{\mathrm{M}}(F)$ for the $\mathrm{M}(F)$-conjugacy classes in $\widetilde{\mathrm{M}}(F)$ consisting of elements that match γ. Our notation concerning the orbital integrals $O(\gamma', f^{\mathrm{H}}, dh/dt_{\gamma'})$ and $O(\delta, f, dm/dt_\delta)$ is as follows. Suppose $\mathcal{G}$ is a unimodular locally compact totally disconnected topological group acting continuously on a locally compact totally disconnected topological space Y. Let $F\in C_c^\infty(Y)$ and $y\in Y$. Suppose the stabilizer $\mathcal{G}_y$ of y in $\mathcal{G}$ is unimodular, and dg and dt_y are Haar measures on $\mathcal{G}$ and $\mathcal{G}_y$, respectively. Then we set

$$O(y,F,dg/dt_y) = \int_{\mathcal{G}_y\backslash\mathcal{G}} F(g^{-1}\cdot y)\,\frac{dg}{dt_y},$$

provided this orbital integral is convergent. In our situation, the convergence of orbital integrals (which presupposes unimodularity) is assured thanks to [RR72]. There are also some compatibilities required of the centralizer measures $dt_{\gamma'}$ and dt_δ in (3.1); we refer the reader to [KS99, Section 5.5] for more details.

Thanks to the work of Waldspurger and Ngô, for every $f\in C_c^\infty(\widetilde{\mathrm{M}}(F))$, there exists $f^{\mathrm{H}}\in C_c^\infty(\mathrm{H}(F))$ such that f and f^{H} have matching orbital integrals. Thus, we get a well defined map $C_c^\infty(\widetilde{\mathrm{M}}(F))\to C_c^\infty(\mathrm{H}(F))/\sim$, where $C_c^\infty(\mathrm{H}(F))/\sim$ is the quotient of $C_c^\infty(\mathrm{H}(F))$ by its subspace consisting of those f^{H} such that the left-hand side of (3.1) vanishes for all strongly $\widetilde{\mathrm{M}}$-regular $\gamma\in\mathrm{H}(F)$. Dually, *i.e.*, pulling back under $f\mapsto f^{\mathrm{H}}$, we get a map at the level of distributions

$$\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}\colon (C_c^\infty(\mathrm{H}(F))/\sim)^*\to C_c^\infty(\widetilde{\mathrm{M}}(F))^*.$$

The space $(C_c^\infty(\mathrm{H}(F))/\sim)^*$ is known as the space of *stable distributions* on $\mathrm{H}(F)$.

It is a well-known consequence of the main result of [Rog80] that (since ϵ is central) $\delta_\epsilon\colon f^{\mathrm{H}}\mapsto f^{\mathrm{H}}(\epsilon)$ is a stable distribution on $\mathrm{H}(F)$, so its pull back $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)\colon f\mapsto f^{\mathrm{H}}(\epsilon)$ under $f\mapsto f^{\mathrm{H}}$ is well defined as a distribution on $\widetilde{\mathrm{M}}(F)$.

Lemma 3.6 (i) *Suppose $\gamma\in\mathrm{H}(F)$ is strongly $\widetilde{\mathrm{M}}$-regular, $\delta\in\widetilde{\mathrm{M}}(F)$ is strongly regular, and $z\in\mathrm{A}(F)$. Then $\Delta(\gamma,z\delta)=\omega(z)\Delta(\gamma,\delta)$.*

(ii) *For $m\in\mathrm{M}_2(F)$, $\gamma\in\mathrm{H}(F)$, and $\delta\in\widetilde{\mathrm{M}}(F)$, $\Delta(\gamma,\mathrm{Int}\,m^{-1}(\delta))=\Delta(\gamma,\delta)$.*

Lemma 3.7 $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)$ *is supported on $\mathcal{O}$, so the support of* $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)$ *is a single $\mathrm{A}(F)\times\mathrm{M}_2(F)$-orbit (with the action as in Remark 3.1).*

Lemmas 3.6 and 3.7 have the following immediate corollary.

Corollary 3.8 $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)$ *is an* $(\mathrm{A}(F) \times \mathrm{M}_2(F), \omega \times \mathbb{1})$*-invariant distribution supported on* $\mathcal{O}$.

3.3 Nonvanishing of the Endoscopic Transfer of a Distribution

Lemma 3.9 $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon) \neq 0$.

Proof Essentially this proof will only use that ϵ is central in H, that it is fixed by all the automorphisms of H, and that H is an endoscopic group to $\widetilde{\mathrm{M}}$.

First, we would like to get rid of the cases where $\mathrm{M}_2 \cong \mathrm{GL}_2$ and $\omega = \mathbb{1}$. In these cases the lemma can be proved using a semisimple descent argument as in the proof of Lemma 4.7, together with [LS87, Theorem 5.5.A]. Alternatively, with only the weaker assumption that ω is unramified, the endoscopic datum $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$ is unramified, so that the twisted fundamental lemma applies (it is now fully proved in arbitrary residual characteristic [LMW15, LW15]), which trivially yields the lemma.

Now assume that either $\mathrm{M}_2 \not\cong \mathrm{GL}_2$ or $\omega \neq \mathbb{1}$. In this case it is easily seen that $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$ is elliptic, *i.e.*, $\widehat{\xi}(\mathcal{H})$ is not contained in any proper parabolic subgroup of ${}^L\mathrm{M}$. Let $\mathcal{I}(\widetilde{\mathrm{M}})$ (resp., $\mathcal{SI}(\mathrm{H})$) denote the quotient of $C_c^\infty(\widetilde{\mathrm{M}}(F))$ (resp., $C_c^\infty(\mathrm{H}(F))$) by the equivalence relation under which two functions having the same orbital integrals (resp., stable orbital integrals) at strongly regular semisimple elements are declared equivalent. We have a subspace $\mathcal{I}_{\mathrm{cusp}}(\widetilde{\mathrm{M}})$ obtained as the image of those functions in $C_c^\infty(\widetilde{\mathrm{M}}(F))$ whose constant terms with respect to proper Levi subspaces $\widetilde{\mathrm{M}}_1$ of $\widetilde{\mathrm{M}}$ are zero as elements of $\mathcal{I}(\widetilde{\mathrm{M}}_1)$ [Wal14, §3.1]. Similarly we have a subspace $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})$ of $\mathcal{SI}(\mathrm{H})$, as defined in [Wal14, §3.1].

We hasten to remark that, strictly speaking, the definition of $\mathcal{SI}(\mathrm{H})$ in [Wal14] (on account of its handling a much more general situation) uses a z-extension [Wal14, §2.5]. However, we take the character λ_1 therein to be trivial, which lets us identify the $\mathcal{SI}(\mathrm{H})$ from [Wal14] with what we are denoting by $\mathcal{SI}(\mathrm{H})$.

By [Wal14, Proposition 4.11(i)], which applies since $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$ is elliptic, the endoscopic transfer map $\mathcal{I}(\widetilde{\mathrm{M}}) \to \mathcal{SI}(\mathrm{H})$ restricts to a *well-defined surjection* from $\mathcal{I}_{\mathrm{cusp}}(\widetilde{\mathrm{M}})$ to the subspace of $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})$ fixed by a certain finite outer automorphism group (we assume fixed invariant measures on $\widetilde{\mathrm{M}}(F)$ and $\mathrm{H}(F)$). Since δ_ϵ (being a stable distribution) factors through $C_c^\infty(\mathrm{H}(F)) \to \mathcal{SI}(\mathrm{H})$, and since the automorphism group of H fixes ϵ and hence δ_ϵ, we are now reduced to showing that the map $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H}) \to \mathbb{C}$ induced by δ_ϵ is nonzero. In other words, it is now enough to show that $C_c^\infty(\mathrm{H}(F))$ contains at least one function f^{H} whose constant terms with respect to proper parabolic subgroups vanish, and which satisfies that $f^{\mathrm{H}}(\epsilon) \neq 0$.

Since the space of such functions is invariant under right translation by ϵ (ϵ being central), we are reduced to showing that there exists a function $\varphi \in C_c^\infty(\mathrm{H}(F))$ whose constant terms with respect to proper parabolic subgroups vanish, and such that $\varphi(1) \neq 0$. Now note that in all our cases H is either semisimple or a torus (2.6). If H is a torus, it has no proper parabolic subgroup, so our assertion is clear. If, on the other hand, H is semisimple, then it has at least one supercuspidal representation, and any suitable matrix coefficient not vanishing at the identity (such clearly exist) does the job (alternatively, see [Beu16, Theorem 4]). ∎

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3.4 Relating $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)$ to the Residue

Remark 3.10 The space of $(\mathrm{A}(F)\times\mathrm{M}_2(F), \omega\times\mathbb{1})$-invariant distributions supported on the $\mathrm{A}(F)\times\mathrm{M}_2(F)$-orbit $\mathcal{O}$ has dimension at most one. To see this, follow the reasoning of [Ber84, Section 1.5, Remark 2]; namely, apply the lemma of [Ber84, Section 1.5] with $\mathrm{A}(F)\times\mathrm{M}_2(F)$ in place of G there, taking the map $p\colon X\to Z$ there to be the identity map $\mathcal{O}\to\mathcal{O}$, χ to be the character $\omega\times\mathbb{1}$, ν to be trivial, and z_0 to be any point in $\mathcal{O}$; since $\mathcal{O}$ is a finite union of $\mathrm{M}(F)$-orbits, the μ of that reference can be taken to be a suitable positive finite linear combination of $\mathrm{M}(F)$-orbital integrals.

Moreover, the notion of orbital integrals provides a way to write down an $(\mathrm{A}(F)\times\mathrm{M}_2(F), \omega\times\mathbb{1})$-invariant distribution on $\mathcal{O}$.

$$
\begin{aligned}
\text{(3.2)}\quad f\mapsto &\int_{(\mathrm{A}(F)\times\mathrm{M}_2(F))_{\widetilde{m}_0}\backslash\mathrm{A}(F)\times\mathrm{M}_2(F)} f(z^{-1}m^{-1}\widetilde{m}_0 m)(\omega\times\mathbb{1})(z,m)\,\overline{dmdz}\\
&= [(\mathrm{A}(F)\times\mathrm{M}_2(F))_{\widetilde{m}_0} : \mathrm{A}(F)^2\times\mathrm{M}_2(F)_{\widetilde{m}_0}]^{-1}\\
&\qquad\int_{\mathrm{A}(F)^2\backslash\mathrm{A}(F)}\int_{\mathrm{M}_2(F)_{\widetilde{m}_0}\backslash\mathrm{M}_2(F)} f(z^{-1}m^{-1}\widetilde{m}_0 m)\omega(z)\,d\dot{m}d\dot{z},
\end{aligned}
$$

where $\widetilde{m}_0$ is a choice of a point on $\mathcal{O}$, $(\mathrm{A}(F)\times\mathrm{M}_2(F))_{\widetilde{m}_0}$ is the stabilizer of $\widetilde{m}_0$ under $\mathrm{A}(F)\times\mathrm{M}_2(F)$, $\mathrm{M}_2(F)_{\widetilde{m}_0}$ that of $\widetilde{m}_0$ under conjugation by $\mathrm{M}_2(F)$, and $\overline{dmdz}$, $d\dot{m}$, and $d\dot{z}$ are choices of appropriately invariant measures. The convergence of the above integral is as a well-known and easy consequence of the convergence of usual orbital integrals [RR72], together with the fact that the images of $\mathrm{M}_1(F)$ in $\mathrm{M}_2(F)$ and $\mathrm{A}(F)^2$ in $\mathrm{A}(F)$ have finite index. Note that by *convention*, the integral in (3.2) makes sense even when $\omega\times\mathbb{1}$ is nontrivial on the stabilizer of some $\widetilde{m}_0\in\mathcal{O}$, since the integral is by convention defined to be zero in that case. If this is not the case, then the distribution (3.2) is nonzero and gives a basis for the one-dimensional space of $(\mathrm{A}(F)\times\mathrm{M}_2(F), \omega\times\mathbb{1})$-invariant distributions on $\mathcal{O}$.

Remark 3.11 Let us recall Shahidi's expression for D (this is due to [Sha92, Yu09] for Cases 2 and 3, and [Yu09] for Case 1). Write $\pi_\circ$ for the representation of $\mathrm{M}(F)$ underlying the representation π of $\widetilde{\mathrm{M}}(F)$. By Lemma 3.3(i) and [Sha00, Theorem 2.5] (extended by [Yu09] to cover all our cases), $s\mapsto A(s\widetilde{\alpha},\pi,w_0)$ has a pole at $s=0$ if and only if for some $f_\circ\in C_c^\infty(\mathrm{M}(F))$ such that

$$
\text{(3.3)}\qquad m\mapsto\int_{\mathrm{A}(F)}\omega(z)f_\circ(z^{-1}m)\,dz
$$

is a matrix coefficient for $\pi_\circ$, we have

$$
\text{(3.4)}\quad \int_{\mathrm{A}(F)^2\backslash\mathrm{A}(F)}\int_{\mathrm{M}_2(F)_{w_0^{-1}m_0}\backslash\mathrm{M}_2(F)} f_\circ(z^{-1}\cdot w_0 m^{-1}w_0^{-1}\cdot m_0\cdot m)\omega(z)\,d\dot{m}d\dot{z}\neq 0,
$$

where $m_0\in\mathrm{M}(F)$ is some element such that $w_0^{-1}m_0$ lies in the image of the map $n\mapsto\widetilde{m}$ considered in (2.3), $\mathrm{M}_2(F)_{w_0^{-1}m_0}$ is the stabilizer of $w_0^{-1}m_0\in\widetilde{\mathrm{M}}(F)$ under conjugation by $\mathrm{M}_2(F)$ (or equivalently, the stabilizer of m_0 under $\mathrm{Int}\,w_0$-twisted conjugation by $\mathrm{M}_2(F)$), $d\dot{m}$ is a choice of an $\mathrm{M}_2(F)$-invariant measure on

$$
\mathrm{M}_2(F)_{w_0^{-1}m_0}\backslash\mathrm{M}_2(F),
$$

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and $d\dot{z}$ may in our case be taken to be the counting measure on $\mathrm{A}(F)^2\backslash \mathrm{A}(F)$. We should mention that the formula in Theorem 2.5 of [Sha00] features $\mathrm{Z}_{\mathrm{M}}(F)$ in place of $\mathrm{A}(F)$ in both (3.3) and (3.4), but these changes cancel each other out ($\mathrm{A}(F)$ has finite index in $\mathrm{Z}_{\mathrm{M}}(F)$).

For $f_\circ \in C_c^\infty(\mathrm{M}(F))$, define $f \in C_c^\infty(\widetilde{\mathrm{M}}(F))$ by $f(\widetilde{m}) = f_\circ(w_0\widetilde{m})$ (note that $w_0\widetilde{m} \in \mathrm{M}(F)$). Then it is easy to see that the expression (3.3) is the matrix coefficient for $\pi_\circ$ attached to vectors $\widetilde{v}, v$ if and only if

$$\widetilde{m} \mapsto \int_{\mathrm{A}(F)} \omega(z) f(z^{-1}\widetilde{m})\, dz \tag{3.5}$$

is the matrix coefficient for π attached to $w_0^{-1}\widetilde{v}$ and v (use that $\omega = \omega^{-1}$). Moreover, we have:

$$\begin{aligned}\int_{\mathrm{A}(F)^2\backslash \mathrm{A}(F)} \int_{\mathrm{M}_{2,w_0^{-1}m_0}(F)\backslash \mathrm{M}_2(F)} & f_\circ(z^{-1}(w_0 m^{-1} w_0^{-1}) m_0 m)\omega(z)\, d\dot{m}\, d\dot{z} \\ &= \int_{\mathrm{A}(F)^2\backslash \mathrm{A}(F)} \int_{\mathrm{M}_{2,\widetilde{m}_0}(F)\backslash \mathrm{M}_2(F)} f(z^{-1}m^{-1}\widetilde{m}_0 m)\omega(z)\, d\dot{m}\, d\dot{z},\end{aligned}$$

where $\widetilde{m}_0 = w_0^{-1}m_0$ lies in the image of the map $n \to \widetilde{m}$ (recall that $\omega = \omega^{-1}$).

Using in particular (3.2), we conclude the following theorem from Remark 3.10.

Theorem 3.12 *Let D be a nonzero $(\mathrm{A}(F) \times \mathrm{M}_2(F), \omega \otimes \mathbb{1})$-invariant distribution supported on $\mathcal{O}$, if it exists, and let $D = 0$ otherwise (see Remark 3.10). Then $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ has a pole at $s = 0$ if and only if $D(f) \neq 0$ for some $f \in C_c^\infty(\widetilde{\mathrm{M}}(F))$ such that the expression (3.5) is a matrix coefficient of π.*

Remark 3.13 In Case 3 when $\mathrm{G} = \mathrm{G}_1$ and $\omega \neq 1$, it is easy to check that $\omega \times \mathbb{1}$ is nontrivial on the stabilizer of some (and hence any) point of $\mathcal{O}$ in $\mathrm{A}(F) \times \mathrm{A}_2(F) \subset \mathrm{A}(F) \times \mathrm{M}_2(F)$. Thus, the space of $(\mathrm{A}(F) \times \mathrm{M}_2(F), \omega \times \mathbb{1})$-invariant distributions on $\mathcal{O}$ vanishes in this case. On the other hand, in all the other cases, Corollary 3.8 furnishes an element of this space, namely $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)$, which is nonzero by Lemma 3.9. Therefore, except when we are in Case 3 with $\mathrm{G} = \mathrm{G}_1$ and $\omega \neq 1$, the distribution of (3.2) is nonzero, so that for a nonzero scalar c,

$$\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon) = c \cdot \int_{\mathrm{A}(F)^2\backslash \mathrm{A}(F)} \int_{\mathrm{M}_2(F)_{\widetilde{m}_0}\backslash \mathrm{M}_2(F)} f(z^{-1}m^{-1}\widetilde{m}_0 m)\omega(z)\, d\dot{m}\, d\dot{z}.$$

We now get the following corollary.

Corollary 3.14 *If we are in Case 3 with $\mathrm{G} = \mathrm{G}_1$ and $\omega \neq \mathbb{1}$, then $D = 0$. In all other situations, up to a nonzero scalar, $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon) = D$, where D is as in Theorem 3.12.*

Now, as in [Sha00], we state a plausible definition for endoscopic transfer of representations so as to state the above corollary in a manner that better illustrates the expectations from endoscopy (see, however, the clarification in Section 3.5).

Definition 3.15 Suppose that we are not in Case 3 with $\mathrm{G} = \mathrm{G}_1$ and $\omega \neq \mathbb{1}$, so that the endoscopic datum $(\mathrm{H} = \mathrm{H}_\omega, \mathcal{H}, s, \widehat{\xi})$ is defined. Then π is said to arise by endoscopic transfer from $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$ if for some $f \in C_c^\infty(\widetilde{\mathrm{M}}(F))$ such that the prescription of (3.5) defines a matrix coefficient of π, some (and hence, equivalently, any) endoscopic transfer f^{H} of f (with respect to the datum $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$) satisfies $f^{\mathrm{H}}(\epsilon) \neq 0$.

According to the above definition, one can rephrase Corollary 3.14.

Corollary 3.16 *If we are not in Case 3 with* $\mathrm{G} = \mathrm{G}_1$ *and* $\omega \neq \mathbb{1}$*, then* $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ *has a pole at* $s = 0$ *(or equivalently,* $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ *is irreducible) if and only if* π *arises by endoscopic transfer from* $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$*. If we are in Case 3 with* $\mathrm{G} = \mathrm{G}_1$ *and* $\omega \neq \mathbb{1}$*,* $s \mapsto A(s\widetilde{\alpha}, \pi, w_0)$ *is holomorphic at* $s = 0$ *(equivalently,* $\mathrm{Ind}_{\mathrm{P}(F)}^{\mathrm{G}(F)} \pi$ *is reducible).*

3.5 Clarification on Definition 3.15

In Cases 2 and 3 when $\mathrm{M} = \mathrm{M}_2$, and more or less in Case 1 when $\mathrm{M} = \mathrm{M}_1$, we already have an alternate definition of what it means for π to come from $(\mathrm{H} = \mathrm{H}_\omega, \mathcal{H}, s, \widehat{\xi})$ by endoscopic transfer [Art13]). Therefore, we need to relate Definition 3.15 to Arthur's definition in these situations (in fact even when $\mathrm{M} = \mathrm{M}_2$ in Case 1, something similar should be possible using the work of [Xu15], but we skip this for now).

One approach to doing this would involve plugging f^{H} into the Plancherel formula for H, using the constancy of the Plancherel measure on L-packets (which follows from Arthur's normalization of intertwining operators [Art13, Chapter 2]), and appealing to endoscopic character identities. Instead, we will show how the equivalence of the two definitions follows from results of [Mœg14]. However, for simplicity, we will skip the cases where $\mathrm{M}_2 \cong \mathrm{GL}_2$ and $\omega = \mathbb{1}$, and assume that our endoscopic datum $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$ is elliptic.

3.5.1 Cases 2 and 3, $\mathrm{G} = \mathrm{G}_2$

Suppose we are in Case 2 or Case 3, with $\mathrm{M} = \mathrm{M}_2$. Let $f \in C_c^\infty(\widetilde{\mathrm{M}}(F))$ be such that (3.5) gives a matrix coefficient $\dot{f}$ of π (we emphasize that π is viewed as a representation of $\widetilde{\mathrm{M}}(F)$, not one of $\mathrm{M}(F)$). Assume that $\operatorname{tr}\pi(\dot{f}) \neq 0$ (we will later justify this assumption). In order to apply results from [Mœg14], we need to first make sure that f may be chosen to be a pseudo-coefficient of π, *i.e.*, $\operatorname{tr}\pi(f) \neq 0$ and $\operatorname{tr}\pi'(f) = 0$ for all irreducible representations π' of $\widetilde{\mathrm{M}}(F)$ such that the representations of $\mathrm{M}(F)$ underlying π and π', which we henceforth denote for clarity as $\pi_\circ$ and $\pi'_\circ$, respectively, are not isomorphic.

It follows from Lemma 3.6 that the dependence of f^{H} on f is only through $\dot{f}$, which allows us to change f without changing $\dot{f}$. In particular, we can assume that $\dot{f}$ is obtained from f as in (3.5) but with $\mathrm{A}(F)$ replaced by $\mathrm{A}(F)^2$, on which ω is trivial (and with measures adjusted accordingly). This choice ensures that for all irreducible tempered representations π' of $\widetilde{\mathrm{M}}(F)$, $\pi'(f) = c \cdot \pi'(\dot{f})$ for a nonzero scalar c, where $\pi'(f)$ is interpreted using an integral over $\widetilde{\mathrm{M}}(F)$ and $\pi'(\dot{f})$ using one over

$\widetilde{\mathrm{M}}(F)/\mathrm{A}(F)^2$. We will use this to prove that f, as we have just chosen, is necessarily a pseudo-coefficient of π.

First, $\operatorname{tr}\pi(f) = c\operatorname{tr}\pi(\dot{f}) \neq 0$. As regards proving that $\operatorname{tr}\pi'(f) \neq 0$ if $\pi_\circ \ncong \pi'_\circ$, note that the extension π' of $\pi'_\circ$ to $\widetilde{\mathrm{M}}(F)$ determines an isomorphism between the spaces of matrix coefficients of π' and $\pi'_\circ$, taking $\widetilde{m} \mapsto \langle u, \pi'(\widetilde{m})v\rangle$ to $m \mapsto \langle u, \pi'_\circ(m)v\rangle$. From this, we see that it suffices to prove that $\pi'(\dot{f}_\circ) = 0$ (as opposed to just $\operatorname{tr}\pi'(\dot{f}_\circ) = 0$), whenever $\pi' \ncong \pi_\circ$ is an irreducible representation of $\mathrm{M}(F)$ whose central character restricts trivially to $\mathrm{A}(F)^2$, and where $\dot{f}_\circ$ is any matrix coefficient of $\pi_\circ$. This follows from the usual proof of the Schur orthogonality relations: to show that $\pi'(\dot{f}_\circ)(v') = 0$ for any given v' in the space of π', assuming without loss of generality that $\dot{f}_\circ$ is given by $m \mapsto \langle u, \pi_\circ(m^{-1})v\rangle$ (recall that $\pi_\circ$ is self-contragredient!), note that the prescription:

$$w \mapsto \int_{\mathrm{A}(F)^2\backslash \mathrm{M}(F)} \langle u, \pi_\circ(m^{-1})w\rangle \cdot \pi'(m)v'\,dm$$

is well defined (thanks to the central characters of $\pi_\circ$ and π' restricting trivially to $\mathrm{A}(F)^2$ and the fact that $m \mapsto \langle u, \pi_\circ(m^{-1})w\rangle$ is compactly supported modulo $\mathrm{A}(F)^2$), and now clearly intertwines $\pi_\circ$ with π'. It is forced to be nonzero unless $\pi'(\dot{f}_\circ)(v') = 0$, as we wanted.

Now that f is a pseudocoefficient for π, it follows that the image $I(f)$ of f in $\mathcal{I}(\widetilde{\mathrm{M}})$ (see the proof of Lemma 3.9 for the notation) necessarily belongs to $\mathcal{I}_{\mathrm{cusp}}(\widetilde{\mathrm{M}})$ [Wal12, §7.1]. Using [Li13, Proposition 3.3.2], $I(f)$, viewed as a function on the strongly regular semisimple set of $\widetilde{\mathrm{M}}(F)$ and up to a nonzero scalar, is the (twisted) trace of π multiplied by the characteristic function of the elliptic subset. This makes $I(f)$ the projection of the twisted trace of π on $\mathcal{I}_{\mathrm{cusp}}(\widetilde{\mathrm{M}})$, up to a nonzero scalar, in the sense of [Mœg14]. Thus, the image $SI(f^{\mathrm{H}})$ of f^{H} in $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})$ is uniquely determined up to a nonzero scalar. The considerations so far also show that if $\operatorname{tr}\pi(\dot{f}) = 0$, then the image of f (chosen as above) in $\mathcal{I}_{\mathrm{cusp}}(\widetilde{\mathrm{M}})$ vanishes, so that $SI(f^{\mathrm{H}}) = 0$, justifying our assumption that $\operatorname{tr}\pi(\dot{f}) \neq 0$.

We have a *stabilization* that is a decomposition defined by endoscopic transfer [Mœg14, §2.3(1)].

$$\mathcal{I}_{\mathrm{cusp}}(\widetilde{\mathrm{M}}) = \bigoplus_{\underline{\mathrm{H}}'}\big(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H}')^{\mathrm{Aut}(\mathrm{H}',\widetilde{\mathrm{M}})}\big), \tag{3.6}$$

where $\underline{\mathrm{H}}'$ runs over elliptic endoscopic data for $\widetilde{\mathrm{M}}$, we have written H' for the endoscopic group underlying $\underline{\mathrm{H}}'$, and $\mathrm{Aut}(\mathrm{H}',\widetilde{\mathrm{M}})$ is a certain finite group of outer automorphisms of H'. That (3.6) is defined by endoscopic transfer means that the projection of $I(f)$ along the $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})^{\mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}})})$-factor in (3.6), corresponding to our endoscopic datum $(\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$, is simply $SI(f^{\mathrm{H}})$. Note, in particular, that by the analogue of (3.6) for H, $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})$ can be viewed as a subspace of $\mathcal{I}_{\mathrm{cusp}}(\mathrm{H})$ (this is needed to interpret the results from [Mœg14] that we are going to use now). By [Mœg14, Corollary 4.11], we may write

$$SI(f^{\mathrm{H}}) = \sum_{\sigma} \sum_{\iota\in\mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}})} SI(f_\sigma \circ \iota), \tag{3.7}$$

where σ runs over a set of representatives for equivalence classes of discrete series representations for the relation of equivalence generated by L-indistinguishability and "being in the same $\mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}})$-orbit," and f_σ is a pseudo-coefficient for σ. Here we have identified $\mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}}) \subset \mathrm{Out}(\mathrm{H})$ with a finite subgroup of $\mathrm{Aut}\,\mathrm{H}$ via a splitting (in fact, for us $\mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}}) = \mathrm{Out}(\mathrm{H})$). By [Mœg14, Theorems 4.9 and 4.14], every σ such that $SI(f_\sigma) \neq 0$ transfers to $\widetilde{\mathrm{M}}$, whereas we know from [Art13] that the set of tempered L-packets of $\mathrm{H}(F)$ that transfer to a given representation of $\widetilde{\mathrm{M}}(F)$ is a union of the members of at most one $\mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}})$-orbit of L-packets. Thus, the number of σ's that contribute nontrivially to the right-hand side of (3.7) is 1 or 0, depending on whether or not π comes by endoscopic transfer from H. It now suffices to show that $f^{\mathrm{H}}(\epsilon) \neq 0$ in the former case. This follows since $f_\sigma(\epsilon) \neq 0$ (as $I(f_\sigma) \neq 0$ and f_σ is a pseudo-coefficient for σ), and since $f_\sigma \circ \iota(\epsilon) = f_\sigma(\epsilon)$ for all $\iota \in \mathrm{Aut}(\mathrm{H},\widetilde{\mathrm{M}})$.

3.5.2 Case 1, $\mathrm{G} = \mathrm{G}_1$

Now assume that we are in Case 1 with $\mathrm{M} = \mathrm{M}_1 = \mathrm{GL}(W) \times \mathrm{Sp}(X)$. In this case, the endoscopic datum $\mathcal{E} = (\mathrm{H}, \mathcal{H}, s, \widehat{\xi})$, whose transfer factors we denote by Δ, can be viewed as a product of two endoscopic data: the usual endoscopic datum $\mathcal{E}'$ for $\mathrm{Sp}(X)$ with H as the underlying group, and the twisted endoscopic datum $\mathcal{E}''$ for $\widetilde{\mathrm{GL}}(W) \cong \widetilde{\mathrm{GL}}_1$ with the trivial group as the underlying endoscopic group, which corresponds to the L-embedding of $W_F = 1 \times W_F$ into $\widehat{\mathrm{GL}(W)} \times W_F = \mathrm{GL}_1(\mathbb{C}) \times W_F = \mathbb{C}^\times \times W_F$ via ω. Write Δ' and Δ'' for the transfer factors with respect to these two endoscopic data. Then we can see that a strongly $\widetilde{\mathrm{M}}$-regular $\gamma \in \mathrm{H}(F)$ matches

$$(a, \delta) \in \widetilde{\mathrm{GL}}(W)(F) \times \mathrm{Sp}(X)(F) = \widetilde{\mathrm{M}}(F)$$

if and only if γ matches δ for $\mathcal{E}'$, and

$$\Delta(\gamma, (a, \delta)) = \Delta'(\gamma, \delta)\Delta''(1, a) = \omega(a)\Delta'(\gamma, \delta),$$

up to a nonzero scalar independent of γ, δ, and a that does not concern us. Thus, for $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{\mathrm{M}}(F))$ with $f_1 \in C_c^\infty(\widetilde{\mathrm{GL}}(W)(F))$ and $f_2 \in \mathrm{Sp}(X)(F)$, it is easy to see that its endoscopic transfer f^{H}, up to a nonzero scalar and up to looking only at stable orbital integrals, equals the product of an integral of f_1 against an ω-equivariant function and an endoscopic transfer of f_2 with respect to $\mathcal{E}'$. Now using a similar argument as in Case 2 with $\mathrm{M} = \mathrm{M}_2$ (in fact, what we want is summarized in [Mœg14, Remark 4.15]), it is easy to see that a supercuspidal representation $\pi = \omega \otimes \pi'$ of $\widetilde{\mathrm{M}}(F)$, the restriction ω of whose central character to $\mathrm{A}(F)$ is quadratic, comes from the endoscopic datum $\mathcal{E}$ if and only if π' comes in the sense of Arthur from the unique endoscopic datum of $\mathrm{Sp}(X)$ whose underlying group is the special orthogonal group in $\dim X = 2n - 2$ variables, which is quasi-split, non-split if $\omega \neq \mathbb{1}$, and split over the degree one/quadratic extension of F determined by ω.

3.5.3 Case 1, $\mathrm{M} = \mathrm{M}_2$

Finally, we consider Case 1 with $\mathrm{M} = \mathrm{M}_2 = \mathrm{GSp}(X)$, but only sketch the arguments in this situation. Our claim in this case is that the condition $f^{\mathrm{H}}(1) \neq 0$ for some f as in Definition 3.15 is equivalent to π arising by endoscopic transfer in the sense of

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[Xu15] from the group $\mathrm{GSO}_{2n-2,\omega}$ which is non-split if $\omega \neq \mathbb{1}$, and splits over the trivial/quadratic extension defined by ω (in this case the group also determines the endoscopic datum). Since [Xu15] has established the relevant endoscopic character identities for GSp_{2n-2}, π comes by endoscopic transfer from $\mathrm{GSO}_{2n-2,\omega}$ in the sense of [Xu15] if and only if its coarse Langlands parameter as in that reference factors through the appropriate embedding of L-groups. From the commutative diagram in the proof of [Xu15, Proposition 2.7], this is equivalent to the image of its coarse Langlands parameter in the L-group of Sp_{2n-2} factoring through the L-group of $\mathrm{SO}_{2n-2,\omega}$, or equivalently [Xu15, §4.1], $\pi|_{\mathrm{Sp}_{2n-2}(F)}$ having an irreducible component coming via endoscopic transfer from $\mathrm{SO}_{2n-2,\omega}$. By what we have seen for Case 1 with $\mathrm{M} = \mathrm{M}_1$, it now suffices to show that $f^{\mathrm{H}}(1) \neq 0$ if and only if $f_1^{\mathrm{H}_1}(1) \neq 0$, where f_1 is the pull back of f to $\widetilde{\mathrm{M}}_1(F)$ and $f_1^{\mathrm{H}_1}$ is an endoscopic transfer of f_1 for the endoscopic datum we have associated with M_1 and the pull back of ω via $\mathrm{A}_1(F) \xrightarrow{\cong} \mathrm{A}_2(F)$. Since f_1 is a matrix coefficient for $\pi|_{\widetilde{\mathrm{M}}_1(F)}$, this can be easily seen from Lemmas 3.6, 3.7, and 3.9 (together with the fact that the sets $\mathcal{O}$ for G_1 and G_2 are isomorphic under the obvious map $\mathrm{G}_1 \to \mathrm{G}_2$).

4 Proofs of Lemmas from Section 3

4.1 Proof of Lemma 3.3

Using the Cayley transform or the exponential map, for (i), we see that it is enough to prove an analogous assertion with N replaced by its Lie algebra $\mathfrak{n}$. The second assertion of (i) follows from the first, the definition of stable conjugacy (recalled in Remark 3.1), and the fact that the kernel of $\mathrm{M}(\overline{F}) \to \mathrm{M}_2(\overline{F})$ is contained in $\mathrm{Z}_{\mathrm{M}}^{\mathrm{Int}\, w_0}(\overline{F})$. In all cases, one has an obvious identification of $\mathfrak{n}(F)$ with the set of elements $(A, B) \in \mathrm{Hom}(X, W') \oplus \mathrm{Hom}(W, W')$ such that $B + B^* = 0$. Under the identification of $\mathrm{Hom}(W, W')$ with the space of bilinear forms on W given by $T \mapsto \langle \cdot, T(\cdot)\rangle$, the condition $B + B^* = 0$ merely says that B is symplectic (in Cases 2 and 3), or orthogonal (in Case 1). For the rest of this proof, we will often use that the exponential map (or, alternatively, an appropriately normalized Cayley transform) takes an element $(A, B) \in \mathfrak{n}(F)$ as above to the element $n(A, B - (1/2)AA^*) \in \mathrm{N}(F)$ in the notation that was introduced shortly following (2.2).

In Cases 1 and 2, the set $\mathfrak{n}''$ of all (A, B) in $\mathfrak{n}$, where $A \in \mathrm{Hom}(X, W') \smallsetminus \{0\}$ and $B \in \mathrm{Isom}(W, W')$, is nonempty, Zariski open, and M-invariant, and hence contains $\mathfrak{n}'$. In Case 3, one takes $\mathfrak{n}''$ to be the set of pairs (A, B) in $\mathfrak{n}$ such that $A \in \mathrm{Hom}(X, W') \smallsetminus \{0\}$ and such that B is a symplectic form on W whose radical $\mathrm{Rad}\, B \subset W$ has rank 1 and is not contained in the kernel of $A^* \in \mathrm{Hom}(W, X)$. Again, $\mathfrak{n}''$ is nonempty, Zariski open (the conditions amount to requiring that $A \neq 0$ and that $B - (1/2)AA^* \in \mathrm{Hom}(W, W')$ is a nondegenerate bilinear form on W), and M-invariant, and hence contains $\mathfrak{n}'$. To prove (i), it suffices to show that $\mathfrak{n}''(F)$ is a single $\mathrm{M}_2(F)$-orbit.

In Case 1, using an identification $W \cong F$, M_2 can be naturally identified with $\mathrm{GSp}(X \otimes_F W, \langle \cdot, \cdot \rangle)$, with its action on N given by $g \cdot (A, B) = (A \circ g^{-1}, \mu(g)^{-1}B)$, μ denoting the similitude character. Then (i) in Case 1 follows from the surjectivity of

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μ: $\mathrm{GSp}(X) \to F^\times$ together with the transitivity of the action of $\mathrm{Sp}(X)$ on

$$\mathrm{Hom}(X, W') \smallsetminus \{0\}.$$

In Case 2, (i) follows from the fact that $\mathrm{GL}(W)(F)$ acts transitively on the space of nondegenerate symplectic forms on W, and that the stabilizer of any such symplectic form acts transitively on $\mathrm{Hom}(X, W') \smallsetminus \{0\}$.

Thus, let us now prove (i) for Case 3, which comes down to showing that $\mathrm{GL}(W)(F)$ acts transitively on the space of symplectic forms B on W with $\dim \mathrm{Rad}_B(W) = 1$, and that any stabilizer of any of these symplectic forms acts transitively on the set of elements $A^* \in \mathrm{Hom}(W, X) \smallsetminus \{0\}$ such that $\ker(A^*)$ does not contain $\mathrm{Rad}_B(W)$. That $\mathrm{GL}(W)(F)$ acts transitively on the space of symplectic forms B with $\dim \mathrm{Rad}\, B = 1$ follows from the fact that $\mathrm{GL}(W)(F)$ acts transitively on the one-dimensional subspaces $W_1 \subset W$, together with the fact that the stabilizer of any such W_1 in $\mathrm{GL}(W)(F)$ surjects onto $\mathrm{GL}(W/W_1)(F)$. To finish (i), therefore, it is enough to show that given a line $W_1 \subset W$ and a symplectic form B on W with radical W_1, the stabilizer of B in $\mathrm{GL}(W)(F)$ is transitive on $\mathrm{Hom}(W, F) \smallsetminus \mathrm{Hom}(W/W_1, F)$. This is easy: given φ, φ' belonging to this latter set, choose ordered symplectic bases for their kernels and take one to the other while fixing W_1 pointwise.

Now let us prove (ii) and (iii). First consider Case 1. Any element $\widetilde{m} \in \mathfrak{t}(\mathrm{N}'(F)) \subset \widetilde{\mathrm{M}}(F)$ is the image of an element of $\widetilde{\mathrm{M}}_1(F) \cong \mathrm{Isom}(W, W') \times \mathrm{Sp}(X)(F)$ of the form $(a, 1 + \xi^* \eta^{-1} \xi)$ (2.3). Now as an element of $\mathrm{End}(X)$, $\xi^* \eta^{-1} \xi$ is nilpotent, since $\mathrm{image}(\xi^*)$, being one-dimensional, is contained in $\mathrm{image}(\xi^*)^\perp = \ker \xi$. Thus, $\widetilde{m}_s$ is the image of some $(a, 1) \in \widetilde{\mathrm{M}}_1(F)$. From here, both (ii) and (iii) for Case 1 are easy to check (use Remark 3.2).

In Cases 2 and 3, by (2.3), an element $\widetilde{m} \in \mathfrak{t}(\mathrm{N}'(F)) \subset \widetilde{\mathrm{M}}(F)$ has image in $\widetilde{\mathrm{M}}_2(F) \cong \mathrm{Isom}(W, W')$ of the form $B - (1/2)AA^*$ for some

$$A \in \mathrm{Hom}(X, W') \quad \text{and} \quad B \in \mathrm{Hom}(W, W').$$

Viewing B and $\widetilde{m}$ as bilinear forms on W, we have a relation

$$\widetilde{m}(w_1, w_2) = B(w_1, w_2) - (1/2) a \lambda(w_1) \lambda(w_2),$$

where $a \in F^\times$ and λ is a linear functional on W (the composite of A^* with some isomorphism $X \cong F$).

Let us consider Case 2 first. (ii) and (iii) for M_1 follow from the corresponding assertions for M_2 (use that the obvious map $\mathrm{A} \to \mathrm{A}_2$ is an isomorphism, and verify, say using the computations that follow, that $\widetilde{\eta}$ cannot be $\mathrm{M}(\overline{F})$-conjugated to $b \cdot \widetilde{\eta}$ if $b \neq 1$ belongs to the kernel of $\mathrm{M}_1(F) \to \mathrm{M}_2(F)$), so it is enough to consider the case where $\mathrm{M} = \mathrm{M}_2$. Choose $e \in W$ such that $\lambda = B(e, \cdot)$ on W. Then for $w_1, w_2 \in W$

$$\begin{aligned} \widetilde{m}(w_1, w_2) &= B(w_1, w_2) - (a/2) B(e, w_1) B(e, w_2) \\ &= B(w_1 - (a/2) B(e, w_1) e, w_2) = B(\widetilde{m}_u(w_1), w_2), \end{aligned} \tag{4.1}$$

where $\widetilde{m}_u \in \mathrm{GL}(W)(F) = \mathrm{M}(F)$ is given by $w \mapsto w - B((a/2)e, w)e$. It is easy to see that $\widetilde{m}_u$ is unipotent, in fact belonging to $\mathcal{O}_{\mathrm{Sp}(W,B)}$ (see Remark 3.2), and that its inverse is given by $w \mapsto w + B((a/2)e, w)e$. Therefore,

$$(4.2) \qquad \begin{aligned} B(w_1, \widetilde{m}_u^{-1} w_2) &= B(w_1, w_2) + (a/2)B(e, w_2)B(w_1, e) \\ &= B(w_1, w_2) - (a/2)B(e, w_1)B(e, w_2) = \widetilde{m}(w_1, w_2). \end{aligned}$$

Thus, by (4.1) and (4.2), the unipotent element $\widetilde{m}_u \in \mathrm{GL}(W)(F)$ and the semisimple element $B \in \widetilde{\mathrm{GL}}(W)(F)$ satisfy $\widetilde{m}_u B = \widetilde{m} = B\widetilde{m}_u$. It follows that $B = \widetilde{\eta}$, giving (ii) as $\widetilde{m}_u \in \mathcal{O}_{\mathrm{Sp}(W,B)} = \mathcal{O}_{\mathrm{Sp}(W,\widetilde{\eta})}$. Further, $\mathcal{O}_s$ is actually a single $\mathrm{M}(F) = \mathrm{M}_2(F)$-conjugacy class, for nondegenerate symplectic forms on W are all $\mathrm{GL}(W)(F)$-conjugate, showing (iii) in Case 2.

Now we move to Case 3. Once again, when $\mathrm{M} = \mathrm{M}_1$, (ii) and (iii) follow from the corresponding assertions for $\mathrm{M} = \mathrm{M}_2$. This time $\mathrm{A}_1 \to \mathrm{A}_2$ is not an isomorphism, but we can instead use that the inverse image of $\mathrm{A}_2(F)$ in $\mathrm{M}_1(F)$ is $\mathrm{A}_1(F)$. Therefore, we now assume $\mathrm{M} = \mathrm{M}_2$. Recall that in this case an element of $\mathfrak{n}'(F)$ is represented by a pair (A, B), where $A \in \mathrm{Hom}(X, W') \smallsetminus \{0\}$ and $B \in \mathrm{Hom}(W, W')$, with the condition that $B - (1/2)AA^* \in \mathrm{Isom}(W, W')$. This latter condition implies that the kernel of A^*, which is a $2N$-dimensional subspace of W, does not contain the one-dimensional subspace $\mathrm{Rad}\, B$ of W, and is hence complementary to $\mathrm{Rad}\, B$ in W. Therefore, writing two elements of W as $v_1 + v_2$ and $w_1 + w_2$ with $v_1, w_1 \in \ker A^*$ and $v_2, w_2 \in \mathrm{Rad}\, B$, we have

$$\begin{aligned} (B - (1/2)AA^*)&(v_1 + v_2, w_1 + w_2) \\ &= B(v_1 + v_2, w_1 + w_2) - \frac{1}{2}\langle A^*(v_1 + v_2), A^*(w_1 + w_2)\rangle_X \\ &= B(v_1, w_1) - \frac{1}{2}\langle A^* v_2, A^* w_2\rangle_X, \end{aligned}$$

so that by (2.3), $\widetilde{m} = B - (1/2)AA^* \in \widetilde{\mathrm{GL}}(W)(F)$ is a nondegenerate bilinear form on W that realizes W as a direct sum of the $2N$-dimensional symplectic space $(\ker A^*, B)$ and the one-dimensional quadratic space $(\mathrm{Rad}\, B, (-1/2)AA^*)$. Hence $\widetilde{m}$ is already semisimple, so $\widetilde{\eta} = \widetilde{m}$ (in particular, yielding (ii)). The centralizer $\mathrm{M}^{\widetilde{m}}$ of $\widetilde{m}$ is thus naturally isomorphic to $\mathrm{Sp}_{2N} \times \mathrm{O}_1$. But one also sees that $\mathrm{M}^{\widetilde{m}} = \mathrm{M}_{\widetilde{m}} \cdot \{\pm 1\} = \mathrm{M}_{\widetilde{m}} \cdot \mathrm{Z}_{\mathrm{M}}^{\widetilde{m}}(\overline{F})$, yielding that the stable conjugacy class of $\widetilde{m}$ is precisely $\mathrm{Int}\, \mathrm{M}(\overline{F}) \cdot \widetilde{m} \cap \widetilde{\mathrm{M}}(F)$. Further, this stable conjugacy class identifies with the space of all nondegenerate bilinear forms on W that split into a direct sum of a $2N$-dimensional symplectic part and a one-dimensional orthogonal part. Thus, it is easy to see that one can bijectively map the set of $\mathrm{M}(F)$-conjugacy classes within the stable conjugacy class of $\widetilde{m}$ to $F^\times/F^{\times 2}$, assigning any representative of such a conjugacy class to the discriminant of the quadratic form on the orthogonal part. This also immediately gives (iii), although it is no longer the case that $\mathrm{M}_2(F)$ acts transitively on the stable conjugacy class $\mathcal{O}_s$ in question. ■

Remark 4.1 In each of our cases, it follows from the above proof that given any $\widetilde{\eta} \in \mathcal{O}_s$, $\mathrm{M}^\eta = \mathrm{M}_\eta \cdot \mathrm{Z}_{\mathrm{M}}^\eta$, and $\mathrm{M}^\eta(F) = \mathrm{M}_\eta(F) \cdot \mathrm{Z}_{\mathrm{M}}^\eta(F)$ ($\eta = \mathrm{Int}\, \widetilde{\eta}$; see Section 2.1). This implies also that the stable conjugacy class $\mathcal{O}_s$ of $\widetilde{\eta}$ equals $\mathrm{Int}\, \mathrm{M}(\overline{F})(\widetilde{\eta}) \cap \widetilde{\mathrm{M}}(F)$.

4.2 Notation to Deal With Endoscopic Transfer

Henceforth fix any $\widetilde{\eta} \in \mathcal{O}_s$.

We wish to fix a base point $\widetilde{\theta} \in \widetilde{\mathrm{M}}(F)$. We will also define a Borel pair (B, T) in M stable under $\theta = \mathrm{Int}\,\widetilde{\theta}$ and an element $\nu \in \mathrm{T}(F)$ such that $\widetilde{\eta} = \nu\widetilde{\theta}$.

Suppose first that we are in Case 1 or Case 2. In these cases, we set $\widetilde{\theta} = \widetilde{\eta}$ and $\nu = 1 \in \mathrm{M}(F)$, so that $\widetilde{\eta} = \nu\widetilde{\theta}$. It is easy to check that $\eta = \mathrm{Int}\,\widetilde{\eta} = \theta$ fixes a splitting of M. Fix a Borel pair (B, T) in M forming part of such a splitting. Let $(\mathrm{B_H}, \mathrm{T_H})$ be a Borel pair in H. Dual to $\widehat{\xi}$ (and realized using the above choices of Borel pairs), we have a homomorphism $\xi\colon \mathrm{T} \to \mathrm{T_H}$ that factors through an isomorphism $\mathrm{T}/(1-\eta)\mathrm{T} = \mathrm{T}/(1-\theta)\mathrm{T} \to \mathrm{T_H}$ (ξ will no longer be used to denote an element of $\mathrm{Hom}(X, W')$ as we have done before). We have abused notation slightly here: if H is not split, ξ is only defined over the quadratic extension that splits H.

In Case 3, slightly more explicit computations will be needed, so that we fix our choices a bit more precisely. Recall that $\widetilde{\eta}$ defines a direct sum decomposition of W into a $2N$-dimensional symplectic space and a one-dimensional quadratic space. Choose an ordered basis $e_1, \dots, e_{2N+1}$ of V such that $e_1, \dots, e_N, e_{N+2}, \dots, e_{2N+1}$ form an ordered symplectic basis for the symplectic part of $\widetilde{\eta}$ and e_{N+1} spans the orthogonal part (in particular, $\widetilde{\eta}(e_i, e_{2N+2-i}) = 1$ for $1 \le i \le N$). This ordered basis gives a Borel pair (B, T) in M. As before, dual to $\widehat{\xi}$ and using the choices of Borel pairs we have made, we have a map $\xi\colon \mathrm{T} \to \mathrm{T_H}$. This time, ξ is defined over F as both M and H are split, so that $\widehat{\xi}$ is trivially W_F-equivariant.

Before proceeding further, assume that $\mathrm{M} = \mathrm{M}_2$. Then our ordered basis gives an identification $\mathrm{T} \cong \mathbb{G}_m^{2N+1}/F$. On the other hand, the choice of the basis $\widehat{e}_1, \dots, \widehat{e}_{2N+1}$ gave a Borel pair $(\widehat{\mathrm{B}}, \widehat{\mathrm{T}})$ in $\widehat{\mathrm{M}}$ together with an identification $\widehat{\mathrm{T}} \cong \mathbb{G}_m^{2N+1}/\mathbb{C}$. We assume our L-group data to be chosen so that the identification of $\widehat{\mathrm{T}}$ as Langlands dual to T (made using the Borel pairs (B, T) and $(\widehat{\mathrm{B}}, \widehat{\mathrm{T}})$), modulo the above identifications, is the usual identification of $\mathbb{G}_m^{2N+1}/\mathbb{C}$ with the Langlands dual of $\mathbb{G}_m^{2N+1}/F$. Let $f_1, \dots, f_{2N}$ be an ordered symplectic basis for a symplectic space that defines $\mathrm{H} = \mathrm{Sp}_{2N}$, and use this ordered basis to define a Borel pair $(\mathrm{B_H}, \mathrm{T_H})$ for H together with an isomorphism $\mathrm{T_H} \cong \mathbb{G}_m^N$. Again we assume that this isomorphism dualizes (using the obvious choices of Borel pairs) to the isomorphism $\widehat{\mathrm{T}}_\mathrm{H} \cong \mathbb{G}_m^N$ defined by the ordered basis $\widehat{e}_1, \dots, \widehat{e}_{2N+1}$ of $\widehat{W}$. It is now easy to see that under the identifications $\mathrm{T} \cong \mathbb{G}_m^{2N+1}$ and $\mathrm{T_H} \cong \mathbb{G}_m^N$, the map ξ is given by

$$(a_1, \dots, a_{2N+1}) \mapsto (a_1 a_{2N+1}^{-1}, \dots, a_N a_{N+2}^{-1}).$$

Let $\widetilde{\theta}$ be the quadratic form on W such that $\widetilde{\theta}(e_i, e_j) = (-1)^i \delta_{i,(2N+2-j)}$ for $i \neq N+1$, and such that $\widetilde{\theta}(e_{N+1}, e_{N+1}) = (-1)^N \widetilde{\eta}(e_{N+1}, e_{N+1})$. Thus, $\widetilde{\eta} = \nu\widetilde{\theta}$ where,

$$\begin{aligned}\nu &= (\underbrace{-1, 1, -1, 1, \dots, (-1)^N}_{N \text{ terms}}, (-1)^N, \underbrace{(-1)^{N-1}, \dots, -1, 1}_{N \text{ terms}}) \in \mathbb{G}_m^{2N+1}(F) \\ &= \mathrm{T}(F) \subset \mathrm{GL}(W)(F).\end{aligned}$$

The choice we have made ensures that $\det \nu = 1 \in F^{\times 2}$, so that $\widetilde{\theta}$ lies in the image of $\widetilde{\mathrm{M}}_1(F) \to \widetilde{\mathrm{M}}_2(F)$. This property allows us to define ν when $\mathrm{G} = \mathrm{G}_1$: simply choose any preimage in $\mathrm{M}_1(F)$ of the corresponding element for M_2. This also lets one define $\widetilde{\theta}$ by requiring that $\widetilde{\eta} = \nu\widetilde{\theta}$.

4.2.1 On Matching of Conjugacy Classes

We use $\widetilde{\theta}$ and ξ to define matching of semisimple conjugacy classes. In other words, semisimple elements $\delta \in \widetilde{\mathrm{M}}(F)$ and $\gamma \in \mathrm{H}(F)$ match if and only if $\xi(t) = t_{\mathrm{H}}$ for some $t \in \mathrm{T}(\overline{F})$ and $t_{\mathrm{H}} \in \mathrm{T}_{\mathrm{H}}(\overline{F})$ such that $t\widetilde{\theta}$ is $\mathrm{M}(\overline{F})$-conjugate to δ and t_{H} is $\mathrm{H}(\overline{F})$-conjugate to γ.

Example 4.2 In all three cases, the conjugacy classes of $\widetilde{\eta}$ and ϵ match because $\xi(\nu) = \epsilon$.

4.3 Semisimple Descent

Let $r > 0$. By H_r we will denote $\mathfrak{q}|^{-1}_{\mathrm{H}(F)}(\mathfrak{q}(\mathrm{T}_{\mathrm{H}}(\overline{F})^{<r}))$, where $\mathfrak{q}\colon \mathrm{H} \to \mathrm{H}/\!/\mathrm{H}$ is the map from H to its quotient under Int H, and

$$\mathrm{T}_{\mathrm{H}}(\overline{F})^{<r} = \{t \in \mathrm{T}_{\mathrm{H}}(\overline{F}) \mid |\chi(t) - 1| < r, \forall \chi \in X^*(\mathrm{T}_{\mathrm{H}})\}.$$

Here $|\cdot|$ denotes the unique extension to $\overline{F}$ of the normalized absolute value on F. One knows that H_r is an open and closed subset of $\mathrm{H}(F)$. Every neighborhood of $1 \in \mathrm{H}(F)$ that is invariant under $\mathrm{H}(F)$-conjugation contains some H_r (this is easy to see from the fact that H has only finitely many $\mathrm{H}(F)$-conjugacy classes of maximal F-tori), and the intersection of all the H_r is the set of unipotent elements in $\mathrm{H}(F)$. Define $\mathfrak{t}_{\mathrm{H}}(\overline{F})^{<r} = \{X \in \mathfrak{t}_{\mathrm{H}}(\overline{F}) \mid |d\chi(X)| < r, \forall \chi \in X^*(\mathrm{T}_{\mathrm{H}})\}$, where $d\chi\colon \mathfrak{t}_{\mathrm{H}} \times_F \overline{F} \to \mathbb{G}_a/\overline{F}$ is the derivative of $\chi\colon \mathrm{T}_{\mathrm{H}} \times_F \overline{F} \to \mathbb{G}_m/\overline{F}$. We can adapt the definition of H_r to talk of $\mathrm{M}_{\eta,r}$, $\mathfrak{h}_r$ and $\mathfrak{m}_{\eta,r}$. For small enough r, one knows that $\exp(\mathfrak{h}_r) = \mathrm{H}_r$, and similarly for the other groups of interest. (This follows from the computations in [HC99, §10.1].) Clearly these sets are closed under stable conjugacy.

Definition 4.3 Given $\widetilde{\eta}' \in \widetilde{\mathrm{M}}(F)$ that is semisimple, and a sufficiently small $r > 0$, define the map $\mathrm{tc}_{\widetilde{\eta}'}\colon \mathrm{M}(F) \times \mathfrak{m}_{\eta',r} \to \widetilde{\mathrm{M}}(F)$ by $(m, X) \mapsto \mathrm{Int}\, m^{-1}(\exp(X)\widetilde{\eta}')$ (here $\eta' = \mathrm{Int}\,\widetilde{\eta}'$). Set $U_{\widetilde{\eta}',r} = \mathrm{tc}_{\widetilde{\eta}'}(\mathrm{M}(F) \times \mathfrak{m}_{\eta',r})$.

Lemma 4.4 *For sufficiently small $r > 0$, the following are true.*

(i) *If $X \in \mathfrak{m}_{\eta',r}$, then X is semisimple (resp., regular semisimple) if and only if*

$$(\exp X)\widetilde{\eta}' \in U_{\widetilde{\eta}',r}$$

is semisimple (resp., regular semisimple) as an element of $\widetilde{\mathrm{M}}(F)$.

(ii) *If $X_1, X_2 \in \mathfrak{m}_{\eta',r}$ and $m \in \mathrm{M}(\overline{F})$ are such that $(\exp X_1)\widetilde{\eta}' = \mathrm{Int}\, m((\exp X_2)\widetilde{\eta}')$, then $m \in \mathrm{M}^{\eta'}(\overline{F})$.*

(iii) *$U_{\widetilde{\eta}',r} \subset \widetilde{\mathrm{M}}(F)$ is open, and the map $\mathrm{tc}_{\widetilde{\eta}'}$ is submersive everywhere on $\mathrm{M}(F) \times \mathfrak{m}_{\eta',r}$.*

(iv) *For all $f \in C_c^\infty(U_{\widetilde{\eta}',r}) \subset C_c^\infty(\widetilde{\mathrm{M}}(F))$, there exists $\phi \in C_c^\infty(\mathfrak{m}_{\eta',r})$ such that for all $X \in \mathfrak{m}_{\eta',r}$, we have an equality of orbital integrals*

$$O(X, \phi, \mathrm{M}_{\eta'}) = O((\exp X)\widetilde{\eta}', f).$$

(See Subsection 3.2.1 for the definition of orbital integrals.) Here we assume that the centralizer measures are chosen compatibly, something that makes sense thanks to (ii).

(v) *Assume that* $\mathrm{M}^{\eta'}(F) = \mathrm{M}_{\eta'}(F)\,\mathrm{Z}_{\mathrm{M}}^{\eta'}(F)$*. Then the association* $f \mapsto \phi$ *of* (iv) *(while not well defined) induces a well-defined bijection between the sets* $\mathcal{I}(U_{\widetilde{\eta}',r})$ *and* $\mathcal{I}(\mathfrak{m}_{\eta',r})$*, where* $\mathcal{I}(U_{\widetilde{\eta}',r})$ *(resp.,* $\mathcal{I}(\mathfrak{m}_{\eta',r})$*) denotes the quotient of* $C_c^\infty(U_{\widetilde{\eta}',r})$ *(resp.,* $C_c^\infty(\mathfrak{m}_{\eta',r})$*) by the equivalence relation that identifies functions with the same strongly regular semisimple orbital integrals.*

Proof Most of this can be found in [Wal08, §2.4]. There X is assumed to be regular semisimple, but the proof goes through to arbitrary X, verbatim, except that one needs to use [RR72] to justify the absolute convergence of not necessarily semisimple (possibly twisted) orbital integrals. Further, on the face of it, [Wal08, Lemma 2.4] only states the existence of an injective map $\mathcal{I}(U_{\widetilde{\eta}',r}) \to \mathcal{I}(\mathfrak{m}_{\eta',r})$, but we claim that the argument there easily gives surjectivity, too. For this, given an element $\varphi \in C_c^\infty(\mathfrak{m}_{\eta',r})$, let $K' \subset \mathrm{M}(F)$ be an open compact subgroup and let

$$\alpha = (\operatorname{meas} K')^{-1}\mathbb{1}_{K'} \otimes \varphi \in C_c^\infty(\mathrm{M}(F)) \otimes C_c^\infty(\mathfrak{m}_{\eta',r}) = C_c^\infty(\mathrm{M}(F) \times \mathfrak{m}_{\eta',r}),$$

and note that we can define $f_\alpha \in C_c^\infty(U_{\widetilde{\eta}',r})$ by requiring that

$$f_\alpha(m^{-1}\exp(X)\widetilde{\eta}'m) = \int_{\mathrm{M}^{\eta'}(F)} \alpha(m_1^{-1}m, m_1^{-1}Xm_1)\,dm_1.$$

Now a computation as in the proof of [Wal08, Lemma 2.4], using that $\mathrm{M}^{\eta'}(F) = \mathrm{M}_{\eta'}(F)\,\mathrm{Z}_{\mathrm{M}}^{\eta'}(F)$, gives that the image of f_α in $\mathcal{I}(U_{\widetilde{\eta}',r})$ maps to the image of φ in $\mathcal{I}(\mathfrak{m}_{\eta',r})$. ∎

Lemma 4.4 will only be applied to $\widetilde{\eta}'$ in the stable conjugacy class $\mathcal{O}_s$ containing $\mathfrak{t}_s(\mathrm{N}'(F))$. For these $\widetilde{\eta}'$, the assumption in (v) of that lemma, that $\mathrm{M}^{\eta'}(F) = \mathrm{M}_{\eta'}(F)\,\mathrm{Z}_{\mathrm{M}}^{\eta'}(F)$, holds (see Remark 4.1).

Lemma 4.5 *Let* $\widetilde{\eta}'$ *belong to the stable conjugacy class* $\mathcal{O}_s$*. Let* r*,* $0 < r < |4|$*, be such that* $|2|^{-1}r$ *is as in Lemma 4.4. Suppose* $\delta \in \widetilde{\mathrm{M}}(F)$ *and* $\gamma \in \mathrm{H}(F)$ *are semisimple elements that match.*

(i) *If* δ *belongs to the union of the* $U_{\widetilde{\eta}',r}$*,* $\widetilde{\eta}'$ *running through* $\mathcal{O}_s$*, then* $\gamma \in \epsilon\,\mathrm{H}_r$*.*
(ii) *If* $\gamma \in \epsilon\,\mathrm{H}_r$*, then* δ *belongs to the union of the* $U_{\widetilde{\eta}',|2|^{-1}r}$*,* $\widetilde{\eta}'$ *running through* $\mathcal{O}_s$*.*

Proof Because $\xi(\nu) = \epsilon$ (see Example 4.2), the isomorphism $(\mathrm{T}/(1-\theta)\mathrm{T}) \times_F \overline{F} \to \mathrm{T_H} \times_F \overline{F}$ induced by ξ sends $\nu \cdot (\mathrm{T}/(1-\theta)\mathrm{T})(\overline{F})^{<r}$ to $\epsilon\,\mathrm{T_H}(\overline{F})^{<r}$. Because ϵ is central in H, the set of semisimple elements of $\mathrm{H}(F)$ that are $\mathrm{H}(\overline{F})$-conjugate to $\epsilon\,\mathrm{T_H}(\overline{F})^{<r}$ is precisely $\epsilon\,\mathrm{H}_r$.

Since $\theta|_{\mathrm{T}}$ has order 2, we get maps $\mathrm{T}_\theta(\overline{F}) \to (\mathrm{T}/(1-\theta)\mathrm{T})(\overline{F}) \to \mathrm{T}_\theta(\overline{F})$, where the first map is induced by the inclusion $\mathrm{T}_\theta \hookrightarrow \mathrm{T}$ and the latter is induced by the map $t \mapsto t\theta(t)$. The latter map, like any homomorphism of tori, takes $(\mathrm{T}/(1-\theta)\mathrm{T})(\overline{F})^{<r}$ to $\mathrm{T}_\theta(\overline{F})^{<r}$, while the composite map, being given by $t \mapsto t^2$, takes $\mathrm{T}_\theta(\overline{F})^{<|2|^{-1}r}$ *isomorphically onto* $\mathrm{T}_\theta(\overline{F})^{<r}$ (because, using $r < |4|$, we see that every $x \in \overline{F}^\times$ with

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$|x-1| < r$ has a square root $\sqrt{x}$ such that $|\sqrt{x}-1| < |2|^{-1}r$, and that such a $\sqrt{x}$ is unique, using that $|2|^{-1}r < |2|$).

Now (ii) will follow if we show that a semisimple element of $\widetilde{\mathrm{M}}(F)$ belongs to some $U_{\widetilde{\eta}',|2|^{-1}r}$, with $\widetilde{\eta}' \in \mathcal{O}_s$, if and only if it is $\mathrm{M}(\overline{F})$-conjugate to an element of $\mathrm{T}_\theta(\overline{F})^{<|2|^{-1}r}\widetilde{\eta}$. Let us prove this assertion. Then the same assertion with $|2|^{-1}r$ replaced by r will also give (i).

The 'only if' part is obvious. Therefore, it is enough to show that any $\widetilde{m} \in \mathrm{M}(F)$ such that $m_1^{-1}\widetilde{m}m_1 = t\widetilde{\eta}$ for some $m_1 \in \mathrm{M}(\overline{F})$ and $t \in \mathrm{T}_\theta(\overline{F})^{<|2|^{-1}r}$ belongs to $U_{\widetilde{\eta}',|2|^{-1}r}$ for some $\widetilde{\eta}' \in \mathcal{O}_s$.

Write $\widetilde{m} = m'\widetilde{\eta}'$, where $m' = m_1 t m_1^{-1}$, $\widetilde{\eta}' = m_1\widetilde{\eta}m_1^{-1}$. If we show that $\widetilde{\eta}'$ belongs to $\widetilde{\mathrm{M}}(F)$, it being $\mathrm{M}(\overline{F})$-conjugate to $\widetilde{\eta}$ will also be stably conjugate to $\widetilde{\eta}$ (see Remark 4.1); and then it is easy to see that $\widetilde{m} = m'\widetilde{\eta}' \in U_{\widetilde{\eta}',|2|^{-1}r}$.

Thus, it remains to show that $m_1\widetilde{\eta}m_1^{-1} \in \widetilde{\mathrm{M}}(F)$, or equivalently, that $m_1 t m_1^{-1} \in \mathrm{M}(F)$. In other words, we want to show that for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, $\mathrm{Int}(\sigma(m_1)^{-1}m_1)$ takes t to $\sigma(t)$. Such a σ necessarily takes $t\widetilde{\eta} = tv\widetilde{\theta}$ to an $\mathrm{M}(\overline{F})$-conjugate of it, so that by [KS99, Lemma 3.2.A] and the sentence just before it, there exists $w \in \mathrm{M}_\theta(\overline{F})$ normalizing T such that $wtvw^{-1}$ and $\sigma(t)v$ have the same image in $(\mathrm{T}/(1-\theta)\mathrm{T})(\overline{F})$. We claim that wtw^{-1} and $\sigma(t)$ have the same image in $(\mathrm{T}/(1-\theta)\mathrm{T})(\overline{F})$. This is because (even when $\mathrm{G} = \mathrm{G}_1$ in Case 3) v is easily checked to be necessarily fixed by w. Since t belongs to $\mathrm{T}_\theta(\overline{F})^{<|2|^{-1}r}$, which maps injectively to $(\mathrm{T}/(1-\theta)\mathrm{T})(\overline{F})$ (this is because $t \mapsto t\theta(t) = t^2$ is injective on $\mathrm{T}_\theta(\overline{F})^{<|2|^{-1}r}$, as observed above), we conclude that $wtw^{-1} = \sigma(t)$. Since $\mathrm{Int}\,\sigma(m_1)^{-1}m_1(t\widetilde{\eta}) = \sigma(t\widetilde{\eta}) = wtw^{-1}\cdot\widetilde{\eta} = w(t\widetilde{\eta})w^{-1}$, conjugation by $w^{-1}\sigma(m_1^{-1})m_1$ preserves $t\widetilde{\eta}$. We have an M-conjugation equivariant map $\widetilde{\mathrm{M}} \to \mathrm{M}$ given by $m\widetilde{\eta} \mapsto m\eta(m)\eta^2(m)\eta^3(m)$ (η has order two in Case 1 and Case 2, and order four in Case 3). Hence we get that $\mathrm{Int}\,w^{-1}\sigma(m_1^{-1})m_1$ preserves t^4, so that (since t is in the image of exp) it also preserves t. This forces $\mathrm{Int}\,\sigma(m_1^{-1})m_1$ to take t to $wtw^{-1} = \sigma(t)$, as needed. ∎

Remark 4.6 Lemma 4.5(ii) and [Wal14, Section 5.1, (1)] together have the consequence that $\mathrm{end}_{\mathrm{H}}^{\widetilde{\mathrm{M}}}(\delta_\epsilon)$ is supported in the set of elements of $\widetilde{\mathrm{M}}(F)$ whose semisimple part belongs to $\mathcal{O}_s$ (this uses that $f^{\mathrm{H}}(\epsilon) = 0$ if the stable strongly regular semisimple orbital integrals of f^{H} in $\epsilon\mathrm{H}_r$ vanish, a consequence of a well-known result of Rogawski on Shalika germs [Rog80]).

Proof of Lemma 3.6 Let us prove the two assertions somewhat simultaneously. Note that (ii) is clear if $\mathrm{M} = \mathrm{M}_2$, so that for the purposes of proving (ii) we assume $\mathrm{M} = \mathrm{M}_1$. For $\mathrm{M}' = \mathrm{M}_1, \mathrm{M}_2$ or M, set $\mathrm{M}'_\sharp = \mathrm{M}'/\mathrm{Z}_{\mathrm{M}'}^\theta$. Set $\mathrm{A}_\sharp = \mathrm{A}/\mathrm{A}^\theta$. We have natural isomorphisms $\mathbb{G}_m \cong \mathrm{A}$ and $\mathbb{G}_m \cong \mathrm{A}_\sharp$, which we write as $z \mapsto a(z)$ and $z \mapsto a_\sharp(z)$, respectively. Waldspurger [Wal14, §2.7] described a character $\omega_\sharp$ of $\mathrm{M}_\sharp(F)$ such that for all $\gamma \in \mathrm{H}(F)$ and $\delta \in \widetilde{\mathrm{M}}(F)$ that match, and for all $m \in \mathrm{M}_\sharp(F)$, $\Delta(\gamma, m^{-1}\delta m) = \omega_\sharp(m)\Delta(\gamma,\delta)$. Considering M_2 in place of M, we get a character $\omega_{2,\sharp}$ of $\mathrm{M}_{2,\sharp}(F)$. Now note the following.

- In all three cases, $\mathrm{Int}\,a_\sharp(z)$ acts on $\widetilde{\mathrm{M}}$ as multiplication by $a(z)$, so (i) will follow from showing that $\omega(a(z)) = \omega_\sharp(a_\sharp(z))$. In Case 3 when $\mathrm{M} = \mathrm{M}_1$, because $\mathrm{M}_2(F)$

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is (explicitly checked to be) generated by $A_\sharp(F)$ and the image of $M_1(F)$, this will also give (ii).

- In Case 1 and Case 2, $M_\sharp = M_{2,\sharp}$, so (ii) will follow from showing that $\omega_\sharp = \omega_{2,\sharp}|_{M_\sharp(F)}$ in these cases.

So now let us recall from [Wal14, §2.7] the prescription for $\omega_\sharp$. From

$$1 \to T / Z_M^\theta \to (1-\theta)\,T \times T_{ad} \to (1-\theta)\,T_{ad} \to 1,$$

(with the map $T / Z_M^\theta \to (1-\theta)\,T$ induced by $t \mapsto t^{-1}\theta(t)$) we have, letting $p\colon \widehat{M}_{sc} \to \widehat{M}$ denote the obvious map, and dualizing and restricting, an exact sequence

$$(4.3) \qquad 1 \to Z_{\widehat{M}_{sc}} / Z_{\widehat{M}_{sc}}^{\widehat{\theta}} \overset{(p,1-\widehat{\theta})}{\to} Z_{\widehat{M}} / (Z_{\widehat{M}} \cap \widehat{T}_{\widehat{\theta}}) \times Z_{\widehat{M}_{sc}} \to Z_{\widehat{M}_\sharp} \to 1.$$

Note that there is an obvious compatibility in the form of a commutative diagram between the above sequence and the analogous one for M_2. Given $w \in W_F$, consider an element of $\widehat{\xi}({}^L H) = \widehat{\xi}(\mathcal{H})$ of the form $(g(w), w)$, *e.g.*, $\widehat{\xi}(1 \times w)$, and write $g(w) = z(w)p(g_{sc}(w))$ with $z(w) \in Z_{\widehat{M}}$ and $g_{sc}(w) \in \widehat{M}_{sc}$. Let s_{sc} be an element of $\widehat{M}_{sc}$ with the same image as s in $\widehat{M}_{ad}$. Note that we may and shall arrange that $g_{sc}(w)$ and s_{sc} are the same as their analogues for M_2, and that $z(w)$ is the image in $\widehat{M}$ of the analogous object for M_2. Thus, if we define $a_{sc}(w) \in \widehat{M}_{sc}$ by

$$s_{sc}\widehat{\theta}(g_{sc}(w))w(s_{sc})^{-1} = a_{sc}(w)g_{sc}(w),$$

then $a_{sc}(w)$ belongs to $Z_{\widehat{M}_{sc}}$ and coincides with the analogous object for M_2 in place of M. According to [Wal14, Lemma 2.7], $\omega_\sharp$ is the character of $M_\sharp(F)$ corresponding to the element of $H^1(W_F, Z_{\widehat{M}_\sharp})$ represented by $w \mapsto z_\sharp(w)$, where $z_\sharp(w)$ is the image of $(z(w), a_{sc}(w))$ in $Z_{\widehat{M}_\sharp}$ under the relevant map in (4.3). In Cases 1 and 2, where $M_\sharp = M_{2,\sharp}$, it is clear that $w \mapsto z_\sharp(w)$ coincides (on the nose) with its analogue for M_2, yielding the assertion sought in (ii) above, namely, that $\omega_\sharp = \omega_{2,\sharp}|_{M_\sharp(F)}$. (i) remains.

What we now need to prove is that under the isomorphim $\iota\colon A_\sharp \to A$ obtained by factoring the isogeny $\theta - 1$ on A, $\omega_\sharp|_{A_\sharp(F)}$ corresponds to $\omega^{-1} = \omega$. Now $\omega_\sharp|_{A_\sharp(F)}$ is given by the composition of $w \mapsto z_\sharp(w)$ with the obvious map $d_\sharp\colon \widehat{M}_\sharp \to \widehat{A}_\sharp$ (this easily follows, *e.g.*, from the construction in [Bor79, §10.2]). It is easy to see that in (4.3) the map from the $Z_{\widehat{M}_{sc}}$ factor to $Z_{\widehat{M}_\sharp}$ is the obvious one, so that its composite with $\widehat{A}_\sharp$ is trivial (as it is induced by a composite homomorphism $\widehat{M}_{sc} \to \widehat{M}_\sharp \to \widehat{A}_\sharp$ that is necessarily trivial as a simply connected group cannot have a torus as a quotient). Thus, using the description of the second map of (4.3), we get that $d_\sharp(z_\sharp(w)) = \widehat{\iota}(d(z(w)))$, where $d\colon \widehat{M} \to \widehat{A}$ is the obvious map. It now suffices to show that $w \mapsto d(z(w))$ is the Langlands parameter of ω. But since there does not exist a nontrivial homomorphism of algebraic groups from a simply connected group to a torus, this is the same as $w \mapsto d(g(w))$, which by Remark 2.3 equals ω. ∎

Let $\widetilde{\eta}'$ be stably conjugate to $\widetilde{\eta}$, *i.e.*, $\widetilde{\eta}' \in \mathcal{O}_s$. Given $\varphi \in C_c^\infty(\mathfrak{m}_{\eta',r}) \subset C_c^\infty(\mathfrak{m}_{\eta'}(F))$ and $t \in \mathfrak{O}_F \smallsetminus \{0\}$, define $\varphi_t \in C_c^\infty(\mathfrak{m}_{\eta',r})$ by $\varphi_t(X) = \varphi(t^{-1}X)$ for all $X \in \mathfrak{m}_{\eta,r}$. Similarly we have a self-map $\varphi^H \mapsto \varphi_t^H$ of $C_c^\infty(\mathfrak{h}_r)$ into itself, which, via the exponential map and multiplication by ϵ, yields a self-map $f^H \mapsto f_t^H$ of $C_c^\infty(\epsilon H_r) \subset C_c^\infty(H(F))$.

Lemma 4.7 *Suppose $r > 0$ is sufficiently small. Fix $\widetilde{\eta}'$ stably conjugate to $\widetilde{\eta}$. If regular semisimple $X \in \mathfrak{m}_{\eta',r}$ and $Y \in \mathfrak{h}_r$ are such that $\exp(Y)\epsilon \in \epsilon\mathrm{H}_r$ and $\exp(X)\widetilde{\eta}' \in U_{\widetilde{\eta}',r}$ match, then for all $t \in F \smallsetminus \{0\}$ such that $t^2X \in \mathfrak{m}_{\eta',r}$ and $t^2Y \in \mathfrak{h}_r$,*

$$\Delta(\exp(t^2Y)\epsilon, \exp(t^2X)\widetilde{\eta}') = |t|^{\dim \mathrm{M}_\eta - \dim \mathrm{H}}\Delta(\exp(Y)\epsilon, \exp(X)\widetilde{\eta}').$$

In particular, for regular semisimple $Y \in \mathfrak{h}_r$ and $X \in \mathfrak{m}_{\eta,r}$, $\exp(Y)\epsilon$ and $\exp(X)\widetilde{\eta}'$ match if and only if $\exp(t^2Y)\epsilon$ and $\exp(t^2X)\widetilde{\eta}'$ match.

Proof The latter assertion follows from the former assertion together with the fact that $\Delta(\gamma, \delta) \neq 0$ if and only if γ and δ match. So let us prove the former. Using Lemmas 3.3 and 3.6, we may assume without loss of generality that $\widetilde{\eta}' = \widetilde{\eta}$.

We now wish to apply Waldspurger's descent for transfer factors [Wal08]. For this we need to define a *diagram* as in [Wal08, §3.2]. By [KS99, Lemma 3.3.B], there exists a maximal F-torus T_0 of M that is θ-stable, and $g_0 \in \mathrm{M}_{\widetilde{\theta},\mathrm{sc}}(\overline{F})$, such that $\mathrm{Int}\, g_0(\mathrm{T}_0) = \mathrm{T}$ and $\xi \circ \mathrm{Int}\, g_0 \colon \mathrm{T}_0 \to \mathrm{T}_\mathrm{H}$ is defined over F (recall that ξ itself is not defined over F if H is not split). In fact, we would like T_0 to be η-stable as well. This is clear in Case 1 and Case 2 where $\widetilde{\eta} = \widetilde{\theta}$. In Case 3, T_H is necessarily split so that we take $\mathrm{T}_0 = \mathrm{T}$ and $g = 1$, so that T_0 is η-stable. Now it is easy to check that $(\mathrm{T}_\mathrm{H}, \mathrm{T}_0, \mathrm{T}_0, \mathrm{T}_{0,\widetilde{\eta}} = \mathrm{T}_{0,\widetilde{\theta}}, 1, g_0, 1, \widetilde{\eta})$ is a diagram coming from ϵ.

Given this diagram, we have an endoscopic datum for M_η as in [Wal08, §3.5] (note that in all our cases we have $\mathrm{M}_\eta = \mathrm{M}_{\eta,\mathrm{sc}}$). Let $\overline{\mathrm{H}}$ denote the group underlying this datum. By [Wal08, §3.6], one has a nonstandard endoscopic triple $(\mathrm{H}_\mathrm{sc}, \overline{\mathrm{H}}_\mathrm{sc}, j_*)$. This defines a bijection between the set of stable conjugacy classes of $\mathfrak{h}_\mathrm{sc}(F) = \mathfrak{h}(F)$ and those of $\overline{\mathfrak{h}}_\mathrm{sc}(F) = \overline{\mathfrak{h}}(F)$. Write $Y \leftrightarrow \overline{Y}$ if the stable conjugacy classes of

$$Y \in \mathfrak{h}(F) = \mathfrak{h}_\mathrm{sc}(F) \quad \text{and} \quad \overline{Y} \in \overline{\mathfrak{h}}_\mathrm{sc}(F) = \overline{\mathfrak{h}}(F)$$

correspond under this bijection. According to [Wal08, Theorem 3.9], we can normalize the transfer factors between $\overline{\mathrm{H}}$ and M_η so that for $r > 0$ sufficiently small, for all $Y \in \mathfrak{h}_r$ and $X \in \mathfrak{m}_{\eta,r}$ (so that $\exp(X)\widetilde{\eta} \in U_{\widetilde{\eta},r}$)

$$\Delta(\exp(Y)\epsilon, \exp(X)\widetilde{\eta}) = \Delta(\overline{Y}, X), \tag{4.4}$$

for any $\overline{Y} \in \overline{\mathfrak{h}}(F) = \overline{\mathfrak{h}}_\mathrm{sc}(F)$ such that $Y \leftrightarrow \overline{Y}$.

Now the required assertion follows by combining (4.4) together with the fact that for all $t \in F^\times$, $\Delta(t^2\overline{Y}, t^2X) = |t|^{\dim \mathrm{M}_\eta - \dim \overline{\mathrm{H}}}\Delta(\overline{Y}, X) = |t|^{\dim \mathrm{M}_\eta - \dim \mathrm{H}}\Delta(\overline{Y}, X)$ (see [Fer07, Lemma 3.2.1] or [Sha90, proof of Lemma 9.7]), and the fact that $Y \leftrightarrow \overline{Y}$ if and only if $t^2Y \leftrightarrow t^2\overline{Y}$. ∎

Now we prove Lemma 3.7.

Proof of Lemma 3.7 Notice that for any $\widetilde{\eta}'$ stably conjugate to $\widetilde{\eta}$, the restriction of the endoscopic transfer $f \mapsto f^\mathrm{H}$ to $C_c^\infty(U_{\widetilde{\eta}',r})$ can, in view of Lemma 4.4 (v), be viewed as a transfer $\varphi \mapsto f^\mathrm{H}$ on $C_c^\infty(\mathfrak{m}_{\eta',r})$. Further, by Lemma 4.5 (i) and the fact that $\epsilon\mathrm{H}_r$ is closed and open in $\mathrm{H}(F)$, we may view this as a transfer from $C_c^\infty(\mathfrak{m}_{\eta',r})$ to $C_c^\infty(\epsilon\mathrm{H}_r)$.

By Remark 4.6 and Lemma 4.4, it now suffices to show that the pull back $D_{\widetilde{\eta}'}$ of $f^\mathrm{H} \mapsto f^\mathrm{H}(\epsilon)$ under $\varphi \mapsto f^\mathrm{H}$ is supported in $\log(\mathcal{O}_{\widetilde{\eta}'})$, for any fixed $\widetilde{\eta}'$ stably conjugate

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to $\widetilde{\eta}$ (see Remark 3.4). Note that by Lemma 4.5 (ii), $D_{\widetilde{\eta}}$ is supported in the intersection of all the $\mathfrak{m}_{\eta,s}(0 < s \le |2|^{-1}r)$, namely, in the nilpotent cone of $\mathfrak{m}_\eta(F)$. Thus, it is a linear combination of nilpotent orbital integrals.

Using Lemmas 3.3 (iii) and 3.6, we may assume without loss of generality that $\widetilde{\eta}' = \widetilde{\eta} \in \mathcal{O}_s$. Since $\log(\mathcal{O}_{\widetilde{\eta}})$ is precisely the union of the nilpotent orbits in $\mathfrak{m}_\eta(F)$ with dimension $\dim \mathrm{M}_\eta - \dim \mathrm{H}$ (Remark 3.5), it suffices to show that $D_{\widetilde{\eta}}(\varphi_{t^2}) = |t|^{\dim \mathrm{M}_\eta - \dim \mathrm{H}} D_{\widetilde{\eta}}(\varphi)$ for all $t \in \mathfrak{O}_F \smallsetminus \{0\}$ (see [HC99, Lemma 3.2]). Since δ_ϵ is invariant under $f^{\mathrm{H}} \mapsto f^{\mathrm{H}}_{t^2}$ for all nonzero $t \in \mathfrak{O}_F$, we are reduced to showing that if $f^{\mathrm{H}} \in C_c^\infty(\epsilon \mathrm{H}_r)$ is a transfer of φ, then $|t|^{\dim \mathrm{M}_\eta - \dim \mathrm{H}} f^{\mathrm{H}}_{t^2}$ is a transfer of φ_{t^2}.

Let us prove this. Let $\gamma \in \mathrm{H}(F)$ be strongly $\widetilde{\mathrm{M}}$-regular. It is enough by Lemma 4.4 to show that

$$\sum_{\gamma'} O(\gamma', f^{\mathrm{H}}_{t^2}) = \sum_X \Delta(\gamma, \exp(X)\widetilde{\eta}) \cdot |t|^{\dim \mathrm{H} - \dim \mathrm{M}_\eta} \cdot O(X, \varphi_{t^2}), \tag{4.5}$$

where γ' runs over a set of representatives for the conjugacy classes stably conjugate to γ, and X over a set of representatives for the $\mathrm{M}^\eta(F)$-conjugacy classes (or equivalently $\mathrm{M}_\eta(F)$-conjugacy classes; see Remark 4.1) of elements $X' \in \mathfrak{m}_{\eta,r}$ such that $\exp(X')\eta$ and γ match.

First suppose $\gamma \notin \epsilon \mathrm{H}_{|t|^2 r}$. Then none of the γ' contributing to (4.5) belongs to $\epsilon \mathrm{H}_{|t|^2 r}$, so that the left-hand side of the equation vanishes. On the other hand, by Lemma 4.5 (i), the right-hand side of (4.5) is an empty sum, so the desired equality trivially holds.

Now assume $\gamma \in \epsilon \mathrm{H}_{|t|^2 r}$, and write $\gamma = \epsilon \exp(t^2 Y)$ with $Y \in \mathfrak{h}_r$. Since f^{H} is a transfer of φ, we have an equation analogous to (4.5):

$$\sum_{\gamma''} O(\gamma'', f^{\mathrm{H}}) = \sum_{X''} \Delta(\exp(Y)\epsilon, \exp(X'')\widetilde{\eta}) O(X'', \varphi), \tag{4.6}$$

with γ'' running over a set of representatives for the conjugacy classes stably conjugate to $\epsilon \exp(Y)$. It is easy to see that the terms γ' contributing to the left-hand side of (4.5) are in one-to-one correspondence with the terms γ'' contributing to the left-hand side of (4.6), under the rule that $Y' = t^2 Y''$ where $\gamma' = \exp(Y')\epsilon$ and $\gamma'' = \exp(Y'')\epsilon$. Then we have $O(\gamma', f^{\mathrm{H}}_{t^2}) = O(\gamma'', f^{\mathrm{H}})$, and the left-hand sides of (4.5) and (4.6) coincide. Similarly, but using Lemma 4.7, the terms X contributing to the right-hand side of (4.5) are in one-to-one correspondence with the terms X'' contributing to the right-hand side of (4.6), under the rule $X = t^2 X''$. Therefore, using Lemma 4.7 again, $\Delta(\gamma, \exp(X)\widetilde{\eta})|t|^{\dim \mathrm{H} - \dim \mathrm{M}_\eta} O(X, \varphi_{t^2}) = \Delta(\gamma'', \exp(X'')\widetilde{\eta}) O(X'', \varphi)$. Thus, the right-hand sides of (4.6) and (4.5) also coincide. Since (4.6) holds, (4.5) also holds, as desired. ∎

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