

COMPLETE SPACELIKE SUBMANIFOLDS IN DE SITTER SPACES WITH $R = aH + b$

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Abstract

In this paper, we give a classification of spacelike submanifolds with parallel normalised mean curvature vector field and linear relation $R = aH + b$ of the normalised scalar curvature R and the mean curvature H in the de Sitter space $S_p^{n+p}(c)$.

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1. Introduction

Let \mathbb{R}_p^{n+p} denote an $(n+p)$ -dimensional real vector space endowed with an inner product of index p given by

$$\langle x, y \rangle = - \sum_{i=1}^p x_i y_i + \sum_{j=p+1}^{n+p} x_j y_j,$$

where $x = (x_1, x_2, \dots, x_{n+p})$ are the natural coordinates of \mathbb{R}_p^{n+p} . The manifold \mathbb{R}_p^{n+p} is called *semi-Euclidean space* and it has constant curvature $c = 0$. We also define the semi-Riemannian manifold $S_p^{n+p}(c)$, with $c > 0$, the so-called *de Sitter space*, as follows:

$$S_p^{n+p}(c) = \left\{ (x_1, x_2, \dots, x_{n+p}) \in \mathbb{R}_p^{n+p+1} : - \sum_{i=1}^p x_i^2 + \sum_{j=p+1}^{n+p+1} x_j^2 = \frac{1}{c} \right\}.$$

A smooth immersion $\phi : M^n \rightarrow S_p^{n+p}(c)$ of an n -dimensional connected manifold M^n is said to be a *spacelike submanifold* if the induced metric via ϕ is a Riemannian metric on M^n . When the codimension $p = 1$, it is called a *spacelike hypersurface*.

Spacelike submanifolds usually appear in the study of questions related to causality in general relativity. More precisely, level sets of a function of global time are

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spacelike submanifolds. Also, spacelike hypersurfaces with constant mean curvature are convenient as initial hypersurfaces for the Cauchy problem in an arbitrary space-time and for studying the propagation of gravitational radiation.

The initial step for the study of spacelike submanifolds in the de Sitter space $S_p^{n+p}(c)$ is due to Goddard's conjecture [11]: *complete spacelike hypersurfaces with constant mean curvature H in $S_1^{n+1}(1)$ must be totally umbilical*. It was proved by Akutagawa [2] (for the case where $n = 2$ and $H^2 \leq 1$ or $n \geq 3$ and $H^2 < 4(n-1)/n^2$) and by Montiel [15] (for the compact case) that the conjecture is true. Furthermore, Montiel [15] exhibited examples of complete spacelike hypersurfaces with constant H satisfying $H^2 \geq 4(n-1)/n^2$ but not umbilical—the so-called hyperbolic cylinders, which are isometric to the Riemannian product $H^1(\sinh r) \times S^{n-1}(\cosh r)$.

Having completely settled Goddard's conjecture, most of the research interest turned to the study of submanifolds in $S_p^{n+p}(c)$ with constant scalar curvature instead of constant mean curvature. In particular, interest focuses on characterising the totally umbilical properties or Riemannian product structures (that is, hyperbolic cylinders) of such submanifolds. We refer to [5, 6, 9, 14, 19] and the references therein.

More generally, Cheng [8], Li [13] and Shu [17] studied the spacelike hypersurface M in $S_1^{n+1}(1)$ under the condition that the normalised scalar curvature R and the mean curvature H of M satisfy the linear relation $R = aH$ for some constant a , instead of the restriction that R or H is a constant. Such a spacelike hypersurface M is called a *Weingarten hypersurface*. They obtained many sufficient conditions for such a Weingarten hypersurface to be totally umbilical. Recently, Hou and Yang [12] extended the linear relation $R = aH$ to the case $R = aH + b$ for some constants a, b , and generalised the corresponding results of Cheng [8, Theorem 1] (when $b = 0$ and a is positive), Li [13, Corollary 4.3] (when $b = 0$) and Zheng [19, Theorem B] (when $a = 0$).

In this paper, we will study the situation of higher codimension under the more general relation $R = aH + b$, and prove the following two theorems.

THEOREM 1.1. *Let M^n ($n \geq 2$) be a complete spacelike submanifold in the de Sitter space $S_p^{n+p}(c)$ with parallel normalised mean curvature vector. Assume that the normalised scalar curvature R and the mean curvature H of M^n satisfy $R = aH + b$ for some constants $a \neq 0$ and $b < c$, and H attains a maximum on M^n . If the squared norm S of the second fundamental form of M^n satisfies $S \leq 2\sqrt{n-1}c$, then either M^n is totally umbilical or $S = 2\sqrt{n-1}c$ and M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.*

THEOREM 1.2. *Let M^n ($n \geq 2$) be a complete spacelike submanifold in the de Sitter space $S_p^{n+p}(c)$ with parallel normalised mean curvature vector. Assume that the normalised scalar curvature R and the mean curvature H of M^n satisfy $R = aH + b$ for some constants $a \neq 0$ and $b < c$, and H attains a maximum on M^n .*

- (1) *When $H^2 \leq c$ for $n = 2$ or $H^2 < 4(n-1)c/((n-2)^2p + 4(n-1))$ for $n \geq 3$, then M^n is totally umbilical.*

- (2) When $H^2 = 4(n - 1)c / ((n - 2)^2p + 4(n - 1))$ for $n \geq 3$, then M^n is totally umbilical, or the codimension $p = 1$ and M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.
- (3) When $4(n - 1)c / ((n - 2)^2p + 4(n - 1)) < H^2 \leq c$ for $n \geq 3$, if the squared norm S of the second fundamental form of M^n satisfies $S \leq nH^2 + (B_H^-)^2$ or $S \geq nH^2 + (B_H^+)^2$, then M^n is totally umbilical, or the codimension $p = 1$ and M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$, where B_H^\pm are the two real roots of the polynomial

$$P_H(x) = \frac{1}{p}x^2 - \frac{n(n - 2)}{\sqrt{n(n - 1)}}Hx + n(c - H^2).$$

REMARK 1.3. When the constant a vanishes identically, the linear relation $R = aH + b$ reduces to $R = b$, that is, R is a constant. Therefore we restrict our attention to $a \neq 0$.

REMARK 1.4. The parallel normalised mean curvature vector assumption was introduced by Chen [7]. Submanifolds with nonzero parallel mean curvature vector also have parallel normalised mean curvature vector. The condition of having a parallel normalised mean curvature vector is much weaker than the condition of having a parallel mean curvature vector. For instance, every hypersurface in a semi-Riemannian manifold always has a parallel normalised mean curvature vector. In this sense, Theorem 1.1, for $p = 1$, generalises [12, Theorem 1.5], and Theorem 1.2 reduces to the result of [17] when the constant b vanishes identically and a is positive.

REMARK 1.5. In our main Theorems 1.1 and 1.2, we assume that the mean curvature H attains a maximum on the submanifold M^n . Use of the Cheng–Yau operator, first used in the same sense by Cheng in [8], and then by Hou and Yang in [12], is clearly a helpful technical condition to impose. We do not yet know whether it is necessary. In other words, we do not have an example of a spacelike hypersurface with parallel mean curvature vector for which the maximum of the mean curvature may occur at infinity.

REMARK 1.6. In fact, the results in Theorem 1.2 partially concern the reduction of codimensions, which is an important problem in the classification of submanifolds. When the ambient spaces are Riemannian space forms, we refer to Barbosa and Araújo’s work [4]; the references therein include the classical results. But as we know, there are few results in semi-Riemannian space forms.

2. Preliminaries

Let $S_p^{n+p}(c)$ be an $(n + p)$ -dimensional de Sitter space of constant sectional curvature $c > 0$, whose index is p , and let M^n be an n -dimensional complete spacelike submanifold immersed in $S_p^{n+p}(c)$. Throughout this paper, we shall make use of the convention on the range of indices:

$$1 \leq A, B, \dots \leq n + p; \quad 1 \leq i, j, \dots \leq n; \quad n + 1 \leq \alpha, \beta, \dots \leq n + p.$$

Choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_{n+p}\}$ such that, restricted to M^n , $\{e_1, \dots, e_n\}$ spans the tangent space of M^n and forms an orthonormal frame there. Let $\{\omega_1, \dots, \omega_{n+p}\}$ be its dual frame field so that the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1, \varepsilon_\alpha = -1$. Then the structure equations of $S_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D,$$

$$K_{ABCD} = c \varepsilon_A \varepsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}).$$

Restricting these forms to M^n , then $\omega_\alpha = 0$. By Cartan’s lemma, we can write $\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, h_{ij}^\alpha = h_{ji}^\alpha$. The connection forms of M^n are characterised by the structure equations

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

and

$$R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.1}$$

where R_{ijkl} are the components of the curvature tensor of M^n . Denote by $R_{ik}, n(n-1)R$ and $R_{\alpha\beta ij}$ the components of the Ricci curvature, the scalar curvature and the components of normal curvature of M^n , respectively. Then, from (2.1),

$$R_{ij} = c(n-1)\delta_{ij} - \sum_{\alpha,k} h_{kk}^\alpha h_{ij}^\alpha + \sum_{\alpha,k} h_{ik}^\alpha h_{kj}^\alpha,$$

$$n(n-1)R = n(n-1)c + S - n^2 H^2, \tag{2.2}$$

$$R_{\alpha\beta ij} = \sum_l (h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta). \tag{2.3}$$

Recall that $h = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$ is the second fundamental form; then the mean curvature vector \vec{H} , its length H , and the squared length S of h are defined by

$$\vec{H} = \frac{1}{n} \sum_{\alpha,i} h_{ii}^\alpha e_\alpha, \quad H = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}, \quad S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2.$$

Define the first and the second covariant derivatives of h_{ij}^α , say h_{ijk}^α and h_{ijkl}^α , by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{jk}^\alpha \omega_{ki} + \sum_k h_{ik}^\alpha \omega_{kj} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha},$$

$$\sum_m h_{ijkm}^\alpha \omega_m = dh_{ijk}^\alpha + \sum_m h_{mj}^\alpha \omega_{mi} + \sum_m h_{imk}^\alpha \omega_{mj} + \sum_m h_{ijm}^\alpha \omega_{mk} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}.$$

We obtain the Codazzi equations and the Ricci identities

$$h_{ijk}^\alpha = h_{ikj}^\alpha = h_{jik}^\alpha, \\ h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{mi}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}. \tag{2.4}$$

The Laplacian of h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. It follows from (2.4) that, for any α ,

$$\Delta h_{ij}^\alpha = \sum_k h_{kkij}^\alpha + \sum_{k,m} h_{im}^\alpha R_{mkjk} + \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,\beta} h_{ik}^\beta R_{\alpha\beta jk}. \tag{2.5}$$

Since the normalised mean curvature vector field is parallel, we choose $\vec{H} = He_{n+1}$, then $nH = \sum_i h_{ii}^{n+1}$, and for any $\alpha > n + 1$, $\sum_i h_{ii}^\alpha = 0$. Denote by H^α the matrix (h_{ij}^α) . Then

$$\text{tr}(H^{n+1}) = nH, \quad \text{tr}(H^\alpha) = 0 \quad \text{for } \alpha \geq n + 2. \tag{2.6}$$

Set $\Phi = \sum_{\alpha,i,j} \Phi_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$, where $\Phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}$, and for any $\alpha \geq n + 2$, $\Phi_{ij}^\alpha = h_{ij}^\alpha$. Let Φ^α denote the matrix (Φ_{ij}^α) . Then

$$\Phi^{n+1} = H^{n+1} - HI, \quad \Phi^\alpha = H^\alpha \quad \text{for } \alpha \geq n + 2, \tag{2.7}$$

and

$$|\Phi^{n+1}|^2 = \text{tr}(H^{n+1})^2 - nH^2, \quad \sum_{\alpha > n+1} |\Phi^\alpha|^2 = \sum_{\alpha > n+1} (h_{ij}^\alpha)^2, \quad \text{tr}(\Phi^\alpha) = 0, \quad \forall \alpha. \tag{2.8}$$

So we arrive at

$$S = \sum_\alpha |\Phi^\alpha|^2 + nH^2 = |\Phi|^2 + nH^2 \tag{2.9}$$

and $\frac{1}{2}\Delta S = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha$. When the normalised mean curvature vector field of M^n is parallel, using (2.1), (2.3), (2.5) and (2.6), a direct calculation yields (see [6])

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + n \sum_{\alpha,i,j} h_{ij}^\alpha H_{ij}^\alpha + nc(S - nH^2) - nH \sum_\alpha \text{tr}(H^{n+1}(H^\alpha)^2) \\ + \sum_{\alpha,\beta} (\text{tr}(H^\alpha H^\beta))^2 + \sum_{\alpha,\beta} N(H^\alpha H^\beta - H^\beta H^\alpha), \tag{2.10}$$

where $N(A) = \text{tr}(AA')$ for an $n \times n$ matrix A , with A' its transpose. By (2.7)–(2.9) we can obtain

$$ncS - n^2cH^2 = nc(|\Phi|^2 + nH^2) - n^2cH^2 = nc|\Phi|^2, \tag{2.11}$$

$$nH \sum_\alpha \text{tr}(H^{n+1}(H^\alpha)^2) = nH \sum_\alpha \text{tr}(\Phi^{n+1}(\Phi^\alpha)^2) + 2nH^2 \text{tr}(\Phi^{n+1})^2 \\ + n^2H^4 + nH^2|\Phi|^2, \tag{2.12}$$

$$\sum_{\alpha,\beta} (\text{tr}(H^\alpha H^\beta))^2 = \sum_{\alpha,\beta} (\text{tr}(\Phi^\alpha \Phi^\beta))^2 + n^2H^4 + 2nH^2 \text{tr}(\Phi^{n+1})^2. \tag{2.13}$$

Following Cheng and Yau [10], we introduce the second-order operator \square , acting on any C^2 function f on M^n ,

$$\square f = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1})f_{ij}.$$

We will need the following algebraic lemmas.

LEMMA 2.1 [3]. *Let $\{\mu_i\}_{i=1}^n$ be a set of real numbers satisfying $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = \beta^2$, $\beta \geq 0$. Then*

$$\left| \sum_i \mu_i^3 \right| \leq \frac{n-2}{\sqrt{n(n-1)}} \beta^3,$$

and equality holds if and only if at least $n-1$ of the μ_i are equal to each other.

LEMMA 2.2 [16]. *Let $A, B: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be two symmetric linear maps such that $AB - BA = 0$ and $\text{tr}(A) = \text{tr}(B) = 0$. Then*

$$|\text{tr}(A^2B)| \leq \frac{n-2}{\sqrt{n(n-1)}} N(A) \sqrt{N(B)}.$$

LEMMA 2.3 [18]. *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be $2n$ ($n \geq 2$) real numbers satisfying $\sum_i b_i = 0$. Then*

$$\sum_{i,j} a_i a_j (b_i - b_j)^2 \leq \frac{n}{\sqrt{n-1}} \sum_i a_i^2 \sum_j b_j^2.$$

Since the parallel normalised mean curvature vector assumption implies that $\omega_{n+1\alpha} = 0$ for all α , it is possible to show that $H^{n+1}H^\alpha = H^\alpha H^{n+1}$, for all α (see [6]). Therefore, by definition, the traceless matrix Φ^{n+1} commutes with the traceless matrices Φ^α , for all α . Hence we can apply Lemma 2.2 and the Cauchy–Schwarz inequality in order to obtain, respectively,

$$\sum_\alpha \text{tr}(\Phi^{n+1}(\Phi^\alpha)^2) \leq \frac{n-2}{\sqrt{n(n-1)}} \sqrt{N(\Phi^{n+1})} |\Phi|^2 \leq \frac{n-2}{\sqrt{n(n-1)}} |\Phi|^3, \quad (2.14)$$

$$\sum_{\alpha,\beta} (\text{tr}(\Phi^\alpha \Phi^\beta))^2 \geq \sum_\alpha (N(\Phi^\alpha))^2 \geq \frac{1}{p} \left(\sum_\alpha N(\Phi^\alpha) \right)^2 \geq \frac{1}{p} |\Phi|^4. \quad (2.15)$$

3. Some lemmas

To prove the main theorems, we need the following key lemmas.

LEMMA 3.1. *Let M^n be an n -dimensional spacelike submanifold immersed in the de Sitter space $S_p^{n+p}(c)$ with $R = aH + b$. If $a \neq 0$ and $b < c$, then the operator $L = \square + \frac{1}{2}(n-1)a\Delta$ is elliptic.*

PROOF. If $H = 0$, then $R = b$. It follows from (2.2) that $S = n(n - 1)(b - c) < 0$, which is impossible. Therefore $H > 0$. From (2.2) and $R = aH + b$, we know that

$$S = n^2H^2 + n(n - 1)(aH + b - c), \tag{3.1}$$

and then

$$a = \frac{1}{n(n - 1)H}(S - n^2H^2 + n(n - 1)(c - b)).$$

For any fixed α , we choose locally an appropriate orthonormal frame $\{e_j\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$. Then $nH = \sum_i h_{ii}^{n+1}$ and $\sum_i h_{ii}^\alpha = 0$ for any $\alpha \geq n + 2$. For any i ,

$$\begin{aligned} nH - \lambda_i^{n+1} + \frac{n - 1}{2}a &= \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} + \frac{1}{2nH}(S - n^2H^2 + n(n - 1)(c - b)) \\ &\geq \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} + \frac{1}{2} \left(\sum_j (\lambda_j^{n+1})^2 - \left(\sum_j \lambda_j^{n+1} \right)^2 + n(n - 1)(c - b) \right) (nH)^{-1} \\ &= \left(\left(\sum_j \lambda_j^{n+1} \right)^2 - \lambda_i^{n+1} \sum_j \lambda_j^{n+1} - \frac{1}{2} \sum_{l \neq j} \lambda_l^{n+1} \lambda_j^{n+1} + \frac{1}{2} n(n - 1)(c - b) \right) (nH)^{-1} \\ &= \left(\sum_{j \neq i} (\lambda_j^{n+1})^2 + \frac{1}{2} \sum_{l \neq j, l, j \neq i} \lambda_l^{n+1} \lambda_j^{n+1} + \frac{1}{2} n(n - 1)(c - b) \right) (nH)^{-1} \\ &= \frac{1}{2} \left(\sum_{j \neq i} (\lambda_j^{n+1})^2 + \left(\sum_{j \neq i} \lambda_j^{n+1} \right)^2 + n(n - 1)(c - b) \right) (nH)^{-1}. \end{aligned}$$

It follows from $b < c$ that $nH - \lambda_i^{n+1} + \frac{1}{2}(n - 1)a > 0$. Thus L is an elliptic operator. This completes the proof of Lemma 3.1. □

LEMMA 3.2. *Let M^n be an n -dimensional spacelike submanifold in the de Sitter space $S_p^{n+p}(c)$ with $R = aH + b$. If $a \neq 0$ and $b < c$, then*

$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq n^2 |\nabla H|^2.$$

PROOF. Taking the covariant derivative of (3.1), for any k ,

$$2 \sum_{i,j,\alpha} h_{ij}^\alpha h_{ijk}^\alpha = (2n^2H + n(n - 1)a)H_k.$$

Hence, by the Cauchy–Schwarz inequality,

$$\sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq \left(n^2H + \frac{1}{2}n(n - 1)a \right)^2 |\nabla H|^2.$$

That is,

$$S \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq \left(n^2 H + \frac{1}{2} n(n-1)a \right)^2 |\nabla H|^2.$$

On the other hand, it follows from (3.1) that

$$\begin{aligned} (n^2 H + \frac{1}{2} n(n-1)a)^2 - n^2 S &= n^2(n^2 H^2 + n(n-1)Ha - S) + \frac{1}{4} n^2(n-1)^2 a^2 \\ &= n^3(n-1)(c-b) + \frac{1}{4} n^2(n-1)^2 a^2 \\ &= \frac{1}{4} n^2(n-1)((n-1)a^2 + 4n(c-b)). \end{aligned}$$

Since $a \neq 0$ and $b < c$, we have $(n^2 H + \frac{1}{2} n(n-1)a)^2 > n^2 S$. Hence

$$S \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq \left(n^2 H + \frac{1}{2} n(n-1)a \right)^2 |\nabla H|^2 \geq n^2 S |\nabla H|^2,$$

which implies that $S = 0$, or $\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq n^2 |\nabla H|^2$. □

LEMMA 3.3. *Let M^n be an n -dimensional complete spacelike submanifold immersed in the de Sitter space $S_p^{n+p}(c)$ with parallel normalised mean curvature vector field. Then*

$$\square(nH) \geq -\frac{1}{2} n(n-1)\Delta R + (S - nH^2) \left(nc - \frac{n}{2\sqrt{n-1}} S \right).$$

PROOF. By the definition of the operator \square ,

$$\begin{aligned} \square(nH) &= \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\ &= \frac{1}{2} \Delta(n^2 H^2) - n^2 |\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1} (nH)_{ij} \\ &= \frac{1}{2} \Delta(S - n(n-1)(R-c)) - n^2 |\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1} (nH)_{ij} \\ &= -\frac{1}{2} n(n-1)\Delta R + \frac{1}{2} \Delta S - n^2 |\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1} (nH)_{ij}. \end{aligned} \tag{3.2}$$

Let

$$\|\mu\|^2 = \sum_{i,j} (h_{ij}^{n+1} - H\delta_{ij})^2, \quad \|\tau\|^2 = \sum_{i,j} \sum_{\alpha>n+1} (h_{ij}^\alpha)^2.$$

It is clear that the functions $\|\mu\|^2$ and $\|\tau\|^2$ are defined on M^n globally; in other words, they do not depend on the choice of the orthonormal frame $\{e_1, \dots, e_n\}$. Also,

$$S = \|\mu\|^2 + \|\tau\|^2 + nH^2, \tag{3.3}$$

which leads to

$$\frac{1}{2}\Delta S = \frac{1}{2}\Delta\|\mu\|^2 + \frac{1}{2}\Delta\|\tau\|^2 + \frac{1}{2}\Delta(nH^2) = \frac{1}{2}\Delta \operatorname{tr}(H^{n+1})^2 + \frac{1}{2}\Delta\|\tau\|^2. \tag{3.4}$$

Together with (2.1), (2.3) and (2.5),

$$\begin{aligned} \frac{1}{2}\Delta \operatorname{tr}(H^{n+1})^2 &= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} \\ &= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1} (nH)_{ij} + nc \operatorname{tr}(H^{n+1})^2 - n^2cH^2 \\ &\quad - nH \operatorname{tr}(H^{n+1})^3 + (\operatorname{tr}(H^{n+1})^2)^2 + \sum_{\beta>n+1} (\operatorname{tr}(H^{n+1}H^\beta))^2. \end{aligned} \tag{3.5}$$

Choose an appropriate orthonormal frame such that $h_{ij}^{n+1} = \lambda_i \delta_{ij}$. Since $\sum_i (\lambda_i - H) = 0$, then

$$\begin{aligned} \sum_i (\lambda_i - H)^2 &= \sum_i \lambda_i^2 - nH^2 = \operatorname{tr}(H^{n+1})^2 - nH^2 = \|\mu\|^2, \\ \sum_i \lambda_i^3 &= \sum_i (\lambda_i - H)^3 + nH^3 + 3H\|\mu\|^2. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} -nH \operatorname{tr}(H^{n+1})^3 &= -n^2H^4 - 3nH^2\|\mu\|^2 - nH \sum_i (\lambda_i - H)^3 \\ &\geq -n^2H^4 - 3nH^2\|\mu\|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\|^3. \end{aligned}$$

Putting the above three formulas into (3.5),

$$\begin{aligned} \frac{1}{2}\Delta \operatorname{tr}(H^{n+1})^2 &\geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1} (nH)_{ij} \\ &\quad + \|\mu\|^2 \left(\|\mu\|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\| + n(c - H^2) \right). \end{aligned} \tag{3.6}$$

Consider the quadratic form

$$F(x, y) = x^2 - \frac{n-2}{\sqrt{n-1}}xy - y^2.$$

By the orthogonal transformation

$$\begin{aligned} u &= \frac{1}{\sqrt{2n}}((1 + \sqrt{n-1})x + (1 - \sqrt{n-1})y), \\ v &= \frac{1}{\sqrt{2n}}((\sqrt{n-1} - 1)x + (\sqrt{n-1} + 1)y), \end{aligned}$$

it is clear that $x^2 + y^2 = u^2 + v^2$ and $F(x, y) = n(u^2 - v^2)/2\sqrt{n-1}$. Taking $x = \|\mu\|$, $y = \sqrt{n}H^2$, we obtain $u^2 + v^2 = x^2 + y^2 = \|\mu\|^2 + nH^2$. Hence

$$\begin{aligned} nc + F(x, y) &= nc - \frac{n}{2\sqrt{n-1}}(u^2 + v^2) + \frac{n}{\sqrt{n-1}}u^2 \\ &\geq nc - \frac{n}{2\sqrt{n-1}}(\|\mu\|^2 + nH^2) \\ &\geq nc - \frac{n}{2\sqrt{n-1}}S, \end{aligned} \tag{3.7}$$

which implies that

$$nc + \|\mu\|^2 - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\| \geq nc - \frac{n}{2\sqrt{n-1}}S.$$

Together with (3.6),

$$\frac{1}{2}\Delta \operatorname{tr}(H^{n+1})^2 \geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} + \|\mu\|^2 \left(nc - \frac{n}{2\sqrt{n-1}}S \right). \tag{3.8}$$

For any $\alpha > n + 1$,

$$-nH \operatorname{tr}((H^\alpha)^2 H^{n+1}) + (\operatorname{tr}(H^{n+1} H^\alpha))^2 = -\frac{1}{2} \sum_{i,j} h_{ii}^{n+1} h_{jj}^{n+1} (h_{ii}^\alpha - h_{jj}^\alpha)^2.$$

Let $a_i = h_{ii}^{n+1}$, $b_i = h_{ii}^\alpha$. Since $\sum_i h_{ii}^\alpha = 0$, $\alpha > n + 1$, by Lemma 2.3,

$$-nH \operatorname{tr}((H^\alpha)^2 H^{n+1}) + (\operatorname{tr}(H^{n+1} H^\alpha))^2 \geq -\frac{n}{2\sqrt{n-1}} \left(\sum_j (h_{jj}^\alpha)^2 \right) \left(\sum_i (h_{ii}^{n+1})^2 \right).$$

Taking the sum on both sides with respect to $\alpha > n + 1$,

$$-nH \sum_{\alpha > n+1} \operatorname{tr}((H^\alpha)^2 H^{n+1}) + \sum_{\alpha > n+1} (\operatorname{tr}(H^{n+1} H^\alpha))^2 \geq -\frac{n}{2\sqrt{n-1}} \|\tau\|^2 S. \tag{3.9}$$

Then, using (2.1), (2.3) and (2.5),

$$\begin{aligned} \frac{1}{2}\Delta \|\tau\|^2 &= \sum_{i,j,k,\alpha > n+1} (h_{ijk}^\alpha)^2 + nc \|\tau\|^2 + \sum_{\alpha > n+1} (\operatorname{tr}(H^\alpha H^{n+1}))^2 \\ &\quad + \sum_{\alpha,\beta > n+1} (\operatorname{tr}(H^\alpha H^\beta))^2 - nH \sum_{\alpha > n+1} (\operatorname{tr}(H^\alpha)^2 H^{n+1}) \\ &\quad + \sum_{\alpha,\beta > n+1} N(H^\alpha H^\beta - H^\beta H^\alpha). \end{aligned} \tag{3.10}$$

Since $\sum_{\alpha,\beta > n+1} (\operatorname{tr}(H^\alpha H^\beta))^2 \geq 0$ and $\sum_{\alpha,\beta > n+1} N(H^\alpha H^\beta - H^\beta H^\alpha) \geq 0$, from (3.9) and (3.10)

$$\frac{1}{2}\Delta \|\tau\|^2 \geq \sum_{i,j,k,\alpha > n+1} (h_{ijk}^\alpha)^2 + \|\tau\|^2 \left(nc - \frac{n}{2\sqrt{n-1}}S \right). \tag{3.11}$$

Putting (3.8) and (3.11) into (3.2), and combining (3.3), (3.4) and Lemma 3.2, we complete the proof of Lemma 3.3. \square

4. Proofs of main theorems

PROOF OF THEOREM 1.1. Applying Lemmas 3.1 and 3.3,

$$L(nH) = \square(nH) + \frac{n-1}{2}a\Delta(nH) = \square(nH) + \frac{1}{2}n(n-1)\Delta R$$

$$\geq (S - nH^2)\left(nc - \frac{n}{2\sqrt{n-1}}S\right).$$

It is clear that $S \geq nH^2$ from (3.3). Together with the assumption $S \leq 2\sqrt{n-1}c$ in Theorem 1.1, we obtain $L(nH) \geq 0$. Since L is elliptic and H obtains a maximum on M^n , we deduce that H is a constant. Hence

$$(S - nH^2)\left(nc - \frac{n}{2\sqrt{n-1}}S\right) = 0.$$

If $S < 2\sqrt{n-1}c$ then $S = nH^2$, and M^n is totally umbilical.

If $S = 2\sqrt{n-1}c$, the inequalities in (3.7) and in Lemma 2.1 become equalities. If the inequalities in (3.7) hold, then $\|\tau\|^2 = 0$. Since e_{n+1} is parallel on the normal bundle $T^\perp(M^n)$ of M^n , we know that M^n lies in a totally geodesic submanifold $S_1^{n+1}(c)$ of $S_p^{n+p}(c)$. If equality holds in Lemma 2.1, then M^n has $n-1$ principal curvatures which are equal and constant. As H is a constant, the other principal curvature is a constant as well. So M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$ from the congruence theorem in [1]. This completes the proof of Theorem 1.1. \square

PROOF OF THEOREM 1.2. By the definition of operator \square , and using Lemma 3.2 and (2.10),

$$\begin{aligned} \square(nH) &= -\frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= -\frac{1}{2}n(n-1)\Delta R + \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} + nc(S - nH^2) \\ &\quad - nH \sum_{\alpha} \text{tr}(H^{n+1}(H^\alpha)^2) + \sum_{\alpha,\beta} (\text{tr}(H^\alpha H^\beta))^2 \\ &\quad + \sum_{\alpha,\beta} N(H^\alpha H^\beta - H^\beta H^\alpha) - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &\geq -\frac{1}{2}n(n-1)\Delta R + nc(S - nH^2) - nH \sum_{\alpha} \text{tr}(H^{n+1}(H^\alpha)^2) \\ &\quad + \sum_{\alpha,\beta} N(H^\alpha H^\beta - H^\beta H^\alpha) + \sum_{\alpha,\beta} (\text{tr}(H^\alpha H^\beta))^2. \end{aligned} \tag{4.1}$$

Notice that $\sum_{\alpha\beta} N(H^\alpha H^\beta - H^\beta H^\alpha) \geq 0$. Together with (2.11)–(2.13), then (4.1) becomes

$$\begin{aligned} \square(nH) &\geq -\frac{1}{2}n(n-1)\Delta R + (nc - nH^2)|\Phi|^2 \\ &\quad - nH \sum_{\alpha} \text{tr}(\Phi^{n+1}(\Phi^\alpha)^2) + \sum_{\alpha\beta} (\text{tr}(\Phi^\alpha \Phi^\beta))^2. \end{aligned} \tag{4.2}$$

Substituting (2.14), (2.15) into (4.2),

$$\square(nH) \geq -\frac{1}{2}n(n-1)\Delta R + |\Phi|^2 \left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + n(c - H^2) \right).$$

Furthermore, note that $R = aH + b$. Then

$$\begin{aligned} L(nH) &= \square(nH) + \frac{n-1}{2}a\Delta(nH) = \square(nH) + \frac{1}{2}n(n-1)\Delta R \\ &\geq |\Phi|^2 \left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + n(c - H^2) \right). \end{aligned} \tag{4.3}$$

Put

$$P_H(|\Phi|) = \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + n(c - H^2).$$

(1) When $n = 2$ and $H^2 \leq c$, $P_H(|\Phi|) \geq 0$. Hence, the right-hand side of (4.3) is nonnegative. Since L is elliptic and H obtains a maximum on M^n , we deduce that H is a constant and (4.3) becomes

$$|\Phi|^2 \left(\frac{|\Phi|^2}{p} + 2(c - H^2) \right) = 0,$$

which implies that $|\Phi|^2 = 0$ and M^n is totally umbilical.

When $n \geq 3$ and $H^2 < 4(n-1)c/((n-2)^2p + 4(n-1))$, the discriminant of $P_H(|\Phi|)$ is negative. Then $P_H(|\Phi|) > 0$ and

$$L(nH) \geq |\Phi|^2 \left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + n(c - H^2) \right) \geq 0.$$

Using again the facts that L is elliptic and H obtains a maximum on M^n , we deduce that H is a constant and then (4.3) yields

$$|\Phi|^2 \left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + n(c - H^2) \right) = 0.$$

Since $P_H(|\Phi|) > 0$, we obtain that $|\Phi|^2 = 0$ and M^n is totally umbilical.

(2) When $n \geq 3$ and $H^2 = 4(n-1)c/((n-2)^2p + 4(n-1))$, the discriminant of $P_H(|\Phi|)$ vanishes and $P_H(|\Phi|) \geq 0$. If $P_H(|\Phi|) > 0$, by making use of the same assertion

as in the proof of (1) above, we infer that $|\Phi|^2 = 0$ and M^n is totally umbilical. If $P_H(|\Phi|) = 0$, then the inequalities in (2.14) and (2.15) become equalities. That is,

$$\sqrt{N(\Phi^{n+1})}|\Phi|^2 = |\Phi|^3, \quad |\Phi|^4 = p \sum_{\alpha} N^2(\Phi^{\alpha}). \tag{4.4}$$

Since

$$|\Phi| = \sqrt{\frac{c}{(n-2)^2p + 4(n-1)} \frac{n(n-2)}{\sqrt{n}} p} > 0,$$

using (4.4) we get $N(\Phi^{n+1}) = |\Phi|^2$. On the one hand, by $|\Phi|^2 = \sum_{\alpha} N(\Phi^{\alpha})$ we have $N(\Phi^{\alpha}) = 0$, $\alpha \geq n + 2$, so $|\Phi|^4 = p \sum_{\alpha} (N(\Phi^{\alpha}))^2 = p(N(\Phi^{n+1}))^2 = p|\Phi|^4$. Hence $p = 1$. On the other hand, if equality holds in Lemma 2.1, then M^n has $n - 1$ principal curvatures which are equal and constant. As H is a constant, the other principal curvature is a constant as well. According to the congruence theorem due to Abe *et al.* [1], M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.

(3) When $n \geq 3$ and $4(n-1)c/((n-2)^2p + 4(n-1)) < H^2 \leq c$, the discriminant of $P_H(|\Phi|)$ is positive. Then $P_H(|\Phi|)$ has two real roots:

$$B_H^{\pm} = p \sqrt{\frac{n}{4(n-1)}} \left((n-2)H \pm \sqrt{\frac{((n-2)^2p + 4(n-1))H^2 + 4(n-1)c}{p}} \right).$$

The assumption $S \leq nH^2 + (B_H^-)^2$ or $S \geq nH^2 + (B_H^+)^2$ on M^n implies that $|\Phi| \leq B_H^-$ or $|\Phi| \geq B_H^+$ on M^n . Therefore we know that $P_H(|\Phi|) \geq 0$.

Clearly, B_H^+ is always positive. On the other hand, $B_H^- > 0$ if and only if $H^2 < c$, while $B_H^- = 0$ if and only if $H^2 = c$.

When $4(n-1)c/((n-2)^2p + 4(n-1)) < H^2 < c$, we have $B_H^- > 0$. If $S < nH^2 + (B_H^-)^2$ or $S > nH^2 + (B_H^+)^2$, then we obtain $P_H(|\Phi|) > 0$. By making use of the same assertion as in the proof of (1) above, we infer that $|\Phi|^2 = 0$ and M^n is totally umbilical. If $S = nH^2 + (B_H^-)^2$ or $S = nH^2 + (B_H^+)^2$, then we obtain $P_H(|\Phi|) = 0$. By the same arguments as in the proof of (2) above, we infer that M^n is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.

When $4(n-1)c/((n-2)^2p + 4(n-1)) < H^2 = c$, we have $B_H^- = 0$. If $S > nH^2 + (B_H^+)^2$, we obtain $P_H(|\Phi|) > 0$; and if $S = nH^2 + (B_H^+)^2$, we get $P_H(|\Phi|) = 0$. The rest of the proof is the same as that in case (2), and this completes the proof of Theorem 1.2. \square

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