This method is an improvement on that arising from the orthodox substitution $x - e = \frac{1}{z}$, in which the final reduction to canonical form with $n = 0$ is frequently tedious[1]. The following example with $n = 3$ shows that the reduction process is not necessary for low values of n.

$$\int \frac{(t+1)^3\,dt}{\sqrt{(t^2+k^2)}} = \int t\sqrt{(t^2+k^2)}\,dt + 3\int \sqrt{(t^2+k^2)}\,dt + (3-k^2) \int \frac{tdt}{\sqrt{(t^2+k^2)}} + (1-3k^2)\int \frac{dt}{\sqrt{(t^2+k^2)}}$$

which may be integrated at sight.

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A dual quadratic transformation associated with the Hessian conics of a pencil

By T. Scott.

1. The invariants and covariants of a system of two conics have been much studied[2] but little has been said about those of three conics. Three conics $f_1 \equiv a_x^2$, $f_2 \equiv b_x^2$, $f_3 \equiv c_x^2$ have a symmetrical invariant Ω_{123}, or in symbolical notation $(a\,b\,c)^2$. According to Ciamberlini[3] the vanishing of this invariant signifies that *the* Φ *conic of any two of* f_1, f_2, f_3 *is inpolar with respect to the third*; and in a previous paper[4] I have

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[1] The integral $\int \frac{dx}{(x+1)\sqrt{(2x-x^2)}} = -\int \frac{dz}{\sqrt{(-1+4z-3z^2)}}$ or $-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{(\frac{1}{4}-t^2)}}$ is a case in point.

[2] See Salmon Conic Sections, Ch. xviii, or Sommerville, Analytical Conics, Ch. xx. Taking point-coordinates x, y, z with corresponding line-coordinates l, m, n, a conic $a_x^2 \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy = 0$ has a tangential equation $A_{11}l^2 + A_{22}m^2 + A_{33}n^2 + 2A_{23}mn + 2A_{31}nl + 2A_{12}lm = 0$. Then the vanishing of the invariant $\Theta = b_{11}A_{11} + b_{22}A_{22} + b_{33}A_{33} + 2b_{23}A_{23} + 2b_{31}A_{31} + 2b_{12}A_{12}$ of the conics $f_1 \equiv a_x^2$, $f_2 \equiv b_x^2$ implies that there are triangles circumscribed to f_1 which are self-polar for f_2, and f_1 is said to be inpolar to f_2. The contravariant conic Φ_{12} is the envelope of a line whose intersections with f_1 harmonically separate its intersections with f_2.

[3] *Giorn. di Mat.*, Napoli, 24 (1886), 141.

[4] *Proc. Ed. Math. Soc.*, 2 iv (1935) 258.

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derived by symbolical methods a more symmetrical result, viz., *if* Ω_{123} *vanishes, then u being any line in the plane,* u_1, u_2, u_3 *are concurrent, where* u_i *is the polar with respect to* f_i *of the pole of u with respect to* Φ_{jk}.

I now apply this result to the conics of a pencil, and show that in any pencil of conics there are two special conics which form with any third member a system for which Ω_{123} vanishes. I derive a dual quadratic transformation connected with the pencil.

2. *The conics of a pencil.* In any pencil of conics there exist two conics, f_1 and f_2, the Hessian conics of the pencil, which are mutually apolar (*i.e.*, f_1 is inpolar in f_2 and f_2 is inpolar in f_1). By taking the self-conjugate triangle of the pencil as base triangle, and by a proper choice of unit point, the general conic of any pencil may be written without loss of generality as

$$f_3 \equiv ax^2 + by^2 + cz^2 = 0$$

with $a + b + c = 0$. The two conics

$$f_1 \equiv x^2 + \omega y^2 + \omega^2 z^2 = 0$$
$$f_2 \equiv x^2 + \omega^2 y^2 + \omega z^2 = 0$$

where ω is a complex cube root of unity, are mutually apolar and belong to the pencil and are hence its Hessian conics[1]. The Φ conic of f_1 and f_2 is $\Phi_{12} \equiv l^2 + m^2 + n^2 = 0$. Hence the condition for Φ_{12} to be inpolar to f_3 is $\Omega_{123} = a + b + c = 0$, which is true. *Thus the two Hessian conics together with any conic of the pencil form a system of three conics having* Ω_{123} *zero.*

3. *The Transformation.* It follows that if we choose any line u in the plane, we get a corresponding point P, the intersection of u_1, u_2, u_3. Now we should expect that as we varied the conic f_3 we should obtain a variable point P, determining a curve locus depending on the fixed line u. It in fact happens that P is uniquely determined by u whatever f_3 from the pencil be chosen, and moreover that the correspondence between P and u is a quadratic transformation in its simplest form.

Writing down the equations of Φ_{31}, Φ_{12} we have

$$\Phi_{31} \equiv \omega (b\omega + c) l^2 + (c + a\omega^2) m^2 + (a\omega + b) n^2 = 0$$
$$\Phi_{12} \equiv l^2 + m^2 + n^2 = 0.$$

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[1] Sommerville *loc. cit.*, p. 278; for a fuller discussion of mutual apolarity, see Strazzeri, *Rend. Circ. Mat. di Palermo*, lxi (1937), 100.

Given the line u with coordinates (l, m, n), the pole of u with respect to Φ_{12} is (l, m, n) and the polar of this point with respect to f_3 is

$$u_3 \equiv alx + bmy + cnz = 0. \tag{1}$$

Similarly the pole of u with respect to Φ_{31} is $[\omega (b\omega + c)\, l, (c + a\omega^2)\, m, (a\omega + b)\, n]$ and the polar of this point with respect to f_2 is

$$u_2 \equiv (b\omega + c)\, lx + (c\omega + a)\, my + (a\omega + b)\, nz = 0 \tag{2}$$

P is the point of intersection of u_2 and u_3. Adding, and remembering that $a + b + c = 0$, we have

$$blx + cmy + anz = 0. \tag{3}$$

And again using $a + b + c = 0$, equations (1) and (3) are

$$a\,(lx - nz) + b\,(my - nz) = 0 = b\,(lx - nz) + c\,(my - nz)$$

i.e. $$lx = my = nz$$

or, if we prefer, $\sigma x = mn,\ \sigma y = nl,\ \sigma z = lm$ (4)

the standard equations of a quadratic dual transformation the triangle of reference being that self-conjugate to the pencil and so providing the exceptional elements.

Since a, b, c do not enter into these results, *P is uniquely determined by u independently of the choice of f_3.*

In conclusion I should like to express my gratitude to Professor Turnbull who was kind enough to read and provide helpful criticism of the above result.

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Collapsible circular sections of quadric surfaces

By H. W. TURNBULL.

Cardboard or wire models of ellipsoids and hyperboloids exist which consist of two sets of circular sections. They cover the quadric surface with curvilinear quadrilaterals, whose sides remain constant in length when the model alters in shape. In fact the models admit of one degree of freedom—they are collapsible—and the angle between the two sets of circular sections can be varied.

Such models, with their explanation, are found at the South