Can. J. Math. Vol. **43** (1), 1991 pp. 3–18

VERTICAL SUBGROUPS OF PRIMARY ABELIAN GROUPS

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ABSTRACT. Motived by an intrinisic necessary condition for the purifiability of subgroups of primary abelian groups due to K. Benabdallah and T. Okuyama we introduce new functors on the category of pairs (G, A), where A is a subgroup of G, to the category of $\mathbf{Z}/p\mathbf{Z}$-vector spaces. The vanishing of these functors leads to the notion of vertical subgroup which is a weakening of purity but also an essential component of the latter. In fact, a vertical subgroup is pure if and only if it is neat. We establish various facts about vertical subgroups and "maximal" vertical subgroups and apply the resulting theory to the problem of purifiability. We show that the class of quasi-complete groups is precisely the class of reduced groups in which every subgroup satisfying the intrinsic necessary condition for purifiability is in fact purifiable. This is also the class of reduced p-groups in which the maximal vertical subgroups are precisely the pure subgroups.

Introduction. Prüfer's notion of a pure subgroup in abelian group theory proved to be an important weakening of the concept of a direct summand. One reason is that purity is an inductive property while that of being a direct summand is not. However, purity suffers from two "pathologies":

(1) There exist subsocles of primary groups which do not support pure subgroups.

(2) The p-adic closures of pure subgroups are not in general pure.

K. Honda's concept of neatness is a weakening of purity which eliminates pathology (1) but still has pathology (2). A study of the existence of pure hulls led Benabdallah and Okuyama [3] to the concept of "overhang" which in turn suggested the following notion. A subgroup A of a primary abelian group G is *vertical* (*eventually-vertical*) in G if $(A + p^nG)[p] = A[p] + p^nG[p]$ for all $n \geq 1$ (for almost all $n \geq 1$). Verticality is an interesting weakening of purity inasmuch as it is still an inductive property but has neither of the two pathologies. Furthermore, verticality combined with neatness is precisely purity.

Since verticality is inductive, every subsocle S supports a vertical subgroup which is maximal among the vertical subgroups with socle S. Such a subgroup is called *maximal vertical*. We devote Section 3 to the study of maximal verticality and its relation to purity and topological closure. We show that bounded maximal vertical subgroups of separable groups are pure. However, since the topological closures of pure subgroups are maximal vertical subgroups, maximal vertical subgroups of separable groups are not in general pure. In fact, in Section 4 we show that the class of groups in which maximal verticality is equivalent to purity is the class of quasi-complete groups.

Received by the editors May 25, 1989.

AMS subject classification: Primary 46L55, secondary 46L40.

In Section 5 we characterize those groups in which being purifiable and being eventually-vertical are equivalent. These turn out to be again the quasi-complete groups.

Verticality owes its existence to the more general concept of overhang introduced in [3]. Let A be a subgroup of a p-group G. The n^{th} *overhang* of A in G is the vector space

$$V_n(G,A) = \frac{(A+p^{n+1}G)\cap p^nG[p]}{A\cap p^nG[p]+p^{n+1}G[p]}.$$

The subgroup A is vertical (eventually vertical) in G if and only if $V_n(G,A) = 0$ for all $n \geq 1$ (for almost all $n \geq 1$). Furthermore, V_n defines a functor on the category $\mathcal{G}$ of pairs (G,A) of p-groups where A is a subgroup of G, to the category of vector spaces over $\mathbb{Z}/p\mathbb{Z}$. It was shown in [3] that a subgroup A of a p-group G which is purifiable (see Definition 5.1) is eventually-vertical. This fact suggested the study of vertical and eventually-vertical subgroups.

Section 1 is devoted to properties of the overhang functor which are essential for the remainder of the paper and of independent interest. In addition an interesting connection of the overhang with relative Ulm invariants $U(G,A)$ is explained:

If

$$U_n(G,A) = \frac{p^nG[p]}{(A+p^{n+1}G)\cap p^nG[p]},$$

then

$$0 \longrightarrow V_n(G,A) \longrightarrow U_n(G,A[p]) \longrightarrow U_n(G,A) \longrightarrow 0$$

is exact.

Our notation is standard and follows [5] where all unexplained concepts and unreferenced facts can be found.

1. **The category of pairs (G,A) and the overhang functors.** Let $\mathcal{G}$ be the category whose objects are the pairs $(G,A), A \leq G$, and whose morphisms on (G,A) to (H,B) are the usual homomorphisms $f\colon G \longrightarrow H$ with $f(A) \subset B$. In $\mathcal{G}$ the morphism f is monic if and only if f is a group monomorphism and f is epic if and only if f is a group epimorphism. $\mathcal{G}$ is an additive category (there exists a null object, $\mathcal{G}\big((G,A),(H,B)\big)$ is an abelian group, the composition of morphisms is bilinear and coproducts exist). Moreover, $\mathcal{G}$ is a preabelian category since kernels and cokernels exist. In fact, $(\ker f, A\cap \ker f) \longrightarrow (G,A)$ is a kernel diagram in $\mathcal{G}$ and $(H,B) \longrightarrow (H/f(G), \big(B+f(G)\big)/f(G))$ is a cokernel diagram. The category $\mathcal{G}$ is not abelian. Indeed, if A and B are subgroups of G such that $A \underset{\neq}{\subset} B \subset G$ then $\mathrm{Id}_G : (G,A) \longrightarrow (G,B)$ is a bijective morphism which is not an isomorphism in $\mathcal{G}$.

We now introduce a family of functors on $\mathcal{G}$ to the category of vector spaces over $\mathbb{Z}/p\mathbb{Z}$ as follows.

For a pair $(G,A) \in \mathcal{G}$ and $n \geq 0$ we set

$$A_G^n = (A\cap p^nG+p^{n+1}G)[p] = (A+p^{n+1}G)\cap p^nG[p],$$
$$A_n^G = (A\cap p^nG)[p]+p^{n+1}G[p] = (A[p]+p^{n+1}G[p])\cap p^nG.$$

Clearly $A_n^G \subset A_G^n$.

DEFINITION 1.1. Let $(G,A),(H,B),f\colon (G,A) \longrightarrow (H,B)$ be in $\mathcal{G}$ and let n be an integer ≥ 0. We define the functor $V_n\colon \mathcal{G} \longrightarrow \mathrm{Mod}_{\mathbb{Z}/p\mathbb{Z}}$ by

$$(i) \qquad V_n(G,A) = A_G^n / A_n^G,$$

$$(ii) \qquad V_n(f)\colon V_n(G,A) \longrightarrow V_n(H,B) : V_n(f)(g + A_n^G) = f(g) + B_n^H.$$

$V_n(G,A)$ is called the n^{th} *overhang* of A in G. This notion was introduced first in [3] in connection with the problem of minimal pure subgroups. Some of its fundamental properties are given in [3] although the functorial aspect was not explicitly utilized. For the convenience of the reader we reproduce here with full credit some of the results of [3] which are needed for our presentation. The fact that the V_n are indeed functors on $\mathcal{G}$ to the category of $\mathbb{Z}/p\mathbb{Z}$-vector spaces is readily verified and we leave it to the reader. We begin with an obvious but useful technical criterion.

LEMMA 1.2. *Let $f\colon (G,A) \longrightarrow (H,B)$, and $(G,A),(H,B)$ be in $\mathcal{G}$. Then*

$$(i) \qquad V_n(f) \text{ is injective if and only if } A_G^n \cap f^{-1}(B_n^H) \subset A_n^G,$$

$$(ii) \qquad V_n(f) \text{ is surjective if and only if } B_H^n \subset f(A_G^n) + B_n^H.$$

The next results give various instances where $V_n(f)$ is in fact an isomorphism. The first two appear in [3] and the proofs are omitted.

THEOREM 1.3. ([3, Theorem 1.6]). *Let K be a pure subgroup of G containing A and let $f\colon (K,A) \longrightarrow (G,A)$ be the morphism induced by the inclusion map of K into G. Then*

$$V_n(f)\colon V_n(K,A) \longrightarrow V_n(G,A)$$

is an isomorphism for all $n \geq 0$.

THEOREM 1.4. ([3, Theorem 2.1]). *Let B be a pure and dense subgroup of A and let $f\colon (G,B) \longrightarrow (G,A)$ be the morphism induced by the identity map of G. Then*

$$V_n(f)\colon V_n(G,B) \longrightarrow V_n(G,A)$$

is an isomorphism for all $n \geq 0$.

THEOREM 1.5. *Let K be a pure subgroup of G contained in A and let $f\colon (G,A) \longrightarrow (G/K, A/K)$ be the morphism induced by the canonical epimorphism $f\colon G \longrightarrow G/K$. Then*

$$V_n(f) : V_n(G,A) \longrightarrow V_n(G/K, A/K)$$

is an isomorphism for all $n \geq 0$.

PROOF. We will use repeatedly the fact that $(H/K)[p] = (H[p] + K)/K$ for any subgroup intermediate between K and G. This fact is an easy consequence of the purity of K. Note further that $p^n(G/K)[p] = (p^nG[p] + K)/K$ for any $n \geq 0$.

We first show that $V_n(f)$ is surjective. In fact,

$$\begin{aligned} f(A^n_G) &= f\big((A \cap p^nG + p^{n+1}G)[p]\big) = \frac{(A \cap p^nG + p^{n+1}G)[p] + K}{K} \\ &= \frac{A \cap p^nG + p^{n+1}G + K}{K}[p] = \left(\frac{A \cap p^nG + K}{K} + \frac{p^{n+1}G + K}{K}\right)[p] \\ &= \left(\frac{A \cap (p^nG + K)}{K} + p^{n+1}\frac{G}{K}\right)[p] = \left(\frac{A}{K} \cap p^n\frac{G}{K} + p^{n+1}\frac{G}{K}\right)[p] \\ &= \left(\frac{A}{K}\right)^n_{G/K} \end{aligned}$$

showing the desired surjectivity according to Lemma 1.2(ii).

To verify that $V_n(f)$ is injective we compute the preimage M of the denominator of $V_n(G/K, A/K)$.

$$\begin{aligned} M &= f^{-1}\big((A/K)^{G/K}_n\big) = f^{-1}\left(\left(\frac{A}{K} \cap p^n\frac{G}{K}\right)[p] + p^{n+1}\frac{G}{K}[p]\right) \\ &= A \cap (p^nG[p] + K) + p^{n+1}G[p] + K = A^G_n + K. \end{aligned}$$

Thus $A^n_G \cap M = A^G_n + (A^n_G \cap K) = A^G_n + (A \cap p^nG + p^{n+1}G)[p] \cap K \subset A^G_n + A \cap p^nG[p] \subset A^G_n$ and so $V_n(f)$ is injective by Lemma 1.2(i). ∎

In all three preceding isomorphism theorems the groups G and A were fixed while the third group involved had various properties with respect to A and G which implied that $V_n(f)$ was an isomorphism. It would be interesting to find general necessary and sufficient conditions for $V_n(f)$ to be an isomorphism.

Our next two results give one instance where $V_n(f)$ is injective and one where $V_n(f)$ is surjective.

PROPOSITION 1.6. *Let B be an essential subgroup of A and $f: (G, B) \longrightarrow (G, A)$ the morphism induced by the identity map of G. Then $V_n(f): V_n(G, B) \longrightarrow V_n(G, A)$ is injective for all $n \geq 0$.*

PROOF. Since B is essential in A we have $B[p] = A[p]$ and $A^G_n = B^G_n$. Now

$$B^n_G \cap f^{-1}(A^G_n) = B^n_G \cap A^G_n = B^n_G \cap B^G_n = B^G_n.$$

In view of Lemma 1.2(i), $V_n(f)$ is injective. ∎

PROPOSITION 1.7. *Let B be a subgroup of G such that $A \subset B \subset \bar{A}$ where $\bar{A}$ is the closure of A in the p-adic topology of G. Let $f: (G, A) \longrightarrow (G, B)$ be the morphism induced by the identity map of G. Then $V_n(f): V_n(G, A) \longrightarrow V_n(G, B)$ is surjective for all $n \geq 0$.*

PROOF. We use again the criterion of Lemma 1.2. Note that $A \subset B \subset A + p^{n+1}G$ since $B \subset \bar{A} = \bigcap_{n=0}^{\infty}(A + p^nG)$. Therefore $A + p^{n+1}G = B + p^{n+1}G$. Now

$$f(A^n_G) = A^n_G = (A + p^{n+1}G) \cap p^nG[p] = (B + p^{n+1}G) \cap p^nG[p] = B^n_G.$$

Therefore $V_n(f)$ is an epimorphism. ∎

The n^{th} overhang functor V_n on $\mathcal{G}$ is related to the well-known n^{th} Ulm functor U_n on $\mathcal{G}$. We recall the definition of U_n.

Let (G,A), (H,B), $f\colon (G,A) \longrightarrow (H,B)$ be in $\mathcal{G}$ and let n be a non-negative integer. Then

$$U_n(G,A) = \frac{p^nG[p]}{(A+p^{n+1}G)\cap p^nG[p]} = \frac{p^nG[p]}{A^n_G},$$
$$U_n(f)\colon U_n(G,A) \longrightarrow U_n(H,B) : U_n(f)(g+A^n_G) = f(g)+B^n_H$$

($U_n(G,A)$ is usually called the *relative Ulm invariant* of A in G.) It follows immediately from the definitions that we have the short exact sequence

$$0 \longrightarrow V_n(G,A) \longrightarrow U_n(G,A[p]) \longrightarrow U_n(G,A) \longrightarrow 0.$$

Furthermore, if $A[p]$ supports a pure subgroup K of G then $U_n(G,A[p])$ is isomorphic to the n^{th} Ulm vector space of G/K, i.e., $p^n(G/K)[p]/p^{n+1}(G/K)[p]$. These facts were used implicitly in [3] in order to show that the so-called residual subgroups determined by purifiable subgroups are isomorphic.

We conclude this section with the observation that $V_n(G,A)$ can be defined in two other apparently different ways, namely,

$$V_n'(G,A) = \frac{(A+p^nG[p])\cap p^{n+1}G}{A\cap p^{n+1}G+p^{n+1}G[p]}, \quad V_n''(G,A) = \frac{(p^nG[p]+p^{n+1}G)\cap A}{A\cap p^nG[p]+(p^{n+1}G)\cap A},$$

but it is easy to see that $V_n(G,A) \cong V_n'(G,A) \cong V_n''(G,A)$. This is a general fact which can be stated thus: Let A, B, C be subgroups of a group G, and let $f(A,B,C) = \frac{(A+B)\cap C}{A\cap C+B\cap C}$. Then $f(A,B,C)$ is invariant up to isomorphism under any permutation of A, B, C.

2. **Vertical subgroups.** In this section we examine closely the consequences of the vanishing of the overhang functors. This gives rise, in any primary group, to the interesting family of "vertical subgroups". We give several characterizations of vertical subgroups and establish basic properties.

Throughout this section and the rest of the article A denotes a subgroup of a p-primary group G.

DEFINITION 2.1. A is said to be *vertical* in G if $V_n(G,A) = 0$ for all $n \geq 0$.

PROPOSITION 2.2. *If A is pure in G or if $A \subset G[p]$ then A is vertical in G.*

PROOF. If A is pure in G then we have $V_n(G,A) = V_n(A,A)$ by Theorem 1.3 and clearly $V_n(A,A) = 0$. Therefore A is vertical. If $A \subset G[p]$ then clearly $A^n_G = A^G_n$ and again A is vertical. ∎

Next we give two characterizations of verticality.

PROPOSITION 2.3. *The following properties are equivalent.*

(i) A is vertical in G.

(ii) $(A+p^nG)[p] = A[p]+p^nG[p]$ for all $n \geq 1$.

(iii) $p(A\cap p^nG) = pA\cap p^{n+1}G$ for all $n \geq 1$.

PROOF. (i)$\Rightarrow$(ii). If A is vertical then $(A+pG)[p] = A[p]+pG[p]$ because $A^0_G = A^G_0$. Suppose inductively that A satisfies (ii) for all $n \leq m$. Then $(A+p^{m+1}G)[p] \subset (A+p^mG)[p] = A[p]+p^mG[p]$ hence $(A+p^{m+1}G)[p] = (A+p^{m+1}G)\cap(A[p]+p^mG[p]) = A[p]+(A+p^{m+1}G)\cap p^mG[p] = A[p]+A^m_G = A[p]+A^G_m = A[p]+p^{m+1}G[p]$.

(ii) $\Rightarrow$(i). If A satisfies the condition (ii) for all $n \geq 0$ then $A^n_G = (A+p^{n+1}G)\cap p^nG[p] = (A+p^{n+1}G)[p]\cap p^nG = (A[p]+p^{n+1}G[p])\cap p^nG = A^G_n$ and A is vertical.

(i) $\Leftrightarrow$(iii). Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A[p]\oplus p^nG[p] & \xrightarrow{\text{ins}} & A\oplus p^nG & \xrightarrow{p} & pA\oplus p^{n+1}G & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & (A+p^nG)[p] & \xrightarrow{\text{ins}} & A+p^nG & \xrightarrow{p} & pA+p^{n+1}G & \longrightarrow & 0
\end{array}$$

The vertical maps are all defined by $(x,y)\longmapsto x+y$. The maps f' and f'' are surjective. Hence, by the Snake Lemma,

$$\operatorname{Coker} f \cong \frac{\operatorname{Ker} f''}{p\operatorname{Ker} f'}$$

It is easily checked that $\operatorname{Ker} f' \cong A\cap p^nG$ and $\operatorname{Ker} f'' \cong pA\cap p^{n+1}G$ and therefore

$$\frac{(A+p^nG)[p]}{A[p]+p^nG[p]} \cong \frac{pA\cap p^{n+1}G}{p(A\cap p^nG)}.$$

The claim is an immediate consequence. ∎

The following result appears in [3]. We reproduce it here with our terminology and a proof using the characterization 2.3(iii).

THEOREM 2.4. *A is pure in G if and only if A is neat and vertical in G.*

PROOF. If A is pure in G then A is neat in G and by Proposition 2.2 it is also vertical in G. Conversely if A is neat and vertical in G suppose by induction that $A\cap p^nG = p^nA$. Now $p^{n+1}A = p(A\cap p^nG) = pA\cap p^{n+1}G = (A\cap pG)\cap p^{n+1}G = A\cap p^{n+1}G$. Therefore $A\cap p^nG = p^nA$ for all $n\geq 1$ and A is pure in G. ∎

PROPOSITION 2.5. *If $A[p]$ is dense in $G[p]$ then A is vertical in G.*

PROOF. That $A[p]$ is dense in $G[p]$ means that $A[p]+p^nG[p] = G[p]$ for all $n\geq 1$. Thus $(A+p^nG)[p]\subset A[p]+p^nG[p]$ and, by Proposition 2.3(ii), A is vertical in G. ∎

COROLLARY 2.6. ([9], [5, 66.3]). *If S is a dense subsocle of G, then any subgroup H with $H[p] \subset S$ can be extended to a pure subgroup K of G such that $K[p] = S$.*

PROOF. Let K be a neat subgroup of G containing H and such that $K[p] = S$. Then, by Proposition 2.5, K is vertical. Thus K is neat and vertical in G, and, by Theorem 2.4, K is pure in G. ∎

LEMMA 2.7. *Let A be a subgroup of G. Then*

(i) $V_{n+m}(G,A) = V_m(p^nG, A \cap p^nG)$ for all $n, m \geq 0$.

(ii) $V_s(G,A) = 0$ for $s = 0, \ldots, n$ if and only if $(A + p^iG)[p] = A[p] + p^iG[p]$ for $i = 1, \ldots, n+1$.

(iii) If A is vertical in G then, for all n, $A \cap p^nG$ is vertical in p^nG and in G. Conversely, if for some $n \geq 1$, $A \cap p^nG$ is vertical in p^nG and $(A + p^iG)[p] = A[p] + p^iG[p]$ for $i = 1, \ldots, n$ then A is vertical in G.

PROOF. (i) It is easy to check that $A_G^{n+m} = A_{p^nG}^m$ and $A_{n+m}^G = A_m^{p^nG}$.

(ii) If $(A + p^{s+1}G)[p] = A[p] + p^{s+1}G[p]$ then $A_G^s = A_s^G$ therefore $V_s(G,A) = 0$ and we have $V_s(G,A) = 0$ for $s = 0, \ldots, n$. Conversely, if $V_s(G,A) = 0$ for $s = 0, \ldots, n$ then $(A + pG)[p] = A_G^0 = A_0^G = A[p] + pG[p]$. By induction suppose that $(A + p^kG)[p] = A[p] + p^kG[p]$ for $k < n+1$. Then $(A + p^{k+1}G)[p] \subset (A + p^kG)[p] = A[p] + p^kG[p]$ and the same argument as in the proof of Proposition 2.3 shows that $(A + p^{k+1}G)[p] = A[p] + p^{k+1}G[p]$.

(iii) If A is vertical in G then by (i) $A \cap p^nG$ is vertical in p^nG and it follows easily that $A \cap p^nG$ is vertical in G also. Conversely, if $A \cap p^nG$ is vertical in p^nG then $V_{n+m}(G,A) = 0$ for all $m \geq 0$ and by (ii) $V_s(G,A) = 0$ for $s = 0, \ldots, n-1$ so that A is vertical in G. ∎

Since verticality and neatness are intimately connected to purity we list here a useful fact involving neat subgroups.

LEMMA 2.8. *Let K be a subgroup of G and H a neat subgroup of K. If A is a subgroup of G maximal with respect to $A \cap K = H$ then A is neat in G and $A + K[p] \supset G[p]$.*

PROOF. A/H is K/H-high in G/H. It is well known that A/H is neat in G/H and that $(A/H)[p] \oplus (K/H)[p] = (G/H)[p]$. Now H being neat in K implies $(K/H)[p] = (K[p]+H)/H$. Therefore $G[p] \subset A + K[p]$. Let $pg \in A$ for some $g \in G$. Then there exists $a \in A$ and $h \in H$ such that $pg = pa + h$. Thus $p(g-a) = h$ and $g - a + H \in (G/H)[p]$. Therefore there exists $a_1 \in A$ and $k_1 \in K[p]$, such $(g-a) + H = k_1 + H + a_1 + H$ and $g = a + a_1 + k_1 + h'$ for some $h' \in H$. It follows that $pg = p(a + a_1 + h') \in pA$, and A is neat. ∎

PROPOSITION 2.9. *Let H be a pure subgroup of p^nG, $n \geq 0$. Then every subgroup A of G maximal with respect to $A \cap p^nG = H$ is pure in G.*

PROOF. From Lemma 2.8 A is neat in G and $A + p^nG[p] \supset G[p]$. Consequently $(A + p^iG)[p] = A[p] + p^iG[p] = G[p]$ for $i = 1, \ldots, n$. Moreover, $A \cap p^nG$ is vertical in p^nG. By Lemma 2.7(iii), A is vertical in G. Therefore A is neat and vertical and, by Theorem 2.4, A is pure in G. ■

We conclude this section by stating explicitly the consequences for verticality of the general theorems on the overhang functors. From Theorems 1.3, 1.4 and 1.5 we obtain:

PROPOSITION 2.10. *Let A be a subgroup of a group G.*

(1) If A is vertical in G then A is vertical in any pure subgroup K of G which contains A.

(1′) If A is vertical in a pure subgroup K of G then A is vertical in G.

(2) If A is vertical in G then every pure dense subgroup of A is vertical in G.

(2′) If some pure dense subgroup of A is vertical in G then A is vertical in G.

(3) If A is vertical in G then A/K is vertical in G/K for every K pure in G and contained in A.

(3′) If A/K is vertical in G/K for some pure subgroup K of G then A is vertical in G.

From Propositions 1.6 and 1.7 we have:

PROPOSITION 2.11. *Let A be a vertical subgroup of G. Then*

(i) every essential subgroup of A is vertical in G;

(ii) every subgroup of G between A and $\bar{A}$ is vertical in G, in particular $\bar{A}$ is vertical in G.

3. **Maximal vertical subgroups.** Recall that a *maximal vertical subgroup* is one which is maximal among the vertical subgroups supported by its socle. This section is devoted to maximal vertical subgroups. We characterize the closed vertical subgroups which are maximal and show that the p-adic closures of maximal vertical subgroups are also maximal vertical. Furthermore, we give several general instances where maximal vertical subgroups are in fact pure.

PROPOSITION 3.1. *Every subsocle of G supports a maximal vertical subgroup of G.*

PROOF. Let S be a subsocle of G and let $P = \{A < G \mid A[p] = S$ and A is vertical in $G\}$. Then $S \in P$. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a chain of elements in P. We will show that $A = \bigcup_{\lambda\in\Lambda} A_\lambda \in P$. Clearly $A[p] = S$. It remains to show that A is vertical. But $(A+p^nG)[p] = (\bigcup A_\lambda + p^nG)[p] = \bigcup(A_\lambda + p^nG)[p] = \bigcup(A_\lambda[p]+p^nG[p]) = A[p]+p^nG[p]$. Thus A is vertical in G and, by Zorn's Lemma, P contains maximal elements. ■

Clearly pure subgroups are maximal vertical subgroups. But since there exist non-purifiable subsocles, maximal vertical subgroups are not necessarily pure.

Our next results show that the p-adic closure of pure subgroups are maximal vertical subgroups. We first establish this fact for $p^\omega G$ which is the closure of $\{0\}$.

LEMMA 3.2. *If A is vertical in G and $A[p] \subset p^\omega G = \bigcap_{n=1}^{\infty} p^nG$ then $A \subset p^\omega G$.*

PROOF. Suppose by induction that $A[p^n] \subset p^\omega G$ and let $x \in A[p^{n+1}]$. Then $px \in p^\omega G$. Thus for every $m \geq 0, px = p^{m+1}g_m$ for some $g_m \in G$, so $x - p^m g_m \in (A+p^mG)[p]$.

Now A is vertical in G; therefore $x - p^m g_m = a_0 + p^m g_0 \in A[p] + (p^m G)[p], a_0 \in A[p], p^m g_0 \in G[p]$, but $a_0 \in p^\omega G$, it follows that $x \in p^m G$ for all $m \geq 0$. Thus $A[p^{n+1}] \subset p^\omega G$. Therefore $A \subset p^\omega G$. ∎

COROLLARY 3.3. *The subgroup $p^\omega G$ is the unique maximal vertical subgroup of G supported by $p^\omega G[p]$.*

PROPOSITION 3.4. *Let K be a pure subgroup of G. Then the closure $\bar{K}$ of K in the p-adic topology of G is a maximal vertical subgroup of G supported by $\bar{K}[p] = \overline{K[p]}$. It is the unique maximal vertical subgroup of G supported by $\bar{K}[p]$ which contains K.*

PROOF. $\bar{K}/K = p^\omega(G/K)$; therefore $\bar{K}/K$ is maximal vertical in G/K on $(\bar{K}/K)[p]$. But $(\bar{K}/K)[p] = (\bar{K}[p] + K)/K$ because K is pure in G. By Theorem 1.5 $\bar{K}$ is a maximal vertical subgroup of G supported by $\bar{K}[p]$. Suppose M is a maximal vertical subgroup of G containing K such that $M[p] = \bar{K}[p]$. Then M/K is vertical in G/K and, by Lemma 3.2, $M/K \subset \bar{K}/K$ so that $M \subset \bar{K}$. The maximality of M implies that $M = \bar{K}$. ∎

In view of Proposition 3.4 and the fact that the closures of pure subgroups are not in general pure, there are maximal vertical closed subgroups which are not pure. In the sequel we develop further general properties of maximal vertical subgroups. This leads to some important cases where maximal verticality implies purity. But first we establish two technical lemmas.

LEMMA 3.5. *Let A be a maximal vertical subgroup of G. Let $x \in G$ be such that $px \in A \backslash pA$. Then $h_{G/A}(x + A) = m < \infty$ and $V_m(G, A + \langle x \rangle) \neq 0$, while $V_n(G, A + \langle x \rangle) = 0$ for all $n \neq m$.*

PROOF. Note that the condition $px \in A\backslash pA$ is equivalent to saying that $(A + \langle x \rangle)[p] = A[p]$ and $x \notin A$. By the maximality of A, $A + \langle x \rangle$ is not vertical. From Proposition 2.11(ii) $x \notin \bar{A}$. Therefore $h_{G/A}(x + A) = m < \infty$. We show that $V_n(G, A + \langle x \rangle) = 0$ for all $n \neq m$. If $V_n(G, A + \langle x \rangle) \neq 0$ then there exist $a \in A, \alpha \in \mathbb{Z}$ and $g \in G$ such that $z = a + \alpha x + p^{n+1} g \in p^n G[p]$ but $z \notin (A + \langle x \rangle) \cap p^n G[p] + p^{n+1} G[p] = A \cap p^n G[p] + p^{n+1} G[p] = A_n^G$. Note that α must be relatively prime to p; otherwise $z \in A_G^n = A_n^G$. Thus without loss of generality we may assume that $\alpha = 1$. Now $h_G(x + a) \leq h_{G/A}(x + A) = m$; therefore $n \leq m$. That is to say $V_n(G, A + \langle x \rangle) = 0$ for all $n > m$. Suppose then that $n < m$. Since there exists $a_1 \in A$, such that $a_1 + x \in p^m G \subset p^{n+1} G$, we have $z = (a - a_1') + (a_1 + x) + p^{n+1} g \in A_G^n = A_n^G$. This is a contradiction. Therefore, $V_n(G, A + \langle x \rangle) = 0$ for all $n \neq m$. This means that $V_m(G, A + \langle x \rangle) \neq 0$ for otherwise $A + \langle x \rangle$ would be vertical in G. ∎

LEMMA 3.6. *Let A be a vertical subgroup of G. Then A is maximal vertical in G if and only if $A + \langle x \rangle$ is not vertical in G for every $x \in G$ such that $px \in A \backslash pA$.*

PROOF. If A is maximal vertical then $A + \langle x \rangle$ is not vertical for every $x \in G$ such that $px \in A\backslash pA$ because $(A + \langle x \rangle)[p] = A[p]$ and $x \notin A$. Conversely, suppose that $A + \langle x \rangle$ is not vertical for all $x \in G$ such that $px \in A \setminus pA, H \supset A, H[p] = A[p]$, and H is vertical. Then by Proposition 2.11(i) every subgroup of H containing $A[p]$ is vertical. If $h \in H$ and $ph \in A$ then $ph \in pA$; otherwise $A + \langle h \rangle$ is not vertical. It follows that $h \in A$. Therefore $H = A$ and A is maximal vertical. ∎

The next result is a fundamental property of maximal vertical subgroups.

THEOREM 3.7. *Let A be a maximal vertical subgroup of G. Then $A \cap pG \subset \overline{pA}$, or equivalently, $(G[p] + A)/A$ is dense in $(G/A)[p]$.*

PROOF. Suppose $a \in A \cap pG \setminus pA$. We will show that $h_{G/pA}(a + pA) = \infty$. Since $a \in pG$, we have $h_{G/pA}(a + pA) \geq m + 1$ for some $m \geq 0$. So $a = pa_1 + p^{m+1}g_1$ for some $g_1 \in G$ and $a_1 \in A$. Now let $x = a_1 + p^m g_1$; then $px = a \in A \setminus pA$. Therefore $(A + \langle x \rangle)[p] = A[p]$ and from Lemma 3.5 $h_{G/A}(x + A) = n < \infty$. Note that $n \geq m$, and $V_n(G, A + \langle x \rangle) \neq 0$. Therefore there exist, as in the proof of Lemma 3.5, $a_2 \in A$ and $g_2 \in G$ such that $a_2 + x + p^{n+1}g_2 \in p^n G[p] \setminus A_n^G$. Now $p(a_2 + x + p^{n+1}g_2) = 0$ and $a = px = -pa_2 - p^{n+2}g_2$. Therefore $h_{G/pA}(a + pA) \geq n + 2 > m + 1$. Thus $h_{G/pA}(a+pA) = \infty$ and $a \in \overline{pA}$. It is easy to see that in general the property $A \cap pG \subset \overline{pA}$ is equivalent to the condition that $(G[p] + A)/A$ be dense in $(G/A)[p]$. ∎

COROLLARY 3.8. *If G is a bounded group then the maximal vertical subgroups of G are precisely the pure subgroups of G.*

PROOF. If G is bounded then $p^\omega(G/pA) = 0$. Thus $A \cap pG = pA$. So A is neat and vertical and as such it is pure by Theorem 2.4. ∎

In general, if a group G has the property that multiplication by p maps closed subgroups onto closed subgroups then closed maximal vertical subgroups of G are pure. The class of groups in which multiplication by p is a closed endomorphism has been characterized in [2]. It is immediate from Proposition 3.4 that this class is contained in the class of quasi-closed primary groups.

PROPOSITION 3.9. *Let A be closed and vertical in G. Then A is maximal vertical in G if and only if $A \cap pG = \overline{pA}$.*

PROOF. If A is maximal vertical then by Theorem 3.7 $A \cap pG \subset \overline{pA}$. But $pA \subset A \cap pG$ and $A \cap pG$ is closed in G. Therefore $\overline{pA} \subset A \cap pG$ and $A \cap pG = \overline{pA}$. Conversely, suppose that $A \cap pG = \overline{pA}$ and let H be a vertical subgroup of G containing A such that $H[p] = A[p]$. Let $h \in H$ such that $ph \in A$. Then $ph \in A \cap pG = \overline{pA}$, and for every $i > 0$ there exist $a_i \in A, g_i \in G$ such that $ph = pa_i + p^i g_i$. Thus $h - a_i - p^{i-1}g_i \in (H + p^{i-1}G)[p] = H[p] + p^{i-1}G[p]$ since H is vertical. Thus $h - a_i - p^{i-1}g_i = h_i - p^{i-1}y_i$ where $h_i \in H[p]$ and $p^{i-1}y_i \in p^{i-1}G[p]$. However, $H[p] = A[p]$ therefore $h = a_i + h_i + p^{i-1}(g_i + y_i) \in A + p^{i-1}G$ for all $i \geq 1$. This means that $h \in \bar{A} = A$ since A is closed in G. We have shown that $(H/A)[p] = 0$. Hence $H = A$. ∎

https://doi.org/10.4153/CJM-1991-001-0 Published online by Cambridge University Press

COROLLARY 3.10. *The closure of a maximal vertical subgroup of G is again maximal vertical in G.*

PROOF. Let A be a maximal vertical subgroup of G. Then $\bar{A}$ is vertical and closed in G. Now $\overline{p\bar{A}} \subset \bar{A} \cap pG = \overline{A \cap pG}$ and since A is maximal vertical in G it follows from Theorem 3.7 that $A \cap pG \subset \overline{pA}$. Thus $\overline{A \cap pG} \subset \overline{pA}$ and

$$\overline{p\bar{A}} \subset \bar{A} \cap pG = \overline{A \cap pG} \subset \overline{pA} \subset \overline{p\bar{A}}.$$

Therefore

$$\overline{p\bar{A}} = \bar{A} \cap pG,$$

and, by Proposition 3.9, $\bar{A}$ is maximal vertical in G. ∎

We conclude this section with three general situations where maximal verticality implies purity. We need the following transfer property of maximal verticality.

PROPOSITION 3.11. *Let A be a maximal vertical subgroup of G. Then $A \cap p^nG$ is a maximal vertical subgroup of p^nG.*

PROOF. Clearly it suffices to show that $A \cap pG$ is a maximal vertical subgroup of pG. Let $x \in pG$ be such that $px \in (A \cap pG) \setminus p(A \cap pG)$. Then $px \in A$ and $px \notin pA$ since $p(A \cap pG) = pA \cap p^2G$ by Proposition 2.3(iii). Thus $(A + \langle x \rangle)[p] = A[p]$ and $x \notin A$. Therefore $A + \langle x \rangle$ is not vertical in G. Note that $h(x+A) \geq 1$ since $x \in pG$. From Lemma 3.5, $h(x + A) = m \geq 1$ and $V_m(G, A + \langle x \rangle) \neq 0$, where m is an integer. From Lemma 2.7(i) we get $V_m(G, A + \langle x \rangle) = V_{m-1}(pG, (A + \langle x \rangle) \cap pG)$ but $\langle x \rangle \subset pG$. Therefore $m - 1 \geq 0$, and $V_{m-1}(pG, A \cap pG + \langle x \rangle) = V_m(G, A + \langle x \rangle) \neq 0$. So that $A \cap pG + \langle x \rangle$ is not vertical in pG for all such x in pG. Therefore, by Lemma 3.6, $A \cap pG$ is maximal vertical in pG. Repeating the preceding argument n times shows that $A \cap p^nG$ is maximal vertical in p^nG. ∎

THEOREM 3.12. *Bounded maximal vertical subgroups of separable groups are pure.*

PROOF. Let A be a maximal vertical subgroup of G supported by $A[p]$. Suppose that $p^\omega G = 0$ and $p^nA = 0$. We show that $p^{n-1}(A \cap pG) = 0$. From Theorem 3.7 $A \cap pG \subset \overline{pA}$ but $pA \subset G[p^{n-1}]$ which is a closed subgroup of G. Thus $\overline{pA} \subset G[p^{n-1}]$. Therefore $p^{n-1}(A \cap pG) = 0$. If $n = 1$ then $A \cap pG = 0 = pA$ and A is pure in G. Suppose by induction that p^{n-1}-bounded maximal vertical subgroups of separable groups are pure. Then, in view of Proposition 3.11, and the fact that pG is separable if G is separable, it follows that $A \cap pG$ is pure in pG. Thus, by Proposition 2.9, A can be extended to a subgroup K of G maximal with respect to $K \cap pG = A \cap pG$ and this K is pure in G. Now $pK = K \cap pG = A \cap pG$. Thus $p^nK = 0$. From Proposition 2.9, A is maximal vertical in K and by Corollary 3.8, A is pure in K. Therefore A is pure in G. ∎

THEOREM 3.13. *Let A be maximal vertical in G and suppose that there exists a pure subgroup K of G such that $K \supset A \supset p^n K[p]$ for some $n \geq 0$. Then A is a pure subgroup of G.*

PROOF. A is also maximal vertical in K. $(A \cap p^n K)[p] = p^n K[p]$ and $A \cap p^n K$ is maximal vertical in $p^n K$. Therefore $A \cap p^n K = p^n K$. Indeed $A \cap p^n K$ is essential in $p^n K$ and $p^n K$ is obviously vertical in $p^n K$. It follows that $p^n(K/A) = (p^n K + A)/A \simeq p^n K/A \cap p^n K = 0$. Thus K/A is bounded. This implies that K/pA is also bounded. In fact, $p^{n+1}(K/pA) = 0$ and consequently $p^\omega(K/pA) = 0$. By Theorem 3.7, $A \cap pK = pA$ and A is neat and vertical in K. By Theorem 2.4, A is pure in K. Therefore A is pure in G. ∎

PROPOSITION 3.14. *Let S be a dense subsocle of G. Then the maximal vertical subgroups of G supported by S are pure.*

PROOF. The claim follows immediately from Proposition 2.5 and its corollary. ∎

We collect in the following proposition some additional facts about maximal vertical subgroups.

PROPOSITION 3.15. *Let A be a maximal vertical subgroup of G. Then the following hold.*

(i) A is neat in $\bar{A}$.

(ii) $\bar{A}$ is pure in G if and only if $p\bar{A}$ is closed in G.

(iii) If $\bar{A}$ is pure in G then A is also pure in G.

PROOF. (i) follows from the fact that every subgroup between A and $\bar{A}$ is vertical in G (Proposition 2.11 (ii)).

(ii) If $p\bar{A}$ is closed in G then $p\bar{A} = \overline{p\bar{A}}$. But, by Corollary 3.10, $\overline{p\bar{A}} = \bar{A} \cap pG$. Therefore $\bar{A}$ is neat and vertical. By Theorem 2.4, $\bar{A}$ is pure in G. The converse is clear.

(iii) By (i) A is neat in $\bar{A}$ and if $\bar{A}$ is pure in G then A is neat in G. Therefore A is neat and vertical in G. It follows that A is pure in G. ∎

In the next section we examine the situation when all maximal vertical subgroups are pure.

4. **Characterization of the groups in which maximal vertical subgroups are pure.** As an application of the theory of maximal verticality developed in the preceding section, we propose to determine here the class $\mathcal{M}$ of groups in which the maximal vertical subgroups are all pure. In view of Proposition 3.4 it is clear that the class of such groups is a subclass of the class $\mathcal{H}$ of groups in which the closures of pure subgroups are pure. The class $\mathcal{H}$ was first considered in [6], and the reduced groups in $\mathcal{H}$ were studied in [8] under the name of quasi-closed primary groups. An account of the results of [8], as well as additional facts about these groups can be found in [5, Section 74] under the name of *quasi-complete* groups. It is not too difficult to see that a group G belongs to $\mathcal{H}$ if and

only if its reduced part is in $\mathcal{H}$ We follow [5] here and use the appellation quasi-complete for the reduced members of $\mathcal{H}$and show that $\mathcal{M} = \mathcal{H}$. The following fact is essential.

PROPOSITION 4.1. *Let S be a subsocle of G which supports a pure subgroup K such that G/K is torsion-complete, and let A be a closed vertical subgroup of G such that $A[p] = S = K[p]$. Then pA is a closed subgroup of G.*

PROOF. In general if A is a subgroup of G then pA is closed G if and only if $G[p]+A$ is a closed subgroup of G. We show that $G[p]+A$ is closed under the given hypothesis. Let $x \in \overline{G[p]+A}$. Then for each $i \geq 1$, there exist $g_i \in G[p], a_i \in A, y_i \in G$ such that $x = g_i + a_i + p^i y_i$. It follows that

$$g_{i+1} - g_i = a_{i+1} - a_i + p^i(y_i - py_{i+1}) \in (A + p^i G)[p].$$

But A is vertical in G, therefore $(A + p^i G)[p] = A[p] + p^i G[p]$ and $g_{i+1} - g_i = a_i' + p^i z_i$ for some $a_i' \in A[p] = K[p]$ and $p^i z_i \in G[p]$. This means that $\{g_i + K\}$ is a neat Cauchy-sequence in $(G/K)[p] = (G[p]+K)/K$. Since G/K is torsion-complete we have $\lim_{i\to\infty}(g_i + K) = b + K$ where $b \in G[p]$. This sequence $\{g_i + K\}$ converges neatly to $b+K$ (see [5, Vol. I, p. 66]). Thus $b - g_i \in (p^i G + K)[p] = p^i G[p] + K[p] = p^i G[p] + A[p]$. Now $b - x = (b - g_i) - a_i + p^i y_i \in A + p^i G$, for all $i \geq 1$, i.e., $b - x \in \bar{A} = A$ and $x \in A + G[p]$. Therefore $A + G[p]$ is closed in G and pA is closed in G. ∎

Note that the subsocle S in Proposition 4.1 is a closed distinguished subsocle of G in the sense of [9].

LEMMA 4.2. *Let S be a closed subsocle of G Then all maximal vertical subgroups of G supported by S are closed in G.*

PROOF. Let A be a maximal vertical subgroup of G supported by S. Then $\bar{A}[p] = (\cap_{i=0}^{\infty}(A + p^n G))[p] = \cap_{i=0}^{\infty}(A + p^n G)[p] = \cap(A[p] + p^n G[p]) = \bar{S} = S$. But $\bar{A}$ is also vertical (Proposition 2.11 (ii)). By the maximality of A we have $\bar{A} = A$ and A is closed in G. ∎

We are now ready for the characterization.

THEOREM 4.3. *Let G be a reduced group. All maximal vertical subgroups of G are pure if and only if G is quasi-complete.*

PROOF. If all maximal vertical subgroups are pure then, since the closures of pure subgroups are maximal vertical, we see that G must be quasi-complete. Conversely, if G is quasi-complete let S be a subsocle of G and let A be a maximal vertical subgroup of G such that $A[p] = S$. If A is bounded then A is pure by Theorem 3.12. If A is unbounded then $\bar{A}$ is vertical and unbounded and $\bar{A}[p] = \bar{S}$. Let $B \supset \bar{A} \supset B[p]$ be a maximal vertical extension of $\bar{A}$ on $\bar{S}$. From Lemma 4.2, B is closed in G. Now by [5, Theorem 74.1] there exists a pure subgroup K of G such that $K[p] = \bar{S}$ and by [5, Theorem 74.5], G/K is torsion-complete. By Proposition 4.1, pB is closed in G. Now Theorem 3.7 implies that

$B \cap pG = \overline{pB} = pB$, so that B is a neat subgroup of G. From Theorem 2.4, B is a pure subgroup of G. But $A[p] = S$ is dense in $B[p] = \bar{S}$, thus A is a maximal vertical subgroup in B supported by a dense subsocle of B. By Proposition 3.14 A is pure in G.

While it is easy to see that a group G is in $\mathcal{H}$ if and only if its reduced part is in $\mathcal{H}$, it is not obvious that the same observation holds for the class $\mathcal{M}$.

LEMMA 4.4. *Let G be a group. Then $G \in \mathcal{M}$ if and only if the reduced part of G is in $\mathcal{M}$.*

PROOF. Let $G \in \mathcal{M}$ and let $G = D \oplus R$ where D is divisible and R is reduced. Let A be a maximal vertical subgroup of R then $(A \oplus D)/D$ is maximal vertical in $(D \oplus R)/D \simeq R$. By Proposition 2.10(3) $A \oplus D$ is maximal vertical in G. Thus $A \oplus D$ is pure in G and so is A. Conversely, let $G = D \oplus R$ be as above and assume only that $R \in \mathcal{M}$. Let A be a maximal vertical subgroup of G. Then $\bar{A} \supset D$. Extend $\bar{A}$ to a maximal vertical subgroup B such that $B[p] == \bar{A}[p] = \overline{A[p]}$. Then B/D is a maximal vertical subgroup of $(R \oplus D)/D \simeq R$. Therefore B is a pure subgroup of G. Now $A[p]$ is dense in $B[p]$ so that by Proposition 3.14, A is pure in G. ∎

From the preceding result it is easy to derive the following characterizations.

THEOREM 4.5. *The following properties of the group G are equivalent.*

(1) The closures of pure subgroups of G are pure.

(2) Closed maximal vertical subgroups of G are pure.

(3) All maximal vertical subgroups of G are pure.

5. **Verticality and minimal purity.** The study of vertical subgroups was motivated by the problem of purifiability of subgroups. In this section we apply the results obtained for verticality to characterize the groups in which the subgroups which stand a chance of being purifiable are in fact purifiable. First we define what we mean by purifiable subgroups.

DEFINITION 5.1. A subgroup A of a primary group G is said to be *purifiable* in G if there exists a pure subgroup K of G containing A minimal among the pure subgroups of G that contain A. Such a subgroup K is said to be a *pure hull* of A in G.

This notion was first investigated in [4]. Some important facts about minimal purity appear in [7] and [1]. But in [3] a new and intrinsic necessary condition was discovered: for a subgroup A of a group G to be purifiable in G it is necessary that there exists $m \geq 0$, such that $V_n(G, A) = 0$ for all $n \geq m$. This is equivalent to saying that $(p^m G) \cap A$ is vertical in $p^m G$ for some $m \geq 0$.

DEFINITION 5.2. A subgroup A of a group G is said to be *eventually-vertical* in G if there exists $m \geq 0$ such that $A \cap p^m G$ is vertical in $p^m G$.

Thus purifiable subgroups are eventually-vertical. The converse does not hold in general. In fact, we show here that eventually-vertical subgroups of a group G are purifiable if and only if G is in the class $\mathcal{M}$, i.e., the reduced part of G is a quasi-complete group. A key result is the following.

THEOREM 5.3. *Let A be a subgroup of G. If A is purifiable in G then $A \cap p^nG$ is purifiable in p^nG for all $n \geq 0$. Conversely, if $A \cap p^nG$ is purifiable in p^nG for some $n \geq 1$ then A is purifiable in G.*

PROOF. Let K be a pure hull of A in G then $p^nK = (p^nG) \cap K \supset A \cap p^nG$ and p^nK is a pure hull of $A \cap p^nG$ in p^nG. Indeed, if H is a pure subgroup of p^nG and $p^nK \supset H \supset A \cap p^nG = A \cap p^nK$, by Proposition 2.9, we can extend $A + H$ to a pure subgroup M of K such that $M \cap p^nK = H$. (Note $(A + H) \cap p^nK = H + A \cap p^nK = H$.) Thus $M = K$ and $H = p^nK$. Conversely, suppose $A \cap p^nG$ is purifiable in p^nG and let H be a pure hull of $A \cap p^nG$ in p^nG. As above $(A + H) \cap p^nG = H$ and $A + H$ can be extended to a pure subgroup K of G such that $K \cap p^nG = H$. Clearly, $H = p^nK$. By [1, Theorem 2.1] there exists $m \geq 0$ such that $p^mH[p] \subset p^nG$. Thus $p^{m+n}K[p] \subset A \subset K$. Again by [1, Theorem 2.1], A is purifiable. ∎

PROPOSITION 5.4. *Let A be a vertical subgroup of G. A is purifiable in G if and only if some maximal vertical essential extension of A in G is pure in G.*

PROOF. Let A be purifiable, and let K be a pure hull of A. Then, by [1, Theorem 2.1], there exists $m \geq 0$ such that $p^mK[p] \subset A$. Let B be a maximal vertical extension in K of A such that $A[p] = B[p]$. Then, by Theorem 3.13, B is a pure subgroup of K and thus B is pure in G. By the minimality of K it follows that $B = K$. The converse is obvious. ∎

THEOREM 5.5. *The following are equivalent for a group G.*
(i) All eventually-vertical subgroups of G are purifiable in G.
(ii) The reduced part of G is a quasi-complete group.

PROOF. (i)⇒(ii). By hypothesis, maximal vertical subgroups of G are purifiable and since they are maximal, they are pure by Proposition 5.4. Therefore G is a member of the class $\mathcal{M}$. From Theorem 4.5 it follows that G is in the class $\mathcal{H}$ and the reduced part of G is quasi-complete. Conversely, if G is in the class $\mathcal{H}$ then p^nG is also in $\mathcal{H}$ for all n. Indeed, if H is pure in p^nG then there exists by Proposition 2.9 a pure subgroup K of G such that $K \cap p^nG = H$. Now $\bar{K}$ is pure in G and $\bar{K} \cap p^nG$ is the closure of $K \cap p^nG$ in p^nG. Thus $\bar{H} = \bar{K} \cap p^nG$ is pure in p^nG. Let A be an eventually-vertical subgroup of G. Then $A \cap p^nG$ is vertical in p^nG for some $n \geq 0$. Since $p^nG \in \mathcal{H}$. Theorem 4.5 implies that the maximal vertical essential extensions of $A \cap p^nG$ in p^nG are pure. Therefore $A \cap p^nG$ is purifiable in p^nG. By Theorem 5.3, A is purifiable in G. ∎

As a last remark we mention that the connection between class $\mathcal{H}$ and minimal purity was first noticed by T. J. Head in [6]. He characterized the class $\mathcal{H}$ by the property that the closures of pure subgroups are purifiable if and only if these closures are pure. More generally he identified some purifiable subgroups of members of $\mathcal{H}$, namely those subgroups which are between a pure subgroup and its closure. Our results, however, determine all purifiable subgroups of members of $\mathcal{H}$. The example $\bar{P}$ constructed in [6] is an explicit example of a maximal vertical closed subgroup which is not pure.

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https://doi.org/10.4153/CJM-1991-001-0 Published online by Cambridge University Press