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QUADRATIC CONGRUENCES FOR COHEN–EISENSTEIN SERIES

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The notion of quadratic congruences was introduced in the recently published paper [**1**]. In this note we present different, somewhat more conceptual proofs of several results from [**1**]. Our method allows us to refine the notion and to generalize the results quoted. Here we deal only with the quadratic congruences for Cohen–Eisenstein series. Similar phenomena exist for cusp forms of half-integral weight as well; however, as one would expect, in the case of Eisenstein series the argument is much simpler. In particular, we do not make use of techniques other than p-adic Mazur measure, whereas the consideration of cusp forms of half-integral weight involves a much more sophisticated construction. Moreover, in the case of Cohen–Eisenstein series we are able to obtain a full and exhaustive result. For these reasons we present the argument here.

Our result deals with modular forms, but the argument essentially does not. One can think just about the congruences for Cohen numbers $H(r,N)$. These are arithmetically interesting rational numbers defined below.

Our proof relies on the construction of p-adic Mazur measure. We formulate the precise statement as Proposition 1 in the text. After that we present a corollary (Proposition 2) which is sufficient for our purposes.

Let χ be a Dirichlet character modulo $M > 1$, and denote by $L(s,\chi)$ the associated L-series

$$L(s,\chi) = \sum_{n\geqslant 1} \chi(n) n^{-s}$$

The series converges for $\mathrm{Re}(s) > 1$ and admits the analytic continuation over all $s \in \mathrm{C}$. Its values at negative integers essentially coincide with generalized Bernoulli numbers. More precisely, one has, for a positive integer r,

$$L(1-r,\chi) = -\frac{B_{r,\chi}}{r},$$

where the numbers $B_{r,\chi}$ are defined by

$$\sum_{a=1}^{M} \frac{\chi(a) t e^{at}}{e^{Mt}-1} = \sum_{r\geqslant 0} B_{r,\chi} \frac{t^r}{r!}.$$

Fix an integer $r \geq 2$. If $N = 0$, then let $H(r,0) = \zeta(1-2r)$. Here ζ denotes the Riemann ζ-function. If N is a positive integer and $Df^2 = (-1)^r N$, where D is the discriminant of a quadratic field, then define $H(r,N)$ by

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$$H(r, N) = L(1-r, \chi_D) \sum_{d|f} \mu(d)\chi_D(d)d^{r-1}\sigma_{2r-1}(f/d). \tag{1}$$

Here χ_D denotes the quadratic character associated with $Q(\sqrt{D})$ [**2**]. The arithmetic function σ_{2r-1} is defined by $\sigma_{2r-1}(l) = \sum_{d|l} d^{2r-1}$. We denote by μ the Möbius function. In particular, if $D = (-1)^r N$ is the discriminant of a quadratic field, then

$$H(r, N) = L(1-r, \chi_D) = -\frac{B_{r,\chi_D}}{r}.$$

In all other cases let $H(r, N) = 0$.

The rational numbers $H(r, N)$ were introduced by H. Cohen [**3**]. Their significance is connected with the following fact. The series

$$\mathcal{H}_r = \sum_{n \geq 0} H(r, N) \exp(2\pi i N z)$$

is the Fourier expansion of a modular form of half-integral weight $r + \frac{1}{2}$. This is the Cohen–Eisenstein series.

Since we are going to deal with congruences, it will be convenient to introduce some basic notations from p-adic analysis. Let p be a prime. The restriction $p \neq 2$ will somewhat simplify our argument, and we always assume it. Denote by Q_p the field of p-adic numbers, i.e. the completion of the field of rational numbers Q with respect to the p-adic metric given by the p-adic valuation

$$|\cdot|_p : Q \to R_{\geq 0} = \{x \in R | x \geq 0\}, \quad |a/b|_p = p^{\mathrm{ord}_p b - \mathrm{ord}_p a}, \ |0|_p = 0,$$

where $v_p(a) = \mathrm{ord}_p(a)$ is the highest power of p dividing the integer a. We fix an embedding $\iota : Q \hookrightarrow Q_p$, and we will not distinguish between rational numbers and their images under ι. This allows us to consider rational numbers as p-adic as well.

Let us now define the quadratic congruences.

Definition. Let p be an odd prime. Let $\varphi = \sum_{n \geq 0} a(n) \exp(2\pi i n z)$ be a modular form of half-integral weight $r + \frac{1}{2}$. Assume that the Fourier coefficients $a(n)$ are rational numbers. For a positive integer d, denote by ξ_d the quadratic Dirichlet character associated with $Q(\sqrt{-d})$.

(i) We say that φ satisfies *type a quadratic congruences modulo p* if $r - 1 \equiv 0(\mathrm{mod}(p-1)p^{\alpha-1})$ for an integer $\alpha \geq 1$, and

$$\xi_d(p) = 1 \text{ implies } v_p(a(d)) \geq \alpha.$$

(ii) We saw that φ satisfies *type b quadratic congruences modulo p* if $r - 1 \equiv 0(\mathrm{mod}(p-1)p^{\alpha-1}/2)$ for an integer $\alpha \geq 1$, $r-1 \not\equiv 0 \bmod p-1$, and

$$\xi_d(p) = 1 \text{ implies } v_p(a(pd)) \geq \alpha.$$

Remarks. 1. Our definition is sufficient since we are going to consider only Cohen–Eisenstein series. In general, one should consider cusp forms of half-integral weight as well. Let us indicate, without explaining the details, the necessary changes in the definition above. One should assume ϕ to be a Hecke eigenform [7]. Then $a(n)$ are no longer rational, but algebraic numbers. We extend our fixed embedding ι to an embedding $\iota : \overline{Q} \hookrightarrow C_p$. Here $\overline{Q}$ is the algebraic closure of the field of rational numbers, and C_p is the Tate field (i.e. the completion of the algebraic closure of Q_p [6]). One also should sometimes change the condition $\xi_d(p) = 1$ to $\xi_d(p) = -1$.

2. Our definition is slightly different from the original one given in [1]. After the changes described in the remark above, all the quadratic congruences for the Fourier coefficients of modular forms of half-integral weight presented there will fit into our definition.

The main result of the present note is the following assertion.

THEOREM. *The Cohen–Eisenstein series $\mathcal{H}_r$ satisfy both a and b types modulo p quadratic congruences as long as* $r-1\equiv 0(\text{mod } (p-1)/2)$.

Several examples of this phenomena are described in [1, Section 2]. To be more specific, put $r=5$ and $p=5$. Theorem 6 of [1] asserts that $N\equiv 1(\text{mod } 5)$ yields $H(5,N)\equiv 0(\text{mod } 5)$. This is (half of) our type a quadratic congruences. The original proof of the theorem quoted is based on the fact that $\mathcal{H}_r$ is a modular form. The argument uses the knowledge of specific structure of the space of modular forms of weight 6 on $\Gamma_1(500)$ and involves a machine computation.

Proof. Recall the construction of the p-adic Mazur measure. We direct the reader to [**10**, Chapter 1]; [**11**, Chapters 5,7,12] and [**4** Chapter 3], where detailed discussions and various interpretations are given. The original approach is contained in [**8**] and [**9**].

PROPOSITION 1. *Let Δ be a positive integer, $(\Delta,p)=1$. Put $Z_\Delta = \varprojlim_m (Z/\Delta p^m Z)^*$, and let $c>1$ be an integer coprime to Δ and p. Then there exists a (integer-valued) measure $\mu_{c,\Delta}$ on Z_Δ such that*

$$\int_{Z_\Delta} \xi(a)a^{r-1}d\mu_{c,\Delta} = -(1-\xi(c)c^r)(1-\xi(p)p^{r-1})L(1-r,\xi). \tag{2}$$

The following corollary is a special case of *abstract Kummer congruences* [**10**, Chapter 3; **5**, p.258]. In order to get it, one notices that the integrals of (p-adically) close functions over a compact set (Z_Δ is compact) are (p-adically) close.

PROPOSITION 2. *Let Δ be a positive integer, $(\Delta,p)=1$. Let $c>1$ be an integer coprime to Δ and p. Let ξ_i for $i=1,2$ be two Dirichlet characters modulo Δp^{m_i} for integers $m_i\geq 0$, and let r_i be two positive integers. Suppose that for every a coprime to Δ and p one has*

$$\xi_1(a)a^{r_1-1} \equiv \xi_2(a)a^{r_2-1} \ (\text{mod } p^\alpha), \qquad \alpha \geq 0. \tag{3}$$

Then the right hand sides of (2) *for $i=1,2$ are also congruent modulo p^α:*

$$\begin{aligned} &v_p((1-\xi_1(c)c^{r_1})(1-\xi_1(p)p^{r_1-1})L(1-r_1,\xi_1) \\ &(1-\xi_2(c)c^{r_2})(1-\xi_2(p)p^{r_2-1})L(1-r_2,\xi_2)) \geq \alpha. \end{aligned} \tag{4}$$

We apply Proposition 2 to the case when both ξ_i are quadratic Dirichlet characters. In particular, we take the absolute value of the discriminant of $Q(\sqrt{-d})$ for Δ, the quadratic Dirichlet character ξ_d associated with $Q(\sqrt{-d})$ for ξ_1, and put $r_1 = 1$. Observe that if $\xi_d(p) = 1$, the right-hand side of (2) vanishes because of the factor $(1-\xi_1(p)p^{r_1-1})$. Put $r_2 = r$.

Let us now pick appropriate ξ_2.

(a) Assume $r-1 \equiv 0 \bmod (p-1)p^{\alpha}$. Put $\xi_2 = \xi_d$. The condition (3) is fulfilled. Pick the number c such that $1-\xi_d(c)c^r \equiv 1-\xi_d(c)c \not\equiv 0 \bmod p$. Therefore by (4) $v_p(L(1-r,\xi)) \geq \alpha$. It now follows from (1) that $\mathcal{H}_r$ satisfies the type a quadratic congruences.

(b) Assume $r-1 \equiv 0 \bmod (p-1)p^{\alpha}/2$ and $r-1 \not\equiv 0 \bmod (p-1)$. Take the quadratic Dirichlet character associated with $Q(\sqrt{(-1)^r dp})$ for ξ_2. Notice that since $-d \equiv (-1)^r dp \bmod 4$, the character ξ_2 is defined modulo Δp, correctly. We claim that for every a coprime to Δ and p

$$\xi_d(a) \equiv \xi_2(a)a^{r-1} \bmod p^{\alpha}.$$

Indeed, both $\xi_d(\cdot)$ and $\xi_2(\cdot)^{r-1} \bmod p^{\alpha}$ are non-trivial (since both characters are odd: $\xi_d(-1) \equiv \xi_2(-1)(-1)^{r-1} = -1$) quadratic characters of the multiplicative group $(\mathrm{Z}/\Delta p^{\alpha}\mathrm{Z})^*$. Since this group is cyclic, they are equal. It follows now from (1) that $\mathcal{H}_r$ satisfies the type b quadratic congruences.

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