As the benefit contains that element of uncertainty which offers an attraction to many, it is matter of surprise that no other Office has before now hit upon this novel mode of dealing with its surplus.

I am, Sir,
Your most obedient servant,
H. AMBROSE SMITH.

5, Lothbury,
London, 7th December, 1867.

P.S.—It is worthy of note that the numerator of the first expression above, may be put into the following form:

$$l_x(1+i)+i\{l_{x+1}(1+i)+l_{x+2}(1+i)^2+l_{x+3}(1+i)^3+\ \ldots\ \&c.\}$$

This form is more elegant than the other, and being quite as easy of computation it might be used if we had to deal with only a single age. But it is unsuited for the formation of a Commutation Table as the terms do not represent *annual* values.

ON THE ADJUSTMENT OF PREMIUMS FOR LIFE ASSURANCE IN REFERENCE TO EXTRA RISKS.

To the Editor.

SIR,—There are two kinds of Extra Risk with which Assurance Companies in the ordinary transaction of their business have to deal, viz., that which arises from deteriorated health on the one hand, and that which consists in exposure to danger from some external cause, such as Climate, Military or Naval service, &c., on the other. The first risk is sometimes met by assuming an addition of a certain number of years to the actual age, and sometimes by a fixed addition to the annual premium, that is, fixed as regards the age and the nature of the assurance, but proportional to the sum assured. The second risk almost invariably by the latter method, except in those cases where the office is possessed of special Tables of Mortality founded upon observations made in the countries in which the parties reside, as India, for instance.

The adjustment of the premium by the former method,—that is, by a fixed addition to the age,—is open in some respects to considerable objection. For instance, if the life be young, and the assurance for a "Short Term," the addition of even several years would have comparatively but little effect upon the rate of premium. Nay, if Mr. Bailey's theory be correct, and it should ever be carried into practice, we should sometimes by this method obtain a *diminished* premium as a provision for a supposed *additional* risk! The second method, viz., that of a constant addition to the yearly premium irrespective of the age and the nature of the assurance, is not open to the particular objection just referred to,—but others, perhaps quite as strong, may be urged against it. In proof of this, it is only necessary to mention the case of "Endowment Assurances,"—to which indeed neither of the methods in question is applicable. There are two distinct benefits comprised in assurances of this description, viz., a Term Assurance and an Endowment; and the extra risk under the former is partly compensated by the diminished risk under the latter. The addition to the age which would be made if the case were that of an ordinary whole

https://doi.org/10.1017/S2046167400003578 Published online by Cambridge University Press

life assurance,—with a corresponding postponement in the age at which the Endowment is supposed to become payable,—would not, in general, sufficiently provide for the extra risk;* while a constant addition to the premium, also corresponding to that which would be made on a whole life assurance, would be greater than the occasion demands,—inasmuch as it ignores the compensatory character of the contingency.

The practice of providing for the extra risk by a constant addition to the premium irrespective of the age, is evidently a sort of rule-of-thumb way of giving effect to the notion that the extra risk acts with equal force at all ages. We shall find reason to conclude that, as regards "Climate risks" at least, the idea in question is a singularly happy one. But whether true or not, if the hypothesis be adopted, there can be no good reason why we should not consistently act upon it, the more especially as our Life Annuity Tables, calculated at different rates of interest, afford us the means of doing so with perfect accuracy, and without any additional trouble whatever.

In a paper on "the law of mortality" read by me before the Institute of Actuaries on the 29th April last, and published in vol. 13 of this *Journal* (p. 325), I showed that if we have distinct Tables of the number of survivors at every age in two or more bodies of individuals subject to distinct causes of mortality, a Table may be formed showing the number of survivors in a body subject to *all* these causes, by simply multiplying the numbers at each age in the several Tables into each other. The demonstration of this theorem given in the paper above referred to has not to my knowledge been impugned, but in the discussion which followed the reading of the paper a very able member of the Institute, Mr. Sprague, expressed a wish for some further elucidation of a proposition which does perhaps at first sight seem somewhat paradoxical. Mr. Sprague probably thinks (as I do myself) that a ray of light is worth a bushel of demonstration; and I therefore make no apology for taking the first opportunity of complying with the suggestion.

Let A and A′ represent two individuals, each of whom is subject to the risk of dying from one of two distinct and independent causes. And suppose that the risk to which A is exposed arises from his being continuously fired at by a marksman, B; and that to which A′ is exposed arises in like manner from his being the target of another marksman, B′;—death from all other causes being in each case suspended. Now the first shot that takes effect, whether fired by B at A or by B′ at A′, will terminate the *joint* existence of the two lives. But it will evidently make no difference as regards the period at which the first shot takes effect if we suppose that B and B′ are both firing at A, instead of directing their fire at different individuals. Hence it follows, that the risk to an individual subject to two independent causes of mortality, acting simultaneously, is the same as the risk to the *joint* existence of two individuals, exposed, respectively, to one of those causes only; and if L_x and L'_x denote the survivors at age x under the latter supposition, then $L_xL'_x$ will correctly represent the survivors at the same age under the former.

This demonstration, if not more conclusive, is certainly more luminous than that given in the paper before referred to. I proceed now to apply the theorem to the matter which we have in hand.

Let the series L_x represent the survivors at successive ages in a body

* For the reasons previously stated it *might* produce a *diminished* premium.

subject to the *ordinary mortality only*, and L'_x those in a body subject to the *extra risk only*. Then if the extra risk be supposed constant we shall have $L_xL'_x=L_xk^x$; since the series L'_x will in that case form a geometrical progression. In the examples which follow I assume the ordinary mortality to be represented by the Carlisle Table,—the interest of money, by taking $v=(1{\cdot}04)^{-1}$,—and the extra mortality, by making $kv=(1{\cdot}06)^{-1}$ or $k=\frac{1{\cdot}04}{1{\cdot}06}=(1{\cdot}019)^{-1}$ nearly. The premium is in each case increased by a loading of 30 per cent.

I.—*Assurances without Extra Risk.*

Age.	Annual Premiums per Cent.			
	Assurances for the term of one Year.	Whole Life Assurances, Premiums payable during		Endowment Assurances. (Age of 60.)
		Life.	Ten Years.	
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	0 17 8	1 14 3	4 1 1	2 4 2
30	1 5 3	2 5 8	5 0 10	3 5 4
40	1 12 7	3 1 9	6 4 9	5 6 8
50	1 13 7	4 7 6	7 13 7	11 9 1
60	4 3 10	7 3 10	10 12 2	

II.—*Assurances with Extra Risk.*

Age.	Annual Premiums per Cent.			
	Assurances for the term of one Year.	Whole Life Assurances, Premiums payable during		Endowment Assurances (Age of 60.)
		Life.	Ten Years.	
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	3 4 5	3 15 3	7 7 3	4 1 11
30	3 11 11	4 5 6	8 0 1	5 0 5
40	3 19 1	5 0 0	8 16 3	6 18 2
50	4 0 2	6 3 7	9 16 2	12 14 10
60	6 9 4	8 19 6	12 8 2	

III.—*Difference, or true Extra Premium.*

Age.	Annual Extra Premiums per Cent.			
	Assurances for the term of one Year.	Whole Life Assurances, Premiums payable during		Endowment Assurances (Age of 60.)
		Life.	Ten Years	
	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	2 6 9	2 1 0	3 6 2	1 17 9
30	2 6 8	1 19 10	2 19 3	1 15 1
40	2 6 6	1 18 3	2 11 6	1 11 6
50	2 6 7	1 16 1	2 2 7	1 5 9
60	2 5 6	1 15 8	1 16 0	

These Tables show the effect of a supposed constant extra risk upon the Annual Premium required for different kinds of assurances on single lives;

from an inspection of which we may judge how far a uniform extra premium is calculated to provide for it. I now give the formulæ used in their computation,—deferring for another occasion the examination of cases involving two or more lives.

In each of the foregoing examples $D_x = L_x \times 1{\cdot}04^{-x}$. Let D'_x denote $L_x \times 1{\cdot}06^{-x}$, and $N'_x = D'_{x+1} + D'_{x+2} + \ldots$ We shall then have:

1. Premium for an assurance for the term of one year

$$1{\cdot}04^{-1} - \frac{D'_{x+1}}{D'_x}.$$

2. Annual Premium, payable during life, for an assurance for the whole term of life

$$\frac{D'_x}{N'_{x-1}} - (1 - 1{\cdot}04^{-1}).$$

3. Annual Premium, payable during 10 years, for an assurance for the whole term of life

$$\frac{D'_x - N'_{x-1}(1 - 1{\cdot}04^{-1})}{N'_{x-1} - N'_{x+9}}.$$

4. Annual Premium, payable during $60 - x$ years, for an Endowment Assurance payable at sixty or death.

$$\frac{D'_x}{N'_{x-1} - N'_{59}} - (1 - 1{\cdot}04^{-1}).$$

The formulæ for other cases may be deduced with equal facility. That for a single premium for a whole life assurance would be

$$1 - \frac{N'_{x-1}}{D'_x}(1 - 1{\cdot}04^{-1}).$$

I do not know how this case would be treated by the rule-of-thumb method; but if any courageous advocate of that system should choose to take up the cudgels for it, we shall doubtless be informed upon this point.

Having seen how the hypothesis of a constant extra force of mortality operates upon the calculation of premiums for different kinds of assurance, there remains now to consider the case of a transfer from one condition to the other; the most frequent and striking instance of which is that of an assurance effected on a life resident in India, who returns after the lapse of many years to reside permanently in Europe.

It is not surprising that the practice of charging a constant extra premium for all ages and for all kinds of assurance should have been accompanied by the corresponding primitive one of simply discontinuing it when the extra risk had ceased. This, I imagine, has led to the curious but very general practice which still obtains in Indian Offices, whose rates are now, and have long been, computed by mortality tables formed exclusively upon observations made in India, viz., to fix the European rate, not by a calculation based upon the actual age at the time of arrival in Europe, but according to the European risk for the age at which the assurance was effected. A practice so utterly void of any rational foundation could only have originated in the very infancy of the science of actuarial computation, and it is little to the credit of the profession that it should have maintained its ground to the present day.

The true mode of procedure presents no difficulty whatever, notwithstanding the formidable array of "Gorgons, and Hydras, and Chimeras dire," which a writer on this subject, in a former volume of the *Journal*, so unnecessarily conjured up in his path; the principle to be kept in view being simply that the estimated liability of the Office, under its contract, shall be the same after the change has been effected as it is at the time the application for the transfer is made. This is the utmost that the most unreasonable Policyholder can possibly expect, and it is no more than the most prudent Office can with perfect safety concede.

To obviate any question as to the mode of adjusting the loading (which would open up another and very wide field of inquiry), I will suppose the assurances to be affected at pure premiums. Let a_x represent the value of an annuity at the ordinary (or European) risk, and a'_x the same at the increased (or Indian) risk. To determine p the future reduced premium we have the following equation of condition:

$$1-\frac{1+a'_{x+n}}{1+a'_x}=\{1-(1-v)(1+a_{x+n})\}-p(1+a_{x+n})$$

whence we obtain

$$p=\frac{1+a'_{x+n}}{(1+a'_x)(1+a_{x+n})}-(1-v).$$

This formula enables us to see at a glance the relation which the reduced premium bears, 1st to the premium for the original age by the Table on which the assurance was effected, 2ndly to the premium for the original age according to the Table for the reduced risk, and 3rdly to the premium for the reduced risk for the present or actual age of the life assured. For the first is:

$$\frac{1}{1+a'_x}-(1-v),\quad \text{while } p=\frac{1}{1+a'_x}\cdot\frac{1+a'_{x+n}}{1+a_{x+n}}-(1-v).$$

The second is:

$$\frac{1}{1+a_x}-(1-v),\quad \text{while } p=\frac{1}{1+a_x}\cdot\frac{(1+a_x)(1+a'_{x+n})}{(1+a'_x)(1+a_{x+n})}-(1-v).$$

And the third is:

$$\frac{1}{1+a_{x+n}}-(1-v),\quad \text{while } p=\frac{1}{1+a_{x+n}}\cdot\frac{1+a'_{x+n}}{1+a'_x}-(1-v).$$

Hence we draw the following conclusions, viz.:

1. If $1+a'_{x+n}=1+a_{x+n}$, or the annuities for the actual age by the two Tables are equal in value (which might easily be the case coexistently with very different values at earlier ages) no reduction would take place in the premium. This result is evidently perfectly consistent with the supposition.

2. If $\frac{1+a_x}{1+a'_x}\cdot\frac{1+a'_{x+n}}{1+a_{x+n}}=1$ or $\frac{1+a_x}{1+a'_x}=\frac{1+a_{x+n}}{1+a'_{x+n}}$, that is, if the values of annuities (in advance) by the two Tables were in a constant proportion

https://doi.org/10.1017/S2046167400003578 Published online by Cambridge University Press

to each other,—the present practice of reducing to the European rate for the original age would be correct.

3. If $1+a'_{x+n}=1+a'_x$,—that is if the value of the annuity on the life (at the increased risk) for the original age were the same as that for the present or actual age,—the reduced rate would be that required for a new assurance. This is precisely what might have been expected, for the supposition implies that the risk had not increased since the commencement of the assurance, and therefore that the Policy had acquired no value.

Mr. Neison's observations on the Bengal Military Mortality are now generally accepted as the best index of the value of European life in India. In the following Table, therefore, designed to test the correctness of the hypothetical equation

$$\frac{1+a'_x}{1+a_x}=\frac{1+a'_{x+n}}{1+a_{x+n}}$$

(which we have seen is virtually assumed in the prevailing practice in determining the reduction of the premium) I have given the results derived from those observations in juxta-position with the results of the supposition of a constant additional force of mortality. The values contained in the first three columns are those of annuities in advance.

Age.	(1.) Carlisle 4 per Cent.	(2.) Carlisle 6 per Cent.	(3) Bengal M. 4 per Cent.	(4.) Ratio of (2) to (1).	(5.) Ratio of (3) to (1).
20	19·362	14·835	15·126	·766	·781
30	17·852	14·020	14·283	·785	·800
40	16·074	13 002	13·201	·809	·821
50	13·869	11·631	11·773	·839	·849
60	10·663	9·304	9·849	·873	·924

These results show us that whether we proceed upon the theory of a constant additional force of mortality, or whether we take the actual mortality among Europeans resident in India, the values of annuities (in advance), with and without extra risk, when computed at the rate of interest most commonly realized, tend to a ratio of equality as the age increases; and hence we infer (from the first of the conclusions drawn from an examination of the formula for the reduced premium) that the reduction to the rate for the original age is greater than the results of computation will warrant.

Not to trespass too much on the patience of your readers, I will now bring my letter to a close, with the intention of resuming the subject in another communication.

I remain, Sir,

Your very obedient servant,

W. M. MAKEHAM.